Freie Universität Berlin

Gödel’s Incompleteness Theorem

Valeria Zahoransky

Index

Some basic notions

Note

For a given language/system $\mathcal{L}$, we mean

- $\mathcal{E}$ to be the set of all expressions over $\mathcal{L}$
- $\mathcal{H} \subset \mathcal{E}$ to be the set of all predicates over $\mathcal{L}$
- $\mathcal{S} \subset \mathcal{E}$ to be the set of all sentences over $\mathcal{L}$
- $\mathcal{P} \subset \mathcal{S}$ to be the set of all provable sentences over $\mathcal{L}$
- $\mathcal{R} \subset \mathcal{S}$ to be the set of all refutable sentences over $\mathcal{L}$
- $\mathcal{T} \subset \mathcal{S}$ to be the set of all true sentences over $\mathcal{L}$

Peano Arithmetic

Example

Let $\mathcal{L}$ be the language of Peano Arithmetic over alphabet $\Sigma = \{0, S, +, \cdot, <\}$. We have

$$\exists x(2 < x), \quad 1 + 1, \quad (y + x) = 3, \quad 2 < x \quad \in \mathcal{E}$$

$$2 < x \in \mathcal{H}$$

$$\exists x(2 < x) \in \mathcal{T}$$

Note

For predicate $H \equiv 2 < x$ we write $H(x)$.

Expressibility in $\mathcal{L}$

Definition

Let $n \in \mathbb{N}$, H a predicate, we say n satisfies H if $H(n)$ is a true sentence.
Set $A \in \mathbb{N}$ is expressed by H iff

$$\forall n \in \mathbb{N} : \; H(n) \in \mathfrak{T} \Leftrightarrow n \in A$$

A is expressible in $\mathcal{L}$ if there is some predicate H_A of $\mathcal{L}$, s.t. A is expressed by H_A.

Example

In PA set $A = \{0, 1\}$ is expressed by $H_A \equiv x < 2$.

Correctness of $\mathcal{L}$

Definition

$\mathcal{L}$ is a correct language if

(i) $\mathcal{P} \subset \mathcal{T}$

(ii) $\mathcal{T} \cap \mathcal{R} = \emptyset$

Note

Correctness is stronger than consistency:

$$\mathcal{L} \text{ consistent } \Leftrightarrow \forall s \in \mathcal{S} : \neg(s \in \mathcal{P} \wedge s \in \mathcal{R})$$

Gödel numbering

Definition

Given an injective function $g: \mathcal{E} \rightarrow \mathbb{N}$ and expression $E \in \mathcal{E}$ we call

$$g(E)$$

the Gödel number of E and g a Gödel numbering for language $\mathcal{L}$.

Note

We have seen Gödel numberings as encodings , e.g. of Turing Machines.

$$\langle TM \rangle = g(TM)$$

Gödel numbering

Example

For Peano Arithmetic with extended alphabet $\Gamma = \Sigma \cup \{\neg, \wedge, \rightarrow, \exists, x, (,)\}$ we can define a suitable g as follows:

- Assign each $\sigma \in \Gamma$ a number:

0	S	$+$	$\cdot$	$<$	$\neg$	$\wedge$	$\rightarrow$	$\exists$	x	(	)
0	1	2	3	4	5	6	7	8	9	A	B

- Create E' by substituting every symbol by its assigned number.
- the Gödel number of E is the number whose base 12 representation is E'.

Gödel numbering

Example

$$
\begin{aligned}
g(1 < x) &= g((S\,0) < x) \\
(S\,0) < x &\longrightarrow A10B49 \\
g(1 < x) &= 10 \cdot 12^5 + 1 \cdot 12^4 + 0 \cdot 12^3 + 11 \cdot 12^2 + 4 \cdot 12^1 + 9 \cdot 12^0 \\
&= A10B49_{12} = 2510697_{10}
\end{aligned}
$$

Diagonalisation

Definition

Let E_n be the unique expression, s.t. $g(E_n) = n$.
We call $E_n(n)$ the diagonalisation of E_n and

$$d\colon \mathbb{N} \to \mathbb{N}, \quad d(n) = g(E_n(n))$$

the diagonal function .

