Some Works Based on Kurt Gödel's "Eine Interpretation des Intuitionistischen Aussagenkalküls"

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Abstract

We will take a look at the proof by McKinsey and Tarski on a translation of intuitionistic predicate calculus into S4 modal logic. This work was heavily influenced by Gödel's translation which didn't provide a proof for the converse translation. Their proof uses the connection of these logics to algebraic definitions of topology. Further, we will take a look at other translations of logic systems and some notions of defining translations themselves.

Introduction

With his 1933 paper ”Eine Interpretation des Intuitionistischen Aussagenkalküls”, Gödel formulated one of the first translations of logical systems. Predated by for example Kolmogorov in 1925 or Glivenko in 1929. Gödel provided a translation of intuitionistic predicate calculus into the modal logic system S4. The novelty within his approach laid in adding axioms and rules to predicate calculus and thereby producing a system equivalent to S4.

This was the basis for many works of various authors to follow. At first we will take a closer look at one of those works: McKinsey and Tarski in their 1948 paper ”Some theorems about the Sentential Calculi of Lewis and Heyting” proved the converse of Gödel's translation. There approach was to show equivalences between the logics and certain algebraic structures, which they established in previous works on topology and algebra. In particular ”The Algebra of Topology” and ”On Closed Elements in Closure Algebras” by both McKinsey and Tarski as well as ”A solution of the Decision Problem for the Lewis systems S2 and S4, with an Application to Topolgy” by McKinsey alone. They had also shown, that these two algebraic structures are equivalent as well and by this method established the full translation.

Many more authors' works were based on Gödel's work and also those of McKinsey and Tarski. With for example Hacking who in his 1963 paper found that the system S4 could be weakened to S3, with a fitting translation still holding. Moreover Solovay found a translation in 1976 from S4 to Peano arithmetic with Goldblatt two years later, correcting one of the translation rules which had led to a system, that contradicted Gödel's second Incompleteness theorem.

By reestablishing Gödel's proof in translation-theoretic terms, Prawitz and Malmnäs founded another field of study: The theoretical study of translations. There many other defintions of translations have been argued and we will finish with a defintion, that involves defining translations with terms introduced in the paper of our main interest of McKinsey and Tarski.

1 Interpretation of IPC into S4 by McKinsey & Tarski

First we define our Lewis system S4: There are infinitely many variables arranged in an infinite sequence.
We denote the nth variable by v_n.
Particulary:

$$p = v_1, q = v_2, r = v_3, s = v_4$$

	Negation	$\sim$
There are three constant symbols:	Conjunction	$\wedge$
	Possibility	$\Diamond$

And we define derivability and provability in terms of the operation of *Detachment*/Modus Ponens:

$$\alpha = (\beta \to \gamma)$$

In favour of brevity we dismiss the definition of the axioms of S4.

We now also have to define the closure operation. A set S is a topological space with respect to a unary closure operations $\mathbf{C}$ when:

1. If $A \subseteq S$ then $A \subseteq \mathbf{C}A = \mathbf{CC}A \subseteq S$
2. If $A, B \subseteq S$ then $\mathbf{C}(A \bigcup B) = \mathbf{C}A \bigcup \mathbf{C}B$
3. If $A \subseteq S$, and A contains at most one point, then $\mathbf{C}A = A$

We also define the interior of an element x as

$$\mathbf{I}x = -\mathbf{C} - x$$

The trained logician can already notice a similarity to a well known pattern:

$$\Box x = \sim \Diamond \sim x$$

Take note that an element x is called closed, if $\mathbf{C}x = x$ and an element x is called open, if $\mathbf{I}x = x$.

McKinsey and Tarski start their proof by showing, that the system S4 can be regarded as a matrix, which is equal to a closure algebra. This closure algebra in return is a boolean algebra with respect to the closure operation. Now, for every formula α of S4, we can also regard this matrix as a closure-algebraic function $f^{(\alpha)}$. And finally for the mapping of a formula of S4 into the closure algebra:

1. If $\alpha = v_n$ for some n, then $f^{(\alpha)}$ is the function determined by the equation

   $$f^{(\alpha)}(x_1, \cdots, x_n) = x_n$$

   for all elements $x_1, \cdots, x_n$ of every closure algebra

2. If α is a formula of index m, β a formula of index n, and $r = \max(m, n)$, then $f^{(\alpha \wedge \beta)}$ is the function defined by the equation

   $$f^{(\alpha \wedge \beta)}(x_1, \cdots, x_r) = f^{(\alpha)}(x_1, \cdots, x_m) \cdot f^{(\beta)}(x_1, \cdots, x_n)$$

3. If α is a formula of index n, then $f^{(\sim\alpha)}$ and $f^{(\Diamond\alpha)}$ are functions defined by the equations

   $$f^{(\sim\alpha)}(x_1, \cdots, x_n) = -f^{(\alpha)}(x_1, \cdots, x_m)$$

   $$f^{(\Diamond\alpha)}(x_1, \cdots, x_n) = \mathbf{C}f^{(\alpha)}(x_1, \cdots, x_m)$$

Where α is a formula of S4 if and only if $f^{(\alpha)}$ is identically equal to 1 in every Closure algebra.

This is the first side of the proof.

For the other side of IPC into a Brouwer algebra we take a similar approach. The defintion of IPC as follows: There are infinitely many variables arranged in an infinite sequence.

We denote the nth variable by v_n.

Particulary:

$$p = v_1, q = v_2, r = v_3, s = v_4$$

There are four constant symbols:

Negation	$\sim$
Conjunction	$\wedge$
Disjunction	$\vee$
Implication	$\rightarrow$

Again we define derivability and provability in terms of the operation of *Detachment*/Modus Ponens:

$$\alpha = (\beta \rightarrow \gamma)$$

We also disregard a definition of the axioms of IPC. As the system IPC can be regarded as a Brouwerian algebra, as shown by McKinsey and Tarski, we can again define a mapping:

1. If $\alpha = v_n$ for some n, then $f^{(\alpha)}$ is the function determined by the equation $f^{(\alpha)}(x_1, \ldots, x_n) = x_n$ for all elements $x_1, \ldots, x_n$ of every Brouwerian algebra

2. If α is a formula of index n, then $f^{(\sim\alpha)}$ is the function defined by the equation

   $$f^{(\sim\alpha)}(x_1, \ldots, x_n) = \neg f^{(\alpha)}(x_1, \ldots, x_m)$$

3. If α is a formula of index m, β a formula of index n, and $r = max(m,n)$, then $f^{(\alpha\wedge\beta)}$, $f^{(\alpha\vee\beta)}$, $f^{(\alpha\rightarrow\beta)}$ are the functions defined by the equations

$$f^{(\alpha\wedge\beta)}(x_1,\ldots,x_r) = f^{(\alpha)}(x_1,\ldots,x_m) + f^{(\beta)}(x_1,\ldots,x_n)$$

$$f^{(\alpha\vee\beta)}(x_1,\ldots,x_r) = f^{(\alpha)}(x_1,\ldots,x_m) \cdot f^{(\beta)}(x_1,\ldots,x_n)$$

$$f^{(\alpha\rightarrow\beta)}(x_1,\ldots,x_r) = f^{(\beta)}(x_1,\ldots,x_m) \dot{-} f^{(\alpha)}(x_1,\ldots,x_n)$$

Now we are finally able to tackle the translation of IPC into S4.
Let T be a function defined over all formulas of the Heyting calculus, assuming as values formulas of the Lewis calculus, and satisfying the following conditions:

1. $T(v_i) = \sim\Diamond\sim v_i$
2. $T(\alpha\vee\beta) = T(\alpha)\vee T(\beta)$
3. $T(\alpha\wedge\beta) = T(\alpha)\wedge T(\beta)$
4. $T(\alpha\rightarrow\beta) = T(\alpha)\supset T(\beta)$
5. $T(\sim\alpha) = \sim\Diamond T(\alpha)$

Then for any formula α of the Heyting calculus, α is provable in the Heyting calculus if and only if $T(\alpha)$ is provable in the Lewis system.

Let α be any formula of the Heyting calculus; suppose that α is of index n. By the closure-algebraic funtion we see that $T(\alpha)$ is provable in the Lewis system if and only if the equation

$$f^{(T(\alpha))}(x_1,\ldots,x_n) = 1$$

is true for all elements $x_1,\ldots,x_n$ of every closure algebra. By condition 1 of the hypothesis of our theorem, it is then seen that $T(\alpha)$ is provable in the Lewis system if and only if the above formula is true for all open elements of every closure algebra, so $T(\alpha)$ is provable in S4 if and only if $T(v_i) = \sim\Diamond\sim v_i$ is true for all open elements. We remember, that an element x is called open, if $\mathbf{I}x = x$ and $\mathbf{I}x = -\mathbf{C} - x$. By means of conditions 2-5 of the translation, and the equivalence of Brouwerian and Closure algebras we then see that $T(\alpha)$ is provable in the S4 if and only if the equation

$$f^{(\alpha)}(x_1,\ldots,x_n) = 0$$

is true for all elements of every Brouwerian algebra. Our theorem now follows from the Brouwerian-algebraic function.
Because of the equivalences of the Closure-algebraic and the Brouwerian-algebraic function as well as the equivalence of those two algebras, we get the translation from IPC into S4 as well as S4 into IPC.

2 Other Translations from S4

Now for some picks of other translations. Rasiowa and Sikorski (1953) proved, that Gödel's translation also holds for first order predicate calculus.
This translation can be achieved by adding the rules for the quantifiers to Gödel's original translation for quantified intuitionistic logic by adding to

$$
\begin{array}{ll}
Gd : IPC \to \mathcal{G} & \\
(p)^{Gd} & =_{df} p \\
(\neg\varphi)^{Gd} & =_{df} \sim B(\varphi^{Gd}) \\
(\varphi \supset \psi)^{Gd} & =_{df} B(\varphi^{Gd}) \to B(\psi^{Gd}) \\
(\varphi \vee \psi)^{Gd} & =_{df} B(\varphi^{Gd}) \vee B(\psi^{Gd}) \\
(\varphi \wedge \psi)^{Gd} & =_{df} \varphi^{Gd} \cdot \psi^{Gd} \\
\text{the following:} & \\
(\exists x \varphi)^{Gd} & =_{df} \exists x \Box \varphi^{Gd} \\
(\forall x \varphi)^{Gd} & =_{df} \forall x \varphi^{Gd}
\end{array}
$$

Hacking (1963) proved by using cut-elimination, that S4 can be weakened to S3 and Gödel's conjecture still holds. For this we define S3:

S3 is an extension of PC

α is a tautology or an axiom, then $\vdash \Box\alpha$

$\Box\,(p \to q) \to \Box(\Box p \to \Box q)$

$\Box\, p \to p$

Another translation, which now not only maps into an underlying logical system, but from S4 modal logic into Peano arithmetic was found by Solovay in 1976:

- $S : \mathbf{S_4} \to \mathbf{PA}$
- $(p_i)^S =_{df} p_i$
- $(\bot)^S =_{df} \bot$
- S commutes with $\sim, \vee, \cdot, \to$
- $(\Box\varphi)^S =_{df} Bew(\ulcorner\varphi^S\urcorner)$

where $\ulcorner\varphi\urcorner$ denotes the numeral of the Gödel number of φ, Bew is the canonically defined predicate expressing arithmetized provability in **PA**.
So $Bew(\ulcorner\varphi^S\urcorner)$ is the formula expressing, that φ^S is the Gödel number of a theorem of **PA**. He then concluded: $\mathbf{S4} \vdash \varphi$ if and only if $\mathbf{PA} \vdash \varphi^S$

But Goldblatt found a problem within his translation. Not every translation $(\Box\varphi \to \varphi)^S$ is a theorem of **PA**. Because it is not the case that $\mathbf{PA} \vdash Bew(\ulcorner\varphi\urcorner)^S \to \varphi^S$, since by the incompleteness of **PA** it is known, that there are true sentences of arithmetic, which are not theorems. So to correct Solovay's translation he changed one translation rule.
Original Formula

$$(\Box\varphi)^S =_{df} Bew(\ulcorner\varphi^S\urcorner)$$

Correction

$$(\Box\varphi)^{GS} =_{df} \varphi \cdot Bew(\ulcorner\varphi\urcorner)$$

3 More Recent Theoretic Translations of Logics

The first general definition of translations between logic systems is given by Prawitz and Malmnäs in 1968. They reproduced the translation from intuitionistic quantified calculus into quantified S4 modal logic as well as some further translations to minimal logic.
For their proofs they used natural deduction systems. They went on defining translations themselves as follows: t is a function, that maps the set of formulas from the logic system $\mathbf{S_1}$ into $\mathbf{S_2}$

$$\mathbf{S_1} \vdash \alpha \text{ if, and only if, } \mathbf{S_2} \vdash t(\alpha)$$

Then $\mathbf{S_1}$ is *interpretable* in $\mathbf{S_2}$ by t.
If $\mathbf{S_1}$ is *interpretable with respect to derivability* in $\mathbf{S_2}$ by t, then

$$\mathbf{S_1} \cup \Gamma \vdash \alpha \text{ if, and only if } \mathbf{S_2} \cup t(\Gamma) \vdash t(\alpha)$$

In recent years the Group for Theoretic and Applied Logic (GTAL) at the State University of Campinas has worked on definitions of translations with respect to category theory and found a strong connection to topology.

- Define a category whose objects are logics and whose morphisms are translation
- A logic $\mathbf{A}$ is a pair $\langle A, C_A \rangle$, where the set A is the domain of $\mathbf{A}$ and C_A is the consequence (closure) operator
- A translation from a logic $\mathbf{A}$ into logic $\mathbf{B}$ is a map $t : A \to B$ preserving consequence relations, for any $X \subseteq A$
$$t(C_A(X)) \subseteq C_B(t(X))$$

They remark, that their consequence operator has the same properties as the closure operator, defined above in McKinsey's and Tarski's paper.

Conclusion

We have just seen a few selected works, that pay respect to Gödel's "Eine Interpretation des Intuitionistischen Aussagenkalküls" but hopefully got an idea on how influencial his very short paper was. There are even some notions missing, with for example his own remarks on incompleteness with regard to his translation. But we could see, that translations are not even possible between logical systems, but also directly into Peano arithmetic, regardless of the underlying logical system. This raises questions on what it means, that all those systems seem to be translateable into one another, which will hopefully be studied in the following years.

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