**Selected Works of Kurt Gödel**

# Gödel's Incompleteness Theorem(s)

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*February 26, 2019*

## Preface

With his *Completeness Theorem* the logician and philosopher Kurt Gödel made a first significant step towards carrying out *Hilbert's Program*, only to then shatter any hopes of a possible fulfilment of the program with his *Incompleteness Theorems*.
Mathematician David Hilbert hoped to put all of mathematics on a sound logical basis by means of introducing an axiomatic logical system with *finitary* methods of deduction. In 1929 Gödel proved that classical predicate first order logic is both sound and complete. Based on this success he set out to proceed likewise for higher order logic and came to the conclusion that it was not possible [4]. He published his results in [1]. Below we shall outline the idea of the proof of the first theorem according to [3].

## 1 The idea

So, how did Gödel go about proving his incompleteness theorem? The general idea is to construct a sentence that states its own unprovability. A system that is expressive enough for such a sentence to be stated can then not be consistent and complete. To be able to formulate the argument, let us first consider some basic concepts.

### 1.1 Basic Notions

For a given language $\mathcal{L}$, we mean

- $\mathcal{E}$ to be the set of all **expressions** over $\mathcal{L}$
- $\mathcal{H} \subset \mathcal{E}$ to be the set of all **predicates** over $\mathcal{L}$
- $\mathcal{S} \subset \mathcal{E}$ to be the set of all **sentences** over $\mathcal{L}$
- $\mathcal{P} \subset \mathcal{S}$ to be the set of all **provable sentences** over $\mathcal{L}$
- $\mathcal{R} \subset \mathcal{S}$ to be the set of all **refutable sentences** over $\mathcal{L}$
- $\mathcal{T} \subset \mathcal{S}$ to be the set of all **true sentences** over $\mathcal{L}$

So if $\mathcal{L}$ is the language of **Peano Arithmetic** over alphabet $\Sigma = \{0, S, +, \cdot, <\}$, then
$\exists x(2 < x),\ 1+1,\ (y+x) = 3$ and $2 < x$ would all be expressions, $2 < x$ a predicate and $\exists x(2 < x)$ a true sentence.

*Remark.* Note that numerals are abbreviated by writing $n$ instead of $\overbrace{SS\ldots S}^{n}0$.
To express that predicates like $H \equiv 2 < x$ depend on $x$, we write $H(x)$.

In the following, natural numbers will be at the centre of the discussion since it makes sense for the definitions used and since it is what Gödel originally worked with. This is in spite of the fact that one could consider languages with quite different domains.

**Definition 1.** Let $n \in \mathbb{N}$, $H \in \mathcal{H}$, we say $n$ **satisfies** $H$ if $H(n)$ is a true sentence.
A set $A \subset \mathbb{N}$ is **expressed** by $H$ iff

$$\forall n \in \mathbb{N}: \; H(n) \in \mathcal{T} \Leftrightarrow n \in A$$

$A$ is **expressible** in $\mathcal{L}$ if there is some predicate $H_A$ of $\mathcal{L}$, s.t. $A$ is expressed by $H_A$.

So if we consider Peano Arithmetic, the set $A = \{0, 1\}$ is expressed by $H_A \equiv x < 2$.

**Definition 2.** $\mathcal{L}$ is a **correct** language if

**(i)** $\mathcal{P} \subset \mathcal{T}$

**(ii)** $\mathcal{T} \cap \mathcal{R} = \emptyset$

*Remark.* Correctness is stronger than consistency:

$$\mathcal{L} \textbf{ consistent} \;\Leftrightarrow\; \forall S \in \mathcal{S}: \; \neg(S \in \mathcal{P} \wedge S \in \mathcal{R})$$

This will be relevant for the proof later on.

## 1.2 Gödel numberings

**Definition 3.** Given an injective function $g\colon \mathcal{E} \to \mathbb{N}$ and expression $E \in \mathcal{E}$, we call

$$g(E)$$

the **Gödel number** of $E$ and $g$ a **Gödel numbering** for language $\mathcal{L}$.

Gödel numberings are a remarkable concept since they allow numbers to express statements about themselves, which is what Gödel used to construct the sentence that states its own unprovability.
Note that the only requirement for a Gödel numbering is that it be injective. Usually a Gödel numbering specific to the examined system is defined, s.t. it works well with the proof one wants to develop, so we shall not give further requirements here.
Note also that one may assume that for any Gödel number $n \in \mathbb{N}$ there is a unique expression $E$, s.t. $g(E) = n$. This is because we assume the expressions to be recursively enumerable and any enumerable or countable set $M$ has a bijection into the naturals $\mathbb{N}$.

**Example 4.** We can define a suitable Gödel numbering $g$ for Peano Arithmetic as follows. Let $\Gamma = \Sigma \cup \{\neg, \wedge, \rightarrow, \exists, x, (, )\}$ be the extended alphabet of Peano Arithmetic comprising $\Sigma = \{0, S, +, \cdot, <\}$, logical operators, a quantifier and metasymbols. We obtain $g(E)$, $E \in \mathcal{E}$, by doing the following:

- Assign each $\sigma \in \Gamma$ a number:

| $0$ | $S$ | $+$ | $\cdot$ | $<$ | $\neg$ | $\wedge$ | $\rightarrow$ | $\exists$ | $x$ | ( | ) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B |

- Create $E'$ by substituting every symbol by its assigned number.
- The Gödel number of $E$ is the number whose base 12 representation is $E'$.

