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Seminar: Gödel

Implications of Gödel's Incompleteness Theorems

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1 Introduction

> Kurt Gödel's achievement in modern logic is singular and monumental – indeed it is more than a monument, it is a landmark which will remain visible far in space and time. ... The subject of logic has certainly completely changed its nature and possibilities with Gödel's achievement.[1]

Kurt Gödel's work influenced mathematics immensely. He is one of the most significant logicians of all times. This paper focuses on the consequences of Gödel's most famous work: the Incompleteness Theorems. The historical context with emphasis on the philosophy of mathematics is explored. Even though the impact of Kurt Gödel's work is mostly felt in mathematics, his contributions are also considered in other areas. In the second part of the paper, the effect of the Incompleteness Theorems on the debate on human mind is presented to show some implications outside of mathematics.

2 Historical Background

2.1 Mathematics of the 19th Century

The 19th century marks a turning point in the history of mathematics. Before mathematicians are dealing with algorithms in order to solve concrete problems. Now the emphasis lies on the abstract characterisation of mathematical structures. The abstraction leads to a reflection on the basis of mathematical terms. Several mathematicians work on defining structures.[2]
In 1874, Cantor establishes the set theory. Sets are classified according to cardinality. This concept is also applied to the continuum with the result that real numbers are more numerous than natural numbers.[3] In 1884, Frege publishes his work on the foundation of arithmetics. He approaches this concern by reducing

[1] Yasugi and Passell, 2003, p. 36, quote by John von Neumann
[2] Boundas, 2007
[3] Kleene, 1987 p. 141 f

mathematics to logic. In 1889, Peano announces the axioms on natural numbers, which are important for the proof of Incompleteness Theorems. Finally in 1899, Hilbert publishes the axioms on Geometry.[4]
Zermelo and Russell independently discover a weak point in set theory. The Russell's Paradox considers R the set of all sets that are not elements of themselves. The question is whether R is element of itself or not. Both response options lead to a contradiction.
The discovery of the paradox is a shock to the mathematicians at the time and introduces the foundational crisis of mathematics. The reliability of mathematical intuition is doubted and one reflects on the basis of mathematics.[5]

2.2 Schools of Philosophy

Different philosophical schools of mathematics try to solve the problem and end the foundational crisis. The philosophy of mathematics deals with the nature of mathematical objects and whether they exist independently or whether a human being is necessary for their existence. The different schools provide different answers, but common to all is the belief that mathematical proofs are necessary in order to gain mathematical knowledge.[6]
The school of Logicism follows the attempt of Frege. The idea is that truths of mathematics are perceived a priori. One arrives at statements though reasoning and logical thinking. Mathematical statements are nothing but logical truths and mathematical concepts are based on logical concepts. Therefore logic is the foundation of mathematics. This approach is taken up by Whitehead and Russell as the foundation of the Principia Mathematica published between 1910 and 1913. In order to avoid the Paradox, Russell and Whitehead introduce the system of ramification. It allows a definition to quantify only over objects that are logically prior.[7] Due to the weakness of the system, the axiom of reducibility is added. The approach applied to the Principia Mathematica is not widely accepted and

[4] Mancosu, Zach, and Badesa, 2004, p.8
[5] Kleene, 1987 p. 147
[6] Boundas, 2007
[7] Kleene, 1987 p. 148

especially the axiom of reducibility is strongly criticised.[8]

The school of Intuitionism is opposed to the idea that mathematical statements follow axioms of pure logic. Instead of transcending intuition as done in classic mathematics, mathematics should be derived from methods verified by reason.[9]

According to Brouwer, the main representative of Intuitionism, "there are no non-experienced truths and logic is not an absolutely reliable instrument to discover truths". Whereas in classic logic the term $A \vee \neg A$ denotes "it holds either A or not A", in intuitionistic logic it signifies "either A is provable or it is disprovable". As consequence, proof by contradiction is evicted and statements dealing with the infinite are problematic.[10] As imagining the infinite is beyond intuition, Brouwer introduces "potential infinite". To this idea Hilbert replies: "No one shall drive us out of the paradise which Cantor has created for us".[11]

Hilbert is the main representative of a third school of philosophy called Formalism. The aim is to formalise all mathematical theorems in order to gain a formal system. Through combining axioms and rules one arrives at statements.[12] Hilbert introduces the programme of metamathematics. The intention is to analyse mathematics itself by applying mathematical methods to it.

2.3 Gödel and the Hilbert's Program

At the International Congress of Mathematics in 1900 Hilbert announces 23 mathematical problems that should be solved in the coming century. The second problem addresses the consistency of arithmetic axioms which is a predecessor of Hilbert's Program announced in the 1920s. The program demands to find an axiomatic basis for all mathematics and provide a proof of consistency. The trend in the mathematical society is to solve Hilbert's request.[13]

Gödel publishes the Completeness Theorem as his PhD Thesis in 1929 which asserts that semantic truth and syntactic provability are correspondent in first-order logic. It signifies that FOL is consistent and is seen as a step further to

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[8] Coquand, 2018
[9] Kleene, 1987 p. 148 f
[10] Boundas, 2007
[11] Benacerraf and Putnam, 1983, p.191
[12] Kleene, 1987 p. 150 f
[13] Zach, 2015

solving Hilbert's program.[14] In 1930 at the Conference in Königsberg, Gödel announces the First Incompleteness Theorem: It holds that for any consistent, non-trivial formal system, there will be statements that are true but unprovable. Few people are attending Gödel's announcement and only von Neumann recognises the significance of the theorem.[15] Gödel and von Neumann independently arrive at the Second Incompleteness Theorem[16] which Gödel publishes in the Monatsheft für Mathematik in 1932: No system can demonstrate its own consistency.[17]

