Freie Universität Berlin

Kurt Gödel – Selected Topics

Intuitionistic Logic versus Classical Logic

Gödel's Interpretation and Conjectures

Seminar paper

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1 Introduction

In the article "Eine Interpretation des intuitionistischen Aussagenkalküls", presented in 1932 at the Mathematical Colloquium in Vienna and published in 1933, Kurt Gödel described an interpretation of the intuitionistic propositional logic in a system of classical propositional logic enriched with an additional unary operator B (which stands for provability). The main purpose of this seminar paper is to provide an overview on the given interpretation and the involved logical systems, and also to discuss the connection between the enriched system and the modal logic $\mathbf{S}_4$.

2 Intuitionism, Intuitionistic Logic and Heyting's Calculus

2.1 Intuitionism

Intuitionism is a philosophy of mathematics that was introduced by the Dutch mathematician Luitzen E. J. Brouwer in 1908. In contrast to logicism intuitionism treats logic as a part of mathematics rather than as the foundation of mathematics. Thus it is based on the idea that mathematics is a creation of the mind. The truth of a mathematical statement can only be conceived via a mental construction (a *proof* or *verification*) that proves it to be true:

> "It does not make sense to think of truth or falsity of a mathematical statement independently of our knowledge concerning the statement. A statement is *true* if we have proof of it, and *false* if we can show that the assumption that there is a proof for the statement leads to a contradiction." (Troelstra and van Dalen 1988: p. 4)

In summary, intuitionism centers on proof rather than truth. Logical statements are not just *true* or *false*, but *provable*, *not provable* or neither of both when the existence of a proof is unclear.

2.2 Intuitionistic Propositional Calculus ($\mathbf{IPC}$)

1930 Arend Heyting formalized the Intuitionistic Propositional Calculus ($\mathbf{IPC}$) making Brouwer's definition of intuitionistic truth explicit. His calculus describes the rules for the derivation of formulas that are valid from the point of view of an intuitionist.

A logical system is defined by a formal syntax, a formal semantics and a derivation system, and in the following such a triple is characterized for $\mathbf{IPC}$. We start by specifying how intuitionistic formulas can be build and continue with assinging those formulas a meaning by describing the semantics of $\mathbf{IPC}$. As a last step we provide a proof system which is used for logical reasoning within the system.

2.2.1 Syntax

Alphabet. The alphabet of $\mathbf{IPC}$ consists of propositional variables (A, B, C, $\cdots$), logical connectives $\cdot$, $+$, $\supset$ and a constant symbol $\bot$.

The negation of a formula φ, denoted as $\sim \varphi$, is abbreviated by $\varphi \supset \bot$.

Atomic Formulas. Any propositional variable or $\bot$ is an atomic formula.

Well-formed Formulas. The (well-formed) formulas of $\mathbf{IPC}$ are defined inductively as follows:

- Each atomic formula is a well-formed formula.
- If φ and ψ are well-formed formulas, so are $\varphi \cdot \psi$, $\varphi + \psi$ and $\varphi \supset \psi$.
- Nothing else is a well-formed formula.

2.2.2 Semantics

The Brouwer–Heyting–Kolmogorov (BHK) semantics is widely recognized as the official representation of the intuitionistic meaning for the logical connectives of $\mathbf{IPC}$. According to BHK a statement is *true* if it has a proof, and the proof of a logically compound statement depends on the proofs of its components. Employing the unexplained primitive notions of *construction* and *proof* the description of BHK states informally that

- a proof of $\varphi \cdot \psi$ consists of a proof of φ and a proof of ψ,
- a proof of $\varphi + \psi$ is given by presenting either a proof of φ or a proof of ψ,
- a proof of $\varphi \to \psi$ is a construction which, given a proof of φ, returns a proof of ψ, and
- $\bot$ has no proof.

Negation is not included primitively in our syntax, but the corresponding BHK semantics could be defined like that:

- A proof of $\sim \varphi$ is a construction which, given a proof of φ, would return a proof of $\bot$.

2.2.3 Proof System

The following collection of axiom schemes for a Hilbert-style proof system is introduced by Troelstra and van Dalen (1988):

1. $\varphi \supset (\psi \supset \varphi)$
2. $(\varphi \supset (\psi \supset \mu)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \mu))$
3. $(\varphi \cdot \psi) \supset \varphi$
4. $(\varphi \cdot \psi) \supset \psi$
5. $\varphi \supset (\psi \supset (\varphi \cdot \psi))$
6. $\varphi \supset (\varphi + \psi)$
7. $\psi \supset (\varphi + \psi)$
8. $(\varphi \supset \mu) \supset ((\psi \supset \mu) \supset ((\varphi + \psi) \supset \mu))$
9. $\bot \supset \varphi$

where φ, ψ, μ are some arbitrary formulas in **IPC**.

