# Kurt Gödel Seminar

## Further Works based on “Interpretation eines logischen Aussagenkalküls”

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Interpretation of IPC into S4 McKinsey, Tarski

Other Translations from S4

More Recent Theoretic Translations of Logics

# Interpretation of IPC into S4
McKinsey, Tarski

# Proof Idea of McKinsey and Tarski

- The Lewis System S4 can be regarded as a matrix that is equal to a closure algebra
- For every formula $\alpha$ of S4 there is a closure-algebraic function $f^{(\alpha)}$
- $\alpha$ is provable iff $f^{(\alpha)} = 1$
- The Heyting System IPC can be regarded as a matrix, that is equal to a Brouwer algebra
- For every formula $\alpha$ of IPC there is a Brouwerian-algebraic function $f^{(\alpha)}$
- $\alpha$ is provable iff $f^{(\alpha)}$ vanishes
- Closure algebras and Brouwer algebras are equivalent
- Translation from $\mathbf{IPC} \to \mathbf{S4}$

# The Lewis System S4

There are infinitely many variables arranged in an infinite sequence.
We denote the *n*th variable by $v_n$.
Particulary:

$$p = v_1, q = v_2, r = v_3, s = v_4$$

| | | |
|---|---|---|
| | Negation | $\sim$ |
| There are three constant symbols: | Conjunction | $\wedge$ |
| | Possibility | $\Diamond$ |

In general, possibility can define

$$\Box x = \sim \Diamond \sim x$$

# The Lewis System S4

| | | | | |
|---|---|---|---|---|
| | Disjunction | $\alpha \vee \beta$ | for | $\sim (\sim \alpha \wedge \sim \beta)$ |
| | Material Implication | $\alpha \rightarrow \beta$ | for | $\sim (\alpha \wedge \sim \beta)$ |
| Further symbols are: | Material Equivalence | $\alpha \leftrightarrow \beta$ | for | $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$ |
| | Strict Implication | $\alpha \supset \beta$ | for | $\sim \Diamond (\alpha \wedge \sim \beta)$ |
| | Strict Equivalence | $\alpha \equiv \beta$ | for | $(\alpha \supset \beta) \wedge (\beta \supset \alpha)$ |

Our inference rule is *Detachment*/Modus Ponens:

$$\alpha = (\beta \rightarrow \gamma)$$

**Definition**
A set $S$ is a topological space with respect to a unary closure operations $\mathbf{C}$ when:

1. If $A \subseteq S$ then $A \subseteq \mathbf{C}A = \mathbf{C}\mathbf{C}A \subseteq S$
2. If $A, B \subseteq S$ then $\mathbf{C}(A \bigcup B) = \mathbf{C}A \bigcup \mathbf{C}B$
3. If $A \subseteq S$, and $A$ contains at most one point, then $\mathbf{C}A = A$

**Definition**
The Interior of an element $x$ is

$$\mathbf{I}x = -\mathbf{C} - x$$

Consider the following:

$$\Box x = \sim \Diamond \sim x$$

# Closure Operation

### Definition

- An element $x$ is called closed, if $\mathbf{C}x = x$
- An element $x$ is called open, if $\mathbf{I}x = x$

### Definition

- The empty element is denoted as $\bigwedge = x \bigcap -x$
- The universe element is denoted as $\bigvee = x \bigcup -x$

**Definition**
A set $K$ is a closure algebra with respect to the operations $\bigcup, \bigcap, -$ (set minus) and, $\mathbf{C}$ when:

1. $K$ is a boolean algebra with respect to $\bigcup, \bigcap, -$ (set minus)
2. If $x \in K$ then $\mathbf{C}x \in K$
3. If $x \in K$ then $x \subseteq \mathbf{C}x$
4. If $x \in K$ then $\mathbf{C}\mathbf{C}x = \mathbf{C}x$
5. If $x, y \in K$ then $\mathbf{C}(x \bigcup y) = \mathbf{C}x \bigcup \mathbf{C}y$
6. If $\mathbf{C} \bigvee = \bigvee$

# Closure-Algebraic Function

Definition

1. If $\alpha = v_n$ for some $n$, then $f^{(\alpha)}$ is the function determined by the equation

$$f^{(\alpha)}(x_1, \cdots, x_n) = x_n$$

   for all elements $x_1, \cdots, x_n$ of every closure algebra

2. If $\alpha$ is a formula of index $m$, $\beta$ a formula of index $n$, and $r = \max(m, n)$, then $f^{(\alpha \wedge \beta)}$ is the function defined by the equation

$$f^{(\alpha \wedge \beta)}(x_1, \cdots, x_r) = f^{(\alpha)}(x_1, \cdots, x_m) \cdot f^{(\beta)}(x_1, \cdots, x_n)$$

3. If $\alpha$ is a formula of index $n$, then $f^{(\sim\alpha)}$ and $f^{(\Diamond\alpha)}$ are functions defined by the equations

$$f^{(\sim\alpha)}(x_1, \cdots, x_n) = -f^{(\alpha)}(x_1, \cdots, x_m)$$

$$f^{(\Diamond\alpha)}(x_1, \cdots, x_n) = \mathbf{C}f^{(\alpha)}(x_1, \cdots, x_m)$$

**Theorem**
For every formula $\alpha$ of the Lewis calculus, the following conditions are equivalent:

- $\alpha$ is provebale in S4 (the Lewis System)
- $f^{(\alpha)}$ is identically to $1$ in every closure algebra

**Theorem**
If $\alpha$ is provable in S4 (the Lewis System), then $\sim \Diamond \sim \alpha$ is provable in S4

**Proof**
Since $\alpha$ is provable in S4, we see that $f^{(\alpha)}$ is identically equal to $1$ in every closure algebra. From this we easily conclude that $-\mathbf{C} - f^{(\alpha)}$ or $f^{(\sim\Diamond\sim\alpha)}$ is identically equal to $1$ in every closure algebra, from which it follows that $\sim \Diamond \sim \alpha$ is provable in S4.

# Proof of a Primitive Rule

**Theorem**
If $\sim \Diamond \sim \alpha \vee \sim \Diamond \sim \beta$ is provable in the Lewis System, then either $\alpha$ or $\beta$ is provable in the Lewis System.

**Proof**
If $\sim \Diamond \sim \alpha \vee \sim \Diamond \sim \beta$ is provable in S4, we see that $\gamma = \sim \Diamond \sim \alpha \vee \sim \Diamond \sim \beta$, $f^{(\gamma)}$ is identically equal to 1 in every closure algebra. Hence, $-\mathbf{C} - f^{(\alpha)} + -\mathbf{C} - f^{(\beta)}$ is identically equal to 1 in every closure algebra, from which we can conclude, that either $f^{(\alpha)}$ is identically to 1 or $f^{(\beta)}$ is identically to 1. Our theorem follows then from the above definition.

Which gives us the disjunction property for S4.

There are infinitely many variables arranged in an infinite sequence.
We denote the *n*th variable by $v_n$.
Particulary:

$$p = v_1, q = v_2, r = v_3, s = v_4$$

There are four constant symbols:

| | |
|---|---|
| Negation | $\sim$ |
| Conjunction | $\wedge$ |
| Disjunction | $\vee$ |
| Implication | $\rightarrow$ |

# Brouwerian-Algebraic Function

Definition

1. If $\alpha = v_n$ for some $n$, then $f^{(\alpha)}$ is the function determined by the equation $f^{(\alpha)}(x_1, \ldots, x_n) = x_n$ for all elements $x_1, \ldots, x_n$ of every Brouwerian algebra
2. If $\alpha$ is a formula of index $n$, then $f^{(\sim\alpha)}$ is the function defined by the equation

$$f^{(\sim\alpha)}(x_1, \ldots, x_n) = \neg f^{(\alpha)}(x_1, \ldots, x_m)$$

3. If $\alpha$ is a formula of index $m$, $\beta$ a formula of index $n$, and $r = max(m, n)$, then $f^{(\alpha\wedge\beta)}$, $f^{(\alpha\vee\beta)}$, $f^{(\alpha\rightarrow\beta)}$ are the functions defined by the equations

$$f^{(\alpha\wedge\beta)}(x_1, \ldots, x_r) = f^{(\alpha)}(x_1, \ldots, x_m) + f^{(\beta)}(x_1, \ldots, x_n)$$

$$f^{(\alpha\vee\beta)}(x_1, \ldots, x_r) = f^{(\alpha)}(x_1, \ldots, x_m) \cdot f^{(\beta)}(x_1, \ldots, x_n)$$

$$f^{(\alpha\rightarrow\beta)}(x_1, \ldots, x_r) = f^{(\beta)}(x_1, \ldots, x_m) \dot{-} f^{(\alpha)}(x_1, \ldots, x_n)$$

**Theorem**
For every formula $\alpha$ of the Heyting calculus, the following conditions are equivalent:

- $\alpha$ is provebale in the Heyting Calculus
- $f^{(\alpha)}$ vanishes in every Brouwerian algebra

**Theorem**
If $\alpha \vee \beta$ is provable in the Heyting calculus, then either $\alpha$ is provable in the Heyting Calculus or $\beta$ is provable in the Heyting Calculus.

# Translation of the Heyting Calculus into S4

**Theorem**
Let $T$ be a function defined over all formulas of the Heyting calculus, assuming as values formulas of the Lewis calculus, and satisfying the following conditions:

1. $T(v_i) = \sim \Diamond \sim v_i$
2. $T(\alpha \vee \beta) = T(\alpha) \vee T(\beta)$
3. $T(\alpha \wedge \beta) = T(\alpha) \wedge T(\beta)$
4. $T(\alpha \rightarrow \beta) = T(\alpha) \supset T(\beta)$
5. $T(\sim \alpha) = \sim \Diamond T(\alpha)$

Then for any formula $\alpha$ of the Heyting calculus, $\alpha$ is provable in the Heyting calculus if and only if $T(\alpha)$ is provable in the Lewis system.

**Proof**

Let $\alpha$ be any formula of the Heyting calculus; suppose that $\alpha$ is of index $n$. By the Closure- Algebraic Funtion we see that $T(\alpha)$ is provable in the Lewis system if and only if the equation

$$f^{(T(\alpha))}(x_1, \ldots, x_n) = 1$$

is true for all elements $x_1, \ldots, x_n$ of every closure algebra. By condition 1 of the hypothesis of our theorem, it is then seen that $T(\alpha)$ is provable in the Lewis system iff the above formula is true for all open elements of every closure algebra. By means of conditions 2-5 of the hypothesis of our theorem, and the equivalence of Brouwerian and Closure Algebras we then see that $T(\alpha)$ is provable in the Lewis system if and only if the equation

$$f^{(\alpha)}(x_1, \ldots, x_n) = 0$$

is true for all elements of every Brouwerian algebra. Our theorem now follows from the Brouwerian-Algebraic function.

# Translation of the Heyting Calculus into S4

$T(\alpha)$ is provable in S4 iff $T(v_i) = \sim \Diamond \sim v_i$ is true for all open elements

An element $x$ is called open, if $\mathbf{I}x = x$

$$\mathbf{I}x = -\mathbf{C} - x$$

# Other Translations from S4

# First Order Translation

Rasiowa and Sikorski (1953) proved, that Gödel's translation also holds for first order predicate calculus.

## Translation

$$
\begin{aligned}
&RS : \mathbf{IQC} \to \mathbf{QS_4} \\
&(F^k_m(x_{i_1}, \ldots, x_{i_k}))^{RS} && =_{df} \Box(F^k_m(x_{i_1}, \ldots, x_{i_k})) \\
&(\varphi \vee \psi)^{RS} && =_{df} \varphi^{RS} \vee \psi^{RS} \\
&(\varphi \wedge \psi)^{RS} && =_{df} \varphi^{RS} \cdot \psi^{RS} \\
&(\varphi \supset \psi)^{RS} && =_{df} \Box(\sim \varphi^{RS} \vee \psi^{RS}) \\
&(\neg\varphi)^{RS} && =_{df} \Box(\sim \varphi^{RS}) \\
&(\exists x \varphi)^{RS} && =_{df} \exists x \varphi^{RS} \\
&(\forall x \varphi)^{RS} && =_{df} \Box\forall x \varphi^{RS}
\end{aligned}
$$

# First Order Based on Gödel's Translation

Based on Gödel's translation we get the translation for Quantified Intuitionistic Logic by adding to:

$$Gd : IPC \to \mathcal{G}$$

$$(p)^{Gd} =_{df} p$$

$$(\neg\varphi)^{Gd} =_{df} \sim B(\varphi^{Gd})$$

$$(\varphi \supset \psi)^{Gd} =_{df} B(\varphi^{Gd}) \to B(\psi^{Gd})$$

$$(\varphi \vee \psi)^{Gd} =_{df} B(\varphi^{Gd}) \vee B(\psi^{Gd})$$

$$(\varphi \wedge \psi)^{Gd} =_{df} \varphi^{Gd} \cdot \psi^{Gd}$$

the following:

$$(\exists x\varphi)^{Gd} =_{df} \exists x \Box \varphi^{Gd}$$

$$(\forall x\varphi)^{Gd} =_{df} \forall x\varphi^{Gd}$$

# Weakening S4 into S3

Hacking (1963) also proved by using cut-elimination, that S4 can be weakened to S3 and Gödel's conjecture still holds. For this we define S3 as the following:

## Definition of S3

| S3 | S4 |
|---|---|
| S3 is an extension of PC | S4 is an extension of PC |
| $\alpha$ is a tautology or an axiom, then $\vdash \Box\alpha$ | If $\alpha$ then $\vdash \Box\alpha$ |
| $\Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q)$ | $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ |
| $\Box p \rightarrow p$ | $\Box p \rightarrow p$ |
| | $\Box p \rightarrow \Box\Box p$ |

Definition

- $S : \mathbf{S_4} \to \mathbf{PA}$
- $(p_i)^S =_{df} p_i$
- $(\bot)^S =_{df} \bot$
- $S$ commutes with $\sim, \vee, \cdot, \to$
- $(\Box\varphi)^S =_{df} Bew(\ulcorner\varphi^S\urcorner)$

where $\ulcorner\varphi\urcorner$ denotes the numeral of the Gödel number of $\varphi$, $Bew$ is the canonically defined predicate expressing arithmetized provability in $\mathbf{PA}$.
So $Bew(\ulcorner\varphi^S\urcorner)$ is the formula expressing, that $\varphi^S$ is the Gödel number of a theorem of $\mathbf{PA}$.
It is then shown how:

Theorem
$\mathbf{S4} \vdash \varphi$ if and only if $\mathbf{PA} \vdash \varphi^S$

# Goldblatt's Correction (1978)

Goldblatt found a problem: Not every translation $(\Box\varphi \rightarrow \varphi)^S$ is a theorem of $\mathbf{PA}$. Because it is not the case that $\mathbf{PA} \vdash Bew(\ulcorner\varphi\urcorner)^S \rightarrow \varphi^S$, since by the incompleteness of $\mathbf{PA}$ it is known, that there are true sentences of arithmetic, which are not theorems.

**Original Formula**

$(\Box\varphi)^S =_{df} Bew(\ulcorner\varphi^S\urcorner)$

**Correction**

$(\Box\varphi)^{GS} =_{df} \varphi \cdot Bew(\ulcorner\varphi\urcorner)$

# More Recent Theoretic Translations of Logics

# Prawitz' and Malmnäs' Definition of Translation (1968)

The first general definition of translations between logic systems is given.
Translation: $t$ is a function, that maps the set of formulas from the logic system $\mathbf{S_1}$ into $\mathbf{S_2}$

$$\mathbf{S_1} \vdash \alpha \text{ if, and only if, } \mathbf{S_2} \vdash t(\alpha)$$

Then $\mathbf{S_1}$ is *interpretable* in $\mathbf{S_2}$ by $t$.

If $\mathbf{S_1}$ is *interpretable with respect to derivability* in $\mathbf{S_2}$ by $t$, then

$$\mathbf{S_1} \cup \Gamma \vdash \alpha \text{ if, and only if } \mathbf{S_2} \cup t(\Gamma) \vdash t(\alpha)$$

# Epstein's and Krajewsi's Definition of Translation (1990)

Translation is defined in semantical terms as a map $t$ from a propositional logic $\mathbf{L}$ into a propositional logic $\mathbf{M}$, such that

$$\mathbf{L}, \Gamma \models \alpha \text{ if, and only if } \mathbf{M}, t(\Gamma) \models t(\alpha),$$

for every set $\Gamma \cup \{\alpha\}$ of formulas.

# D'Ottaviano's and Feitosa's Definition of Translation (1999)

- Define a category whose objects are logics and whose morphisms are translation
- A logic $\mathbf{A}$ is a pair $\langle A, C_A \rangle$, where the set $A$ is the domain of $\mathbf{A}$ and $C_A$ is the consequence (closure) operator
- A translation from a logic $\mathbf{A}$ into logic $\mathbf{B}$ is a map $t : A \to B$ preserving consequence relations, for any $X \subseteq A$

$$t(C_A(X)) \subseteq C_B(t(X))$$

# Sources


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