So for example one may want to find $g(E)$ where $E \equiv 1 < x$ and $1 < x$ short for $(S\ 0) < x$.

$$\begin{aligned}(S\ 0) < x &\underset{E \to E'}{\rightarrow} A10B49 \\ g(1 < x) &= 10 \cdot 12^5 + 1 \cdot 12^4 + 0 \cdot 12^3 + 11 \cdot 12^2 + 4 \cdot 12^1 + 9 \cdot 12^0 \\ &= A10B49_{12} = 2510697_{10}\end{aligned}$$

### 1.3 Diagonalisation

**Definition 5.** Let $E_n$ be the unique expression, s.t. $g(E_n) = n$.
We call $E_n(n)$ the **diagonalisation** of $E_n$ and

$$d\colon \mathbb{N} \to \mathbb{N}, \quad d(n) = g(E_n(n))$$

the **diagonal function**.

*Remark.* $E_n(n)$ is true iff $E_n$ is satisfied by its own Gödel number $n$ and $d(n)$ is the Gödel number of sentence $E_n(n)$.

So for $E \equiv 1 < x$ with Gödel number 2510697, $E_n(n)$ would be the theorem $1 < 2510697$ and $d(n)$ in turn would be the Gödel number of that expression. We shall refrain from explicitly stating $d(n)$ since this would require 2510697 many $S$ and twice as many bracket symbols with the given alphabet.

**Definition 6.** For any set $A \subset \mathbb{N}$, let $A^*$ be the **preimage** of $A$ under $d$:

$$n \in \mathbb{N} :\ n \in A^* \Leftrightarrow d(n) \in A$$

For any set $A \subset \mathbb{N}$, let $\overline{A}$ be the **complement** of $A$:

$$\overline{A} = \mathbb{N} \setminus A$$

## 2 Gödel's Theorem - simplified

What follows is a simplified version of Gödel's first incompleteness theorem. Since its proof only requires the general concepts introduced above, it is well suited to explain the general outline of Gödel's original proof.

**Theorem** (Little Gödel)**.** *For a given language $\mathcal{L}$, let $\boldsymbol{P} = \{n \mid \exists S \in \mathcal{P} : g(S) = n\}$ be the set of Gödel numbers of all* ***provable sentences****.*
*If set $\overline{P}^*$ is expressible in $\mathcal{L}$ and $\mathcal{L}$ is correct, then there is a true sentence of $\mathcal{L}$ which is not provable in $\mathcal{L}$.*

*Proof.* Let $H$ be the predicate that expresses $\overline{P}^*$. Let $h$ be the Gödel number of $H$.
Since $H$ expresses $\overline{P}^*$, for any $n \in \mathbb{N}$:

$$H(n) \text{ true } \Leftrightarrow n \in \overline{P}^*$$

And thus:

$$\begin{aligned} H(h) \text{ true } &\Leftrightarrow h \in \overline{P}^* \\ &\Leftrightarrow d(h) \in \overline{P} \Leftrightarrow d(h) \notin P \end{aligned}$$

Note that $d(h) = g(H(h))$, so by definition of $P$ we get

$$d(h) \in P \Leftrightarrow H(h) \text{ provable}$$

and thus

$$H(h) \text{ true } \Leftrightarrow d(h) \notin P \Leftrightarrow H(h) \text{ not provable}$$

**Case 1**: $H(h)$ is false and provable. Contradiction by correctness of $\mathcal{L}$.
**Case 2**: $H(h)$ is true and not provable. □

Thus, by the theorem, for any correct system that is powerful enough for $\overline{P}^*$ to be expressible, there will be a sentence both true and unprovable making the system incomplete.
Of course, if the system were not consistent and thus not correct then it could be complete. By requiring the system to be correct, we actually demanded more than was necessary.

## 3 Gödel's Theorems, the actual version - almost

To conclude, we shall state both incompleteness theorems and remark on their connection to theorem *Little Gödel*. They are freely adapted from [2]. For Gödel's originial version one may consult [1].

**Theorem** (Gödel's Incompleteness Theorem I)**.** *For any consistent formal system $F$ which contains elementary arithmetic, there is a sentence, s.t. neither the sentence itself nor its negation can be proved within the system.*

**Theorem** (Gödel's Incompleteness Theorem II)**.** *No formal system $F$ can be consistent and prove its own consistency.*

Note that the first theorem refers to *basic arithmetic*. This is due to the fact that Gödel tried to prove consistency and completeness of a formal system of arithmetic including higher order quantification. The proof would have been an important achievement on the way to giving mathematics a solid foundation. After all, arithmetic constitutes a significant part of mathematics. Since Gödel's version is concerned with arithmetic, it is clear why the Gödel numbering is so important: It makes it possible to state sentences about arithmetic in arithmetic. In *Little Gödel*, we abstained from referring to any concrete system but instead required a given system to be expressive enough. Basic arithmetic is expressive enough and thus enables Gödel to construct the desired sentence.
From the second theorem it follows that a system $F$ expressive enough to state its own consistency as theorem $Cons(F)$ cannot be consistent and contain a proof for $Cons(F)$, this holds in particular for any system containing basic arithmetic.

Albeit closer to the original, *Gödel's Incompleteness Theorems I* and *II* as stated above are still only informal versions of Theorems VI and XI in Gödel's paper from 1931 [1]. The statement of those requires further definitions. We leave it to the inclined reader to dive into the original, if desired.

## References

[1] Gödel, K. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für mathematik und physik 38*, 1 (1931), 173–198.

[2] Raatikainen, P. *Gödel's Incompleteness Theorems*. https://plato.stanford.edu/entries/goedel-incompleteness/#FirIncThe, 2015. Accessed: 2019-03-24.

[3] Smullyan, R. M. *Gödel's Incompleteness Theorems*. Oxford University Press, 1992.

[4] von Plato, J. In search of the sources of incompleteness. In *Proceedings of the International Congress of Mathematicians 2018* (International, 2018), vol. 3, World Scientific, pp. 4043–4061.