To complete the Hilbert-style proof system we add the Modus Ponens as inference rule:

1. From φ and $\varphi \supset \psi$, conclude ψ.

where φ, ψ are some arbitrary formulas in **IPC**.

3 Classical Propositional Logic (CPL)

In this section we briefly discuss a syntax, a semantics and a Hilbert-style proof system for the system **CPL** to see if and how classical logic differs from intuitionistic logic.

3.1 Syntax

Syntactically, classical propositional logic and intuitionistic propositional logic have no difference. However, for reasons of clarity we use different symbols for the logical operators than we used in the definitions for **IPC**.

Alphabet. The alphabet of **CPL** consists of propositional variables (A, B, C, $\cdots$), logical connectives $\wedge$, $\vee$, $\rightarrow$ and a constant symbol $\bot$.

The negation of a formula φ, denoted as $\neg\varphi$, is abbreviated by $\varphi \rightarrow \bot$.

Atomic Formulas. Any propositional variable or $\bot$ is an atomic formula.

Well-formed Formulas. The (well-formed) formulas of $\mathbf{CPL}$ are defined inductively as follows:

- Each atomic formula is a well-formed formula.
- If φ and ψ are well-formed formulas, so are $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \to \psi$.
- Nothing else is a well-formed formula.

3.2 Semantics

The semantics of $\mathbf{CPL}$ are subject to the usual conditions ("truth tables"):

- $\varphi \wedge \psi$ is *true* if and only if φ is *true* and ψ is *true*,
- $\varphi \vee \psi$ is *true* if and only if φ is *true* or ψ is *true* (or both),
- $\varphi \to \psi$ is *false* if and only if φ is *true* and ψ is *false*, and
- $\perp$ is *false*.

Once again, this would be the semantics of negated formulas of $\mathbf{CPL}$:

- A proof of $\neg\varphi$ is *true* if and only if φ is *false*.

3.3 Proof System

A proof system for intuitionistic logic can be turned into a proof system for classical logic by adding a scheme expressing the Principle of Excluded Middle, $\varphi \vee \neg\varphi$. Therefore, our previously introduced collection of axiom schemes can be extended by a tenth scheme to form a proper collection of axiom schemes for $\mathbf{CPL}$'s proof system:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \mu)) \to ((\varphi \to \psi) \to (\varphi \to \mu))$
3. $(\varphi \wedge \psi) \to \varphi$
4. $(\varphi \wedge \psi) \to \psi$
5. $\varphi \to (\psi \to (\varphi \wedge \psi))$
6. $\varphi \to (\varphi \vee \psi)$
7. $\psi \to (\varphi \vee \psi)$
8. $(\varphi \to \mu) \to ((\psi \to \mu) \to ((\varphi \vee \psi) \to \mu))$
9. $\perp \to \varphi$
10. $\varphi \vee (\varphi \to \perp)$

where φ, ψ, μ are some arbitrary formulas in $\mathbf{CPL}$.

The collection of involved inference rules stays the same:

1. From φ and $\varphi \rightarrow \psi$, conclude ψ.

where φ, ψ are some arbitrary formulas in $\mathbf{CPL}$.

4 Gödel's Interpretation

According to Gödel (1933), for defining an interpretation of intuitionistic propositional logic in classical propositional logic a more expressive classical system than $\mathbf{CPL}$ is needed. Such a system $\mathcal{G}$ can be achieved by adapting $\mathbf{CPL}$ as discussed below.

4.1 System $\mathcal{G}$

For Gödel's system $\mathcal{G}$ a new notion 'a formula φ is provable', denoted by $\mathrm{B}\varphi$, is established. Along with the unary operator B some new axiom schemes and a special inference rule are added to the already known definitions of $\mathbf{CPL}$ presented in section 3.

4.1.1 Syntax

Alphabet. The alphabet of $\mathcal{G}$ consists of propositional variables (A, B, C, $\cdots$), logical connectives B, $\wedge$, $\vee$, $\rightarrow$ and a constant symbol $\perp$.

The negation of a formula φ, denoted as $\neg\varphi$, is abbreviated by $\varphi \rightarrow \perp$.

Atomic Formulas. Any propositional variable or $\perp$ is an atomic formula.

Well-formed Formulas. The (well-formed) formulas of $\mathcal{G}$ are defined inductively as follows:

- Each atomic formula is a well-formed formula.
- If φ and ψ are well-formed formulas, so are $\mathrm{B}\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \rightarrow \psi$.
- Nothing else is a well-formed formula.

4.1.2 Proof System

The original collection of axiom schemes of $\mathbf{CPL}$'s Hilbert-style proof system is expanded by some particular schemes 11., 12. and 13., resulting in a new collection of axiom schemes for the proof system of Gödel's system $\mathcal{G}$:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \mu)) \to ((\varphi \to \psi) \to (\varphi \to \mu))$
3. $(\varphi \wedge \psi) \to \varphi$
4. $(\varphi \wedge \psi) \to \psi$
5. $\varphi \to (\psi \to (\varphi \wedge \psi))$
6. $\varphi \to (\varphi \vee \psi)$
7. $\psi \to (\varphi \vee \psi)$
8. $(\varphi \to \mu) \to ((\psi \to \mu) \to ((\varphi \vee \psi) \to \mu))$
9. $\bot \to \varphi$
10. $\varphi \vee (\varphi \to \bot)$
11. $\mathrm{B}\varphi \to \varphi$
12. $\mathrm{B}\varphi \to (\mathrm{B}(\varphi \to \psi) \to \mathrm{B}\psi)$
13. $\mathrm{B}\varphi \to \mathrm{BB}\varphi$

where φ, ψ, μ are some arbitrary formulas in $\mathcal{G}$.

The corresponding collection of inference rules is increased by one to finish $\mathcal{G}$'s proof system:

1. From φ and $\varphi \to \psi$, conclude ψ.
2. From φ, conclude $\mathrm{B}\varphi$.

where φ, ψ are some arbitrary formulas in $\mathcal{G}$.

4.2 Interpretation

Having system $\mathcal{G}$ defined as above Gödel stipulates his interpretation function $g : \mathbf{IPC} \to \mathcal{G}$ as follows:

$$
\begin{aligned}
g(A) &:= A \\
g(\varphi \cdot \psi) &:= g(\varphi) \ \wedge \ g(\psi) \\
g(\varphi + \psi) &:= \mathrm{B}\, g(\varphi) \ \vee \ \mathrm{B}\, g(\psi) \\
g(\varphi \supset \psi) &:= \mathrm{B}\, g(\varphi) \ \rightarrow \ \mathrm{B}\, g(\psi) \\
g(\perp) &:= \perp \\
g(\sim \varphi) &:= \neg\, \mathrm{B}\, g(\varphi).
\end{aligned}
$$

He states, without proving, that

$$
\begin{aligned}
g(\varphi \cdot \psi) &:= \mathrm{B}\, g(\varphi) \ \wedge \ \mathrm{B}\, g(\psi) \\
g(\sim \varphi) &:= \mathrm{B} \neg \mathrm{B}\, g(\varphi)
\end{aligned}
$$

are variant but equally legit translations for the conjunction and the negation respectively.

The general idea of Gödel's translation is to put the operator B before every subformula of a certain formula φ . When the usual procedure of determining classical truth of φ is applied to $g(\varphi)$, it will test the *provability* (and not the *truth*) of each of φ's subformulas in agreement with Brouwer's ideas. Conversely, no formula of the form $\mathrm{B}\varphi \vee \mathrm{B}\psi$ is derivable from $\mathcal{G}$ unless either $\mathrm{B}\varphi$ or $\mathrm{B}\psi$ is derivable from $\mathcal{G}$. Since for some arbitrary formula φ it is the case that $\mathrm{B}\varphi$ or $\mathrm{B}(\varphi \rightarrow \perp)$ or – and that is the important point – none of these holds, the translation of the Law of Excluded Middle is in general not derivable from $\mathcal{G}$.

Gödel also notes that the operator B should be interpreted as 'provable by any correct means' rather than as 'provable in a given formal system' because otherwise this would contradict his second incompleteness theorem.

4.3 Gödel's Results

Gödel claims that if a formula is derivable from intuitionistic propositional logic, then its translation is derivable from $\mathcal{G}$. That is:

$$\text{If } \vdash_{\mathbf{IPC}} \varphi, \text{ then } \vdash_{\mathcal{G}} g(\varphi).$$

One can also say, that his translation preserves theoremhood. He also conjectures that the converse holds, such that we have:

$$\vdash_{\mathbf{IPC}} \varphi \text{ if, and only if } \vdash_{\mathcal{G}} g(\varphi).$$

Given these two directions Gödel's translation can be seen as faithful. While the first implication follows trivially, Gödel conjectures the backward direction without actually showing it. Later on John C. C. McKinsey and Alfred Tarski (1948) show that the converse implication is indeed true. But in their proof, the Lewis modal system $\mathbf{S}_4$ comes into play.

4.4 Lewis Modal System $\mathbf{S}_4$

The system $\mathbf{S}_4$, originally named after its founder Clarence I. Lewis (1918), is a modal propositional logic. Modal logic extends classical logic by an unary necessity operator $\Box$ as shown below.

4.4.1 Syntax

Alphabet. The alphabet of $\mathbf{S}_4$ consists of propositional variables (A, B, C, $\cdots$), logical connectives $\Box$, $\wedge$, $\vee$, $\rightarrow$ and a constant symbol $\perp$.

The negation of a formula φ, denoted as $\neg\varphi$, is abbreviated by $\varphi \rightