FU Berlin, 27.2.2019

Completeness and Quantum theory

From the spectral gap to EPR and back again

Oliver Passon
Bergische Universität Wuppertal
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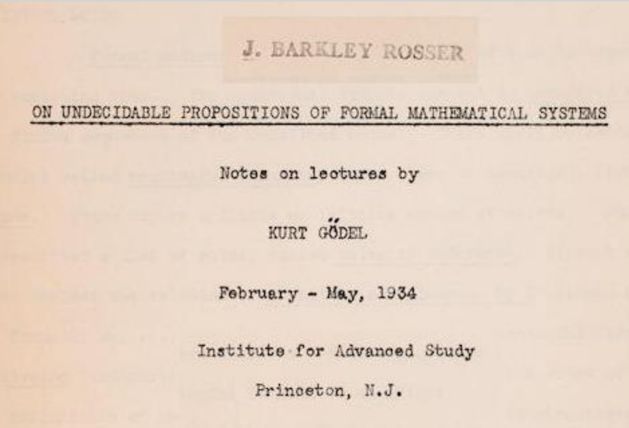

J. BARKLEY ROSSER

ON UNDECIDABLE PROPOSITIONS OF FORMAL MATHEMATICAL SYSTEMS

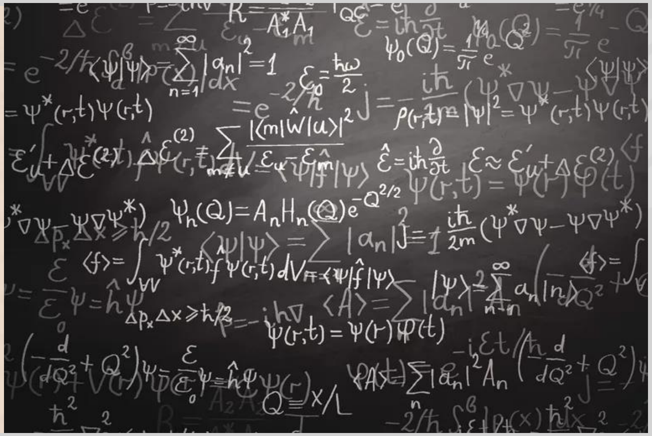
Notes on lectures by

KURT GÖDEL

February - May, 1934

Institute for Advanced Study

Princeton, N.J.



Outline of the talk

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- Three papers and how they relate…

ARTICLE

10 DECEMBER 2015 | VOL 528 | NATURE | 207

doi:10.1038/nature16059

Undecidability of the spectral gap

Toby S. Cubitt[1,2], David Perez-Garcia[3,4] & Michael M. Wolf[5]

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

Physics Vol. 1, No. 3, pp. 195–200, 1964 Physics Publishing Co. Printed in the United States

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†

Department of Physics, University of Wisconsin, Madison, Wisconsin

(*Received* 4 *November* 1964)

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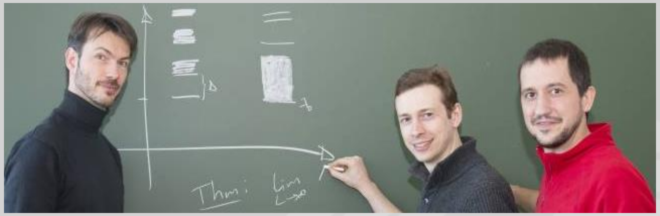
Undecidability of the spectral gap

Toby S. Cubitt[1,2], David Perez-Garcia[3,4] & Michael M. Wolf[5]



VOLUME 64, NUMBER 20 PHYSICAL REVIEW LETTERS 14 MAY 1990

Unpredictability and Undecidability in Dynamical Systems

Cristopher Moore
Department of Physics, Cornell University, Ithaca, New York 14853
(Received 18 August 1989)

VOLUME 54, NUMBER 8 PHYSICAL REVIEW LETTERS 25 FEBRUARY 1985

Undecidability and Intractability in Theoretical Physics

Stephen Wolfram
The Institute for Advanced Study, Princeton, New Jersey 08540
(Received 26 October 1984)

PHYSICAL REVIEW VOLUME 133, NUMBER 2B 27 JANUARY 1964

Undecidability of Macroscopically Distinguishable States in Quantum Field Theory

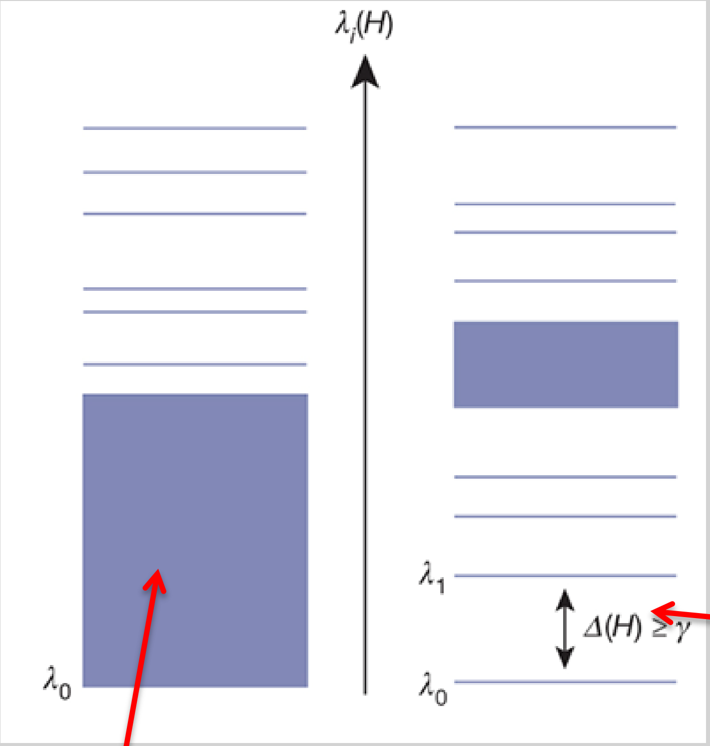
ARTHUR KOMAR*
Syracuse University, Syracuse, New York, and Yeshiva University, New York, New York
(Received 9 September 1963)

4 August 1972, Volume 177, Number 4047 SCIENCE

More Is Different

P. W. Anderson

The problem: Is a given many body system gapped or gapless?



“The main result of this paper is to show that the spectral gap for 2D translationally invariant, nearest-neighbour quantum spin systems on the square lattice, both for open and periodic boundary conditions, is undecidable.In other words, there cannot exist any algorithm – no matter how inefficient – which, given a description of any such system, determines whether it is gapped or not.”

A **gapped system** has a unique ground state $\lambda_0(H)$ and a constant lower-bound γ on the spectral gap $\Delta(H) = \lambda_1 - \lambda_0$ in the thermodynamic limit.

A **gapless system** has continuous energy spectrum $\lambda_i(H)$ above the ground state.

Proof strategy, scope and limitations

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The theorem is proven by reducing the halting problem to the spectral gap problem (via: low energy properties and tiling…)

To this end, define the local Hilbert space to be $\mathcal{H} := \mathcal{H}_c \otimes (\mathcal{H}_e \oplus \mathcal{H}_q) \simeq \mathbb{C}^C \otimes (|0\rangle \oplus \mathbb{C}^Q)$. The Hamiltonian H is defined in terms of the two-body interactions as follows:

$$h^{\text{col}}_{j,j+1} = h^{\text{col}}_c \otimes \mathbb{1}^{(j)}_{eq} \otimes \mathbb{1}^{(j+1)}_{eq} \quad (17a)$$

$$h^{\text{row}}_{i,i+1} = h^{\text{row}}_c \otimes \mathbb{1}^{(i)}_{eq} \otimes \mathbb{1}^{(i+1)}_{eq} \quad (17b)$$

$$+ \mathbb{1}^{(i)}_c \otimes \mathbb{1}^{(i+1)}_c \otimes h_q \quad (17c)$$

$$+ |\square\rangle\langle\square|^{(i)}_c \otimes (\mathbb{1}_{eq} - |\textcircled{<}\rangle\langle\textcircled{<}|)^{(i)} \otimes \mathbb{1}^{(i+1)}_{ceq} \quad (17d)$$

$$+ (\mathbb{1}_c - |\square\rangle\langle\square|_c)^{(i)} \otimes |\textcircled{<}\rangle\langle\textcircled{<}|^{(i)} \otimes \mathbb{1}^{(i+1)}_{ceq} \quad (17e)$$

$$+ \mathbb{1}^{(i)}_{ceq} \otimes |\square\rangle\langle\square|^{(i+1)}_c \otimes (\mathbb{1}_{eq} - |\textcircled{>}\rangle\langle\textcircled{>}|)^{(i+1)} \quad (17f)$$

$$+ \mathbb{1}^{(i)}_{ceq} \otimes (\mathbb{1}_c - |\square\rangle\langle\square|_c)^{(i+1)} \otimes |\textcircled{>}\rangle\langle\textcircled{>}|^{(i+1)} \quad (17g)$$

$$+ \mathbb{1}^{(i)}_c \otimes |0\rangle\langle 0|^{(i)}_e \otimes |\square\rangle\langle\square|^{(i+1)}_c \otimes \mathbb{1}^{(i+1)}_{eq} \quad (17h)$$

$$+ |\square\rangle\langle\square|^{(i)}_c \otimes \mathbb{1}^{(i)}_{eq} \otimes \mathbb{1}^{(i+1)}_c \otimes |0\rangle\langle 0|^{(i+1)}_e \quad (17i)$$

$$+ \mathbb{1}^{(i)}_c \otimes |0\rangle\langle 0|^{(i)}_e \otimes (\mathbb{1}_c - |\square\rangle\langle\square|_c)^{(i+1)} \otimes (\mathbb{1}_{eq} - |0\rangle\langle 0|_e)^{(i+1)} \quad (17j)$$

$$+ (\mathbb{1}_c - |\square\rangle\langle\square|_c)^{(i)} \otimes (\mathbb{1}_{eq} - |0\rangle\langle 0|_e)^{(i)} \otimes \mathbb{1}^{(i+1)}_c \otimes |0\rangle\langle 0|^{(i+1)}_e, \quad (17k)$$

Scope and limitations

- The spectral gap is one of the most important physical properties of a quantum many-body system, determining much of its low energy physics
- The theorem states that the SGP is "algorithmically undecidable" – i.e. for specific cases it may be decidable indeed! (Like: Hilbert's 10^{th} problem and Fermat!)
- The applied model of the many-body system is highly idealized
- The result applies to infinite systems only…
  - …but shows, that finite systems may change abruptly if made larger (→ phase transition driven by size, i.e. "more is different")

Algorithmic undecidability translates directly into axiomatic independence[1], hence:

Cubitt et al.: Quantum mechanics is incomplete (in the sense of Gödel's first theorem)

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

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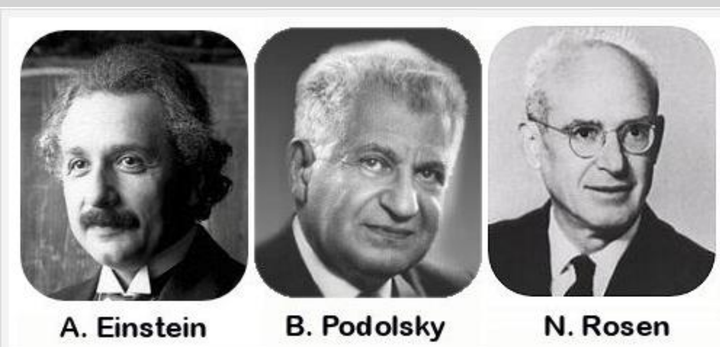
A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*
(Received March 25, 1935)



Question: How do these two strands of the QM (in-)completeness debate relate?

[1]Poonen, B. (2014) Undecidability problems: a sampler. In: J. Kennedy (Ed.) Interpreting Gödel: Critical Essays, Cambridge University Press, pp 211-241.

In 1930 the city of Königsberg hosted the "Tagung für Erkenntnislehre der exakten Wissenschaften" (in connection with the "Physiker-, Mathematik- und Naturforscher-Tagung").

Anmeldungen an den Ortsausschuß, Prof. Reidemeister, Universität Königsberg.

Programm:

5. September, 9 Uhr: 1. R. Carnap-Wien, Die Grundgedanken des Logizismus (60 Min.). — 2. A. Heyting-Enschede, Die intuitionistische Begründung der Mathematik (60 Min.). — 3. J. v. Neumann-Berlin: Die axiomatische Begründung der Mathematik (60 Min.)

6. September, 10 Uhr: 1. H. Reichenbach-Berlin, Der physikalische Wahrheitsbegriff (60 Min.). — 2. W. Heisenberg-Leipzig, Kausalität und Quantenmechanik (60 Min.). — Anschließend Diskussion. — 15 Uhr: 1. O. Neugebauer-Göttingen, Die Geschichte der vorgriechischen Mathematik (60 Min.). — 2. K. Gödel-Wien: Über die Vollständigkeit des Logik-Kalküls (20 Min.). — 3. A. Scholz-Freiburg, Über den Gebrauch des Begriffs Gesamtheit in der Axiomatik (20 Min.). — 4. W. Dubislav-Berlin, Über den sogenannten Gegenstand der Mathematik (20 Min.).

7. September, 10 Uhr: Diskussion über die Grundlagen der Mathematik im Anschluß an die Vorträge von Carnap, Heyting, Neumann. — Wortmeldungen: **H. Härlen-Dordrecht:** Logische und symbolische Grundlegung der Mathematik. — R. Carnap-Wien und H. Hahn-Wien. — Weitere Wortmeldungen möglichst schon vor Beginn der Tagung erbeten.

First informal announcement of the incompleteness result by Gödel!

John von Neumann realized its importance and anticipated the second incompleteness theorem (“the consistency can not be proven”):

> *Lieber Herr Gödel!* 20. November 1930
>
> *Ich habe mich in der letzten Zeit wieder mit Logik beschäftigt, unter Verwendung der Methoden, die Sie zum Aufweisen unentscheidbarer Eigenschaften so erfolgreich benützt haben. Dabei habe ich ein Resultat erzielt, das mir bemerkenswert erscheint. Ich konnte nämlich zeigen, dass die Widerspruchsfreiheit der Mathematik unbeweisbar ist.* [...]
>
> John von Neumann

However, at that time the Gödel paper was completed already…
… used the people to think. But Jan von Plato showed otherwise!!!

All this indicates that the major figures in the development of QM where well aware of Context of Gödel’s discovery…

Any implication of Gödel (1931) for physics?

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Physics applies math – so math being „incomplete“ translates somehow into physics…

More interesting seems the following (Jammer 1985):
“A physical theory is incomplete if there are physically meaningful propositions which can not be proved nor disproved by the theory an yet can be consistently adjoined to it”. (p. 131)

Example (Jammer 1985): The d‘Alembert principle (“constraint forces do no virtual work“) does not follow from Newtonian mechanics – but can be consistently added to it → Newtonian mechanics is incomplete.

Hence: Is the completeness of a theory in physics desirable at all? May be its incompleteness is more fruitful…

However: The question of physics being “complete” or “incomplete” was apparently never asked in physics before the advent of quantum theory…

Jammer, Max (1985) “The EPR problem and its historical development“ In: Lahti & Mittelstaedt (Eds.) Symposium on the Foundation of Modern Physics, World Scientific, Singapore.

“Completeness” and quantum mechanics

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- The claim of quantum theory being “complete” was apparently brought up by Bohr in 1927.

- Already at that time Einstein developed thought experiments to challenge this alleged completeness (QM’s statistical pre-dictions might be grounded in hitherto “unknown elements”).

- In 1930 John von Neumann found a proof of the impossibility of “hidden variables” (published 1932 in his “Mathematische Grundlagen der Quanten-mechanik”, English edition in 1955; apparently translated until 1949):

John von Neumann on completeness and hidden variables

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Furthermore, there will be a detailed discussion of the problem as to whether it is possible to trace the statistical character of quantum mechanics to an ambiguity (i.e., incompleteness) in our description of nature. Indeed, such an interpretation would be a natural concomitant of the general principle that each probability statement arises from the incompleteness of our knowledge. This explanation "by hidden parameters," as well as another, related to it, which ascribes the "hidden parameter" to the observer and not to the observed system, has been proposed more than once. However, it will appear that this can scarcely succeed in a satisfactory way, or more precisely, such an explanation is incompatible with certain qualitative fundamental postulates of quantum mechanics.[3]

MATHEMATICAL FOUNDATIONS OF QUANTUM MECHANICS

By JOHN VON NEUMANN

translated from the German edition by
ROBERT T. BEYER

PRINCETON
PRINCETON UNIVERSITY PRESS
1955

According to Jammer (1985) this was viewed as a proof of the “absolute completeness” of quantum mechanics (in the sense of Tarski, 1930).

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

QM in a nutshell

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(i) A state of a system is described by a wave function $\boldsymbol{\psi}(x)$ or $|\boldsymbol{\psi}\rangle$ (with: $\boldsymbol{\psi}(x)=\langle x|\boldsymbol{\psi}\rangle$). It is the solution of the Schrödinger equation (SE):

$\frac{ih}{2\pi}\frac{\partial\boldsymbol{\psi}}{\partial t} = \mathcal{H}\cdot\,\boldsymbol{\psi}$ time evolution: $\boldsymbol{\psi}(t) = \boldsymbol{\psi}_0 \cdot\, e^{-\frac{i2\pi}{h}\mathcal{H}t}$ ("unitary" ($U^{-1} = U^{\dagger}$))

(ii) Dynamical quantities ("observables") are represented by Hermitian operators: $\mathcal{A}, \mathcal{B}, \ldots$ Their (real) eigenvalues correspond to possible measurement outcomes.

Be $\{|n\rangle\}$ an ON basis of eigenvectors of $\mathcal{A}$. Hence, each state can be expanded as: $|\boldsymbol{\psi}\rangle = \sum c_n |n\rangle$ with: $\mathcal{A}\cdot|n\rangle = n\cdot|n\rangle$

(iii) Born (1926): The probability to measure the eigenvalue n of the observable $\mathcal{A}$ *is* $|c_n|^2$ (→ expectation value and measure of variation can be defined)

In general $\mathcal{AB} \neq \mathcal{BA}$ *, holds i.e.* $\mathcal{AB} - \mathcal{BA} = [\mathcal{A},\mathcal{B}] \neq 0$ (joint measurement not possible)

It holds: $\Delta\mathcal{A}\cdot\Delta\mathcal{B} \geqslant \frac{1}{2}|\langle\boldsymbol{\psi}|[\mathcal{A},\mathcal{B}]|\boldsymbol{\psi}\rangle|$ e. g. $\boldsymbol{\Delta x}\cdot\boldsymbol{\Delta p} \geqslant \frac{\boldsymbol{h}}{\boldsymbol{4\pi}}$

"Heisenberg's uncertainty relation" (HUR) 1927

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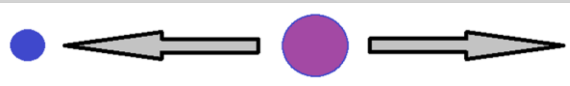
A system „decays“ into two sub-systems



$$\boldsymbol{\psi}(x_1, x_2) = \int \exp\left(\frac{i}{h}(x_1 - x_2 + x_0)p\right) dp$$

Position measurement on system 1 yields $\boldsymbol{x}$ → System 2 has position $\boldsymbol{x} + \boldsymbol{x_0}$

Momentum measurement on sys. 1 yields $\boldsymbol{p}$ → System 2 has momentum $-\boldsymbol{p}$

„without in any way disturbing“

Definite Momentum and position of particle 2 established – in contradiction with HUR!

Reception of the EPR argument and how the story continued…

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OCTOBER 15, 1935 PHYSICAL REVIEW VOLUME 48

Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

N. Bohr, *Institute for Theoretical Physics, University, Copenhagen*

(Received July 13, 1935)

Of course there is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system*. Since these conditions constitute an inherent element of the description of any phenomenon to which the term "physical reality" can be properly attached, we see that the argumentation of the mentioned authors does not justify their conclusion that quantum-mechanical description is essentially incomplete.

Bell (1981): "Indeed I have very little idea what this means."

Form EPR to the Bell inequality

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Physics Vol. 1, No. 3, pp. 195–200, 1964 Physics Publishing Co. Printed in the United States

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†

Department of Physics, University of Wisconsin, Madison, Wisconsin

(*Received* 4 *November* 1964)

BELL'S THEOREM AND WORLD PROCESS*

Henry P. Stapp

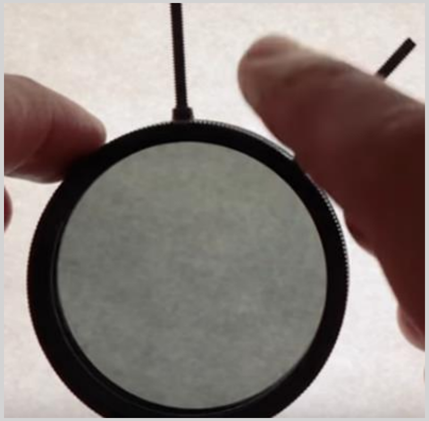
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

March 4, 1975

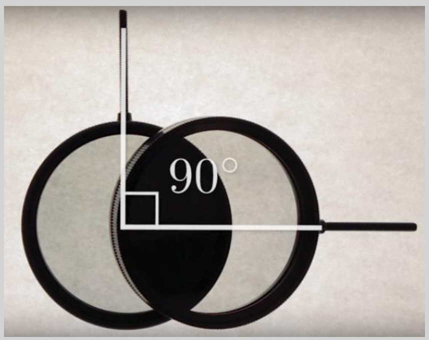
2. Bell's Theorem

Bell's theorem (2) is the most profound discovery of science. It shows that if the statistical predictions of quantum theory are approximately correct then, in certain cases, the principle of local causes must fail.

The three filter paradox



A second polarization filter blocks some light which passes the first – depending on the polarization axis

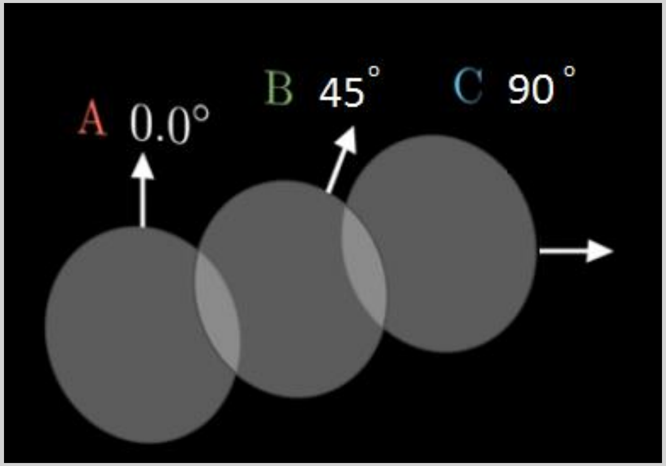


On an angle of 90^o all light gets blocked.

If you put a third filter in between more light gets through!?

This is correctly predicted by classical EM and quantum theory!

A (very much simplified) argument for Bell's theorem

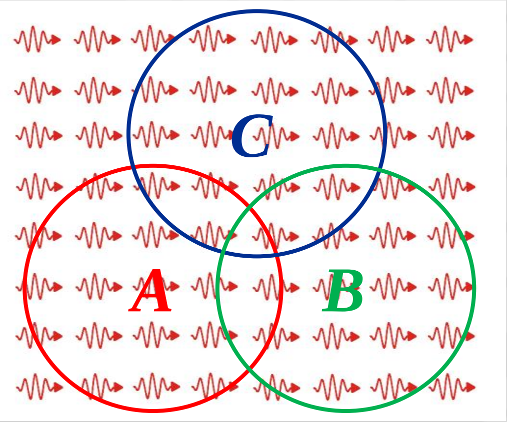


Suppose, that "hidden variables" would determine, whether a "photon" passes through a filter. Let A, B or C denote the set of photons with the property to pass the corresponding filter.

