Implications of Gödel's Incompleteness Theorems

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Seminar: selected works of Kurt Gödel
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"Kurt Gödel's achievement in modern logic is singular and monumental – indeed it is more than a monument, it is a landmark which will remain visible far in space and time. ... The subject of logic has certainly completely changed its nature and possibilities with Gödel's achievement."

- John von Neumann

Agenda

- Impact on Mathematics
    - Historical Background
    - Philosophy of Mathematics
    - Consequences of Gödel’s Theorems
- Impact on Debate on Human Mind
    - Different Philosophers
    - Discussion

Historical Background

- 19th century: emphasis on abstract characterization of mathematical structures instead of algorithmic concerns
- Reflection on basis of mathematical terms:
    - Cantor - Set Theory
    - Frege - Grundgesetze der Arithmetik
    - Peano - Axioms on Natural Numbers
    - Hilbert - Axioms on Geometry

→ Agree on basics of mathematics

Historical Background

Russells Paradoxon: set of all sets of that are not elements of themselves

$$\text{let } R = \{x \mid x \notin x\}$$

$$\text{then } R \in R \iff R \notin R$$

→ Reliability of mathematical intuition is doubted and notion of proof is questioned, reflecting on the basis of mathematics

→ **Foundational crisis of mathematics**

Questions of Philosophy of Mathematics

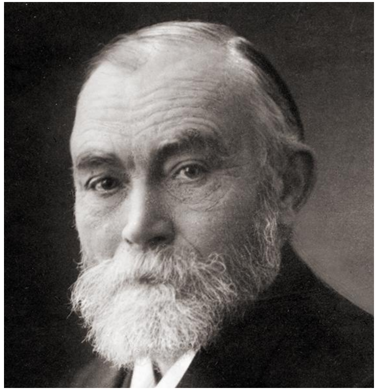
- What are mathematical objects?
- How can human beings know about them?

→ Mathematical proofs are necessary in order to gain mathematical knowledge

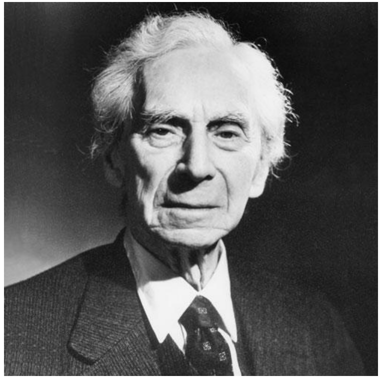
Schools of Philosophy

Logicism:
mathematical statements follow axioms of pure logic

- Truths of mathematics are a priori
- Logic foundation of mathematics → mathematical statements are logical truths, based on logical concepts
- Foundation of Principia Mathematica (1910-1913) by Whiteheat and Russell
  - System of ramification: definition quantifies only over concepts whose definitions are logically prior
  - Axiom of reducibility



Gottlob Frege (1848-1925)



Bertrand Russell (1872-1970)

Schools of Philosophy

Intuitionism:
derive mathematics from methods verified by reason

- Classic mathematics transcends intuition
- “there are no non-experienced truth”
- true statements exist due to thinking and verification:

$A \vee \neg A$	provable or disprovable

Luitzen Brouwer (1881-1966)

Schools of Philosophy

Formalism:
derive mathematics from axiomatic systems

- Formalize all theorems to gain formal system:

axioms + rules → statements

→ no character of truth

→ Program of metamathematics

David Hilbert (1862-1943)

Hilbert’s Program

- International Congress of Mathematics (1900):
  23 mathematical problems

  → 2nd problem addresses consistency of arithmetic axioms

- Hilbert’s Program (1920s):

  find axiomatic basis for all mathematics and provide a proof of consistency

Gödel

- PhD Thesis (1929): **Completeness Theorem**
  Semantic truth and syntactic provability are correspondent in first-order logic

- Conference at Königsberg (1930): **First Incompleteness Theorem**
  For any consistent, non-trivial formal system will be statements that are true but unprovable

- Monatsheft für Mathematik (1932): **Second Incompleteness Theorem**
  No system can demonstrate its own consistency

Consequences

- Every non-trivial formal system is either incomplete or inconsistent

  → Hilbert's program impossible

  → FOL incomplete, HOL inconsistent

  → whole of mathematics can be inconsistent

- There exist true statements which cannot be proved

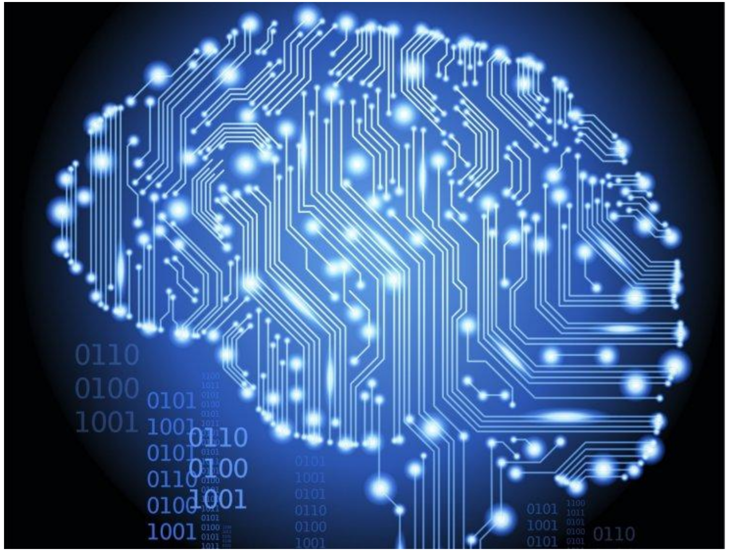
  → could turn out untrue at some point

- Reality cannot be fully addressed through formal means

Artificial Intelligence

- Is strong Artificial Intelligence possible?
- Can a Turing Machine fully represent a human being?

→ Incompleteness theorems would also apply to human beings



Lucas Argument

- Machine is concrete instantiation of a formal system:
  human being capable of enunciating truths of arithmetic
  Gödel formula cannot be proved-in-the-system
  human being recognizes true statement
- Unprovable statements: introduce more powerful machine to solve
- Machine's reaction deterministic → human beings have no free will

→ Human ratio cannot be explained as machine

Rogers Argument

- Machine should be able to perform implications non-deductively
- Machine can judge axiom as true without proving it by keeping a list

↯ Pair of axioms cannot be added: otherwise Gödel formular and negation get accepted

↯ Accepting only one axiom does not work: negation could get accepted

Penrose Argument

- Exploration of truth not based on concrete algorithm: heuristic reasoning, insight, inspiration

- Mathematician seeks for source of errors
  → consistency

- Understanding is essential, which machines cannot

→ Limitation of AI systems

Nagel, Newman

- Machine has corresponding axiomatic system
- Machine can solve a concrete problem, but one machine cannot solve all
- Human brain is limited, but (still) superior in some aspects
- Structure and power of human mind complex and subtle

Open Questions

- Is machine necessarily instantiation of formal system?
- Are human beings consistent?
- Can a machine understand? Can human brain be equivalent to a machine?
- Do human beings have a free will?

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