Note

$E_n(n)$ is true iff E_n is satisfied by its own Gödel number n.
$d(n)$ is the Gödel number of sentence $E_n(n)$.

A^* and $\overline{A}$

Definition

For any set $A \subset \mathbb{N}$, let A^* be the preimage of A under d:

$$n \in \mathbb{N}: \; n \in A^* \Leftrightarrow d(n) \in A$$

For any set $A \subset \mathbb{N}$, let $\overline{A}$ be the complement of A:

$$\overline{A} = \mathbb{N} \setminus A$$

Little Gödel

Theorem

For a given language $\mathcal{L}$, *let* $P = \{n \mid \exists S \in \mathcal{P} : g(S) = n\}$ *be the set of Gödel numbers of all provable sentences .*
If set $\overline{P}^*$ *is expressible in* $\mathcal{L}$ *and* $\mathcal{L}$ *is correct, then there is a true sentence of* $\mathcal{L}$ *which is not provable in* $\mathcal{L}$.

Little Gödel - Proof I

Proof

Let H be the predicate that expresses $\overline{P}^*$. Let h be the Gödel number of H.

Since H expresses $\overline{P}^*$, for any $n \in \mathbb{N}$:

$$H(n) \text{ true } \Leftrightarrow n \in \overline{P}^*$$

And thus:

$$\begin{aligned} H(h) \text{ true } &\Leftrightarrow h \in \overline{P}^* \\ &\Leftrightarrow d(h) \in \overline{P} \Leftrightarrow d(h) \notin P \end{aligned}$$

Little Gödel - Proof II

Proof

Note that $d(h) = g(H(h))$, so by definition of P we get

$$d(h) \in P \Leftrightarrow H(h) \text{ provable}$$

and thus

$$H(h) \text{ true } \Leftrightarrow d(h) \notin P \Leftrightarrow H(h) \text{ not provable}$$

Case 1 $H(h)$ is false and provable. Contradiction by correctness of $\mathcal{L}$.

Case 2 $H(h)$ is true and not provable. □

Gödel's Incompleteness Theorem I

Theorem

Any consistent formal system $\mathcal{L}$ that has a certain expressivity is incomplete.

Note

Gödel used so-called ω-consistency which is stronger than consistency.

Gödel's Incompleteness Theorem II

Theorem

No formal system $\mathcal{L}$ can be consistent and prove its own consistency.

Thank you for your attention!

Questions?

Gödel's Incompleteness Theorem I

Note

Statement according to Gödel:

> *Zu jeder* ω *-widerspruchsfreien rekursiven Klasse* κ *von Formeln gibt es rekursive Klassenzeichen* r *, so daß weder* $v \text{ Gen } r$ *noch* $Neg(v \text{ Gen } r)$ *zu* $Flg(\kappa)$ *gehört (wobei* v *die freie Variable aus* r *ist).*
>
> *For any* ω *-consistent recursive class* κ *of formulae, there are recursive class-signs* r *, s.t. neither* $v \text{ Gen } r$ *nor* $Neg(v \text{ Gen } r)$ *belong to* $Flg(\kappa)$ *(where* v *is the free variable of* r *).*

Gödel's Incompleteness Theorem II

Note

Statement according to Gödel:

> *Sei* κ *eine beliebige rekursive widerspruchsfreie Klasse von Formeln, dann gilt: Die Satzformel, welche besagt, daß* κ *widerspruchsfrei ist, ist nicht* κ *-beweisbar; insbesondere ist die Widerspruchsfreiheit von* P *in* P *unbeweisbar, vorausgesetzt, daß* P *widerspruchsfrei ist [...].*
>
> *If* κ *be a given recursive, consistent class of formulae, then the propositional formula which states that* κ *is consistent is not* κ *-provable; in particular, the consistency of* P *is unprovable in* P *, it being assumed that* P *is consistent [...].*

Bibliography

[1] Raymond M. Smullyan. *Gödel's Incompleteness Theorems*. Oxford University Press, 1992.

[2] Kurt Gödel. *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*. Monatshefte für Mathematik und Physik, Springer, 1931.

[3] Kurt Gödel. *On formally undecidable propositions of Principia Mathematica and related systems I*. Dover Publications, 1992.