The consequences of the Incompleteness theorems are acute and frustrating for many mathematicians. First, it holds that every non-trivial formal system is either incomplete or inconsistent. Therefore Hilbert's program terminates as it is impossible. A formal system is defined as non-trivial if it is strong enough to express the natural numbers, thus higher-order logic is incomplete.[18] One problem that arises is that one cannot be sure that all of mathematics is consistent as it cannot be proved. Second, there exist true statements which cannot be proved. It is problematic, because when dealing with an unprovable statement, it could turn out to be untrue at some point. Third, reality cannot be fully addressed through formal means.

3 Debate on Human Mind

As demonstrated above, the Incompleteness Theorems impact mathematics immensely. Another field that is affected by Gödel's work is Artificial Intelligence. It is debated whether a machine is able to perform the full range of human cognitive abilites.

The philosopher Lucas argues that a machine is necessarily an instantiation of a formal system. Given that human beings are capable of enunciating truths of

[14] Boundas, 2007
[15] Raatikainen, 2015
[16] Standing in contrast with presentations held at the Gödel Seminar: Insights of Gödel's Notebooks give rise to the assumption that von Neumann found the Second Incompleteness Theorem first. Gödel wrote down a proof after receiving a letter by von Neumann that contained his proof.
[17] Kleene, 1987 S. 145
[18] Horsten, 2017

arithmetic, a machine performing human abilities needs to be non-trivial. Therefore the Incompleteness Theorems hold and there exists a Gödel formula that cannot be proved in the system. Human beings on the other hand recognise the true statement, as they are observing the paradox on a meta-level. Alternatively, one could introduce a more powerful machine to solve the unprovable statement. Again, for this machine exists a Gödel formula and the problem is not solved, but more complex machines are needed ad infinitum.
Roger objects to the argument that human ratio cannot be represented by a machine. In order to solve the Gödel formula, a machine should be able to perform implications non-deductively.
One suggested approach is to create a machine that keeps a list of axioms which are judged as true without proving the axiom. The first objection to that is that a pair of axioms cannot be added, otherwise the Gödel formula and its negation would get accepted which leads to a contradiction. The second objection is that accepting just one axiom could lead to treating the negation of the Gödel formula as true.[19]
The mathematician Penrose expands Lucas' approach. The claim is that exploration of truth is not based on concrete algorithms but requires aspects such as reasoning, insight and inspiration. These aspects are essential to human beings but cannot be implemented in machines.[20] If a mathematician is confronted with an error during the process of exploration, one seeks for the source of the error. This can be taken as a hint at the consistency of a human being. A machine on the other hand would continue its work despite the error and is therefore subordinate to human beings. Furthermore, understanding is essential in the uncovering of truth which machines cannot achieve. To conclude, AI systems are limited and cannot perform the full range of human cognitive abilities.[21]
Finally, also Nagel and Newman discuss the problem of human mind. Again it is assumed that any machine has a corresponding axiomatic system. A machine can solve one or more concrete problems, but one machine cannot solve all problems (e.g. the Gödel formula). The human brain is limited and in multiple instances machines are superior. Yet there are aspects in which human beings are more

[19] Lucas, 1961
[20] Penrose, 1991p. 408-422
[21] Penrose, 1994, p. 379 ff

advanced. Nagel and Newman claim that in the future machines will evolve and could eventually be able to perform all human abilities. It is controversial whether the human mind can be fully represented by a machine. The debate hints at the complexity and subtlety of structure and power of human mind.[22]

[22] Nagel and Newman, 2001, p. 109-113

References

Benacerraf, P. & Putnam, H. (1983). *Philosophy of mathematics - selected readings.* Cambridge University Press.

Boundas, C. V. (2007). The edinburgh companion to the 20th century philosophies.

Coquand, T. (2018). Type theory. Retrieved from https://plato.stanford.edu/entries/type-theory/. (retrieved April 1, 2019)

Horsten, L. (2017). Philosophy of mathematics. Retrieved from https://plato.stanford.edu/entries/philosophy-mathematics/#FouSch. (retrieved April 1, 2019)

Kleene, S. C. (1987). *Kurt gödel 1906-1978.* National Academy of Sciences.

Lucas, J. (1961). Minds, machines and gödel.

Mancosu, P., Zach, R., & Badesa, C. (2004). *The development of mathematical logic from russell to tarski: 1900–1935.* The Development of Modern Logic.

Nagel, E. & Newman, J. R. (2001). *Gödel's proof.* New York University Press.

Penrose, R. (1991). *Computerdenken.* Spektrum der Wissenschaft.

Penrose, R. (1994). *Shadows of the mind.* Oxford University Press.

Raatikainen, P. (2015). Gödel's incompleteness theorems. Retrieved from https://plato.stanford.edu/entries/goedel-incompleteness/. (retrieved April 1, 2019)

Yasugi, M. & Passell, N. (2003). *Memoirs of a proof theorist.* World Scientific Publishing.

Zach, R. (2015). Hilbert's program. Retrieved from https://plato.stanford.edu/entries/hilbert-program/. (retrieved April 1, 2019)