"Bell inequality":

$$\underbrace{|A \cap C|}_{= 0} \geq \underbrace{|A \cap B \cap C|}_{> 0}$$

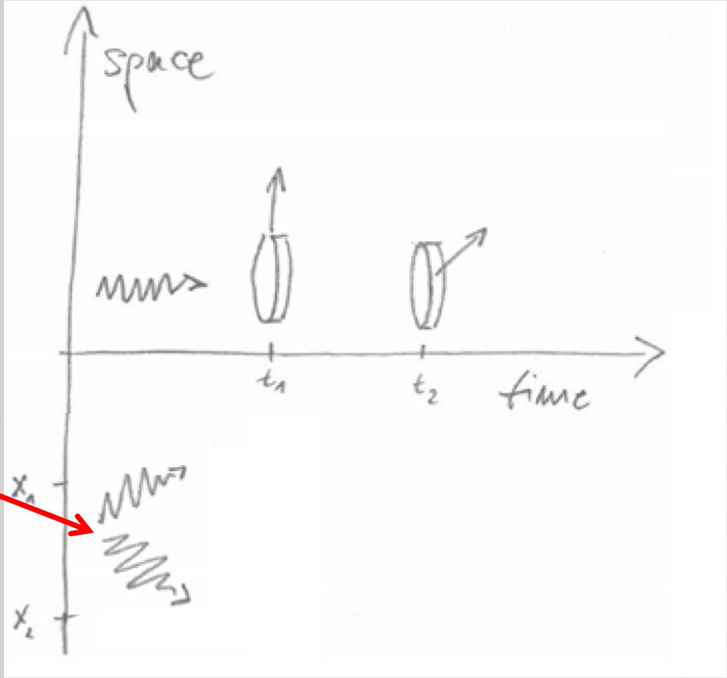
Experiment
(and QM prediction):



→ QM violates Bell's inequality, i.e. such a property assignment (by "hidden variables") is impossible!

However, the consecutive measurement could alter the state.
The situation should be turned into an simultaneous measurement:

This needs an „entangled“ pair of photons



A “properly” derived Bell inequality is still violated by QM (and experiment).

There is apparently a connection between time-like events in QM!

→ QM violates Bell’s inequality, i.e. such a property assignment by (local) hidden variables is impossible.

https://www.reddit.com/r/quantum/comments/707dlt/bells_theorem_the_quantum_venn_diagram_paradox/

The meaning of Bell's theorem

Technically, Bell's theorem ("QM violates the Bell inequality") is a no-go result:

- Local hidden variables can not reproduce the (successfully tested) predictions of QM
- Common narrative: Bell shattered EPR's dream. QM is **complete**!

> While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.

EPR 1935

?????

Cubitt et al.: Quantum mechanics is **incomplete.**

The meaning of Bell's theorem II

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However, this reading of Bell's theorem misses important points:

- EPR (and Bell) are not assuming deterministic hidden variables, but they follow from the premise of "local causality".
- The non-local QM correlations can not be used to communicate faster than light, but…
- …"correlation cry out for an explanation" (Bell 2004, p. 152)

Bell (1964): Quantum mechanics is non-local.

See: Bricmont, J. (2015) "History of Quantum Mechanics or the Comedy of Errors", https://arxiv.org/abs/1703.00294v1

Summary and conclusion

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Cubitt et al.: Quantum mechanics is **incomplete.**

Bell (1964): Quantum mechanics is **non-local**.

Both claims support an anti-reductionist intuition in the sense of “More is different”, since in a non-local theory the whole is “more than just the sum of its parts” either…

More technical: For an entangled EPR state the division into proper parts is not even possible!

Many thanks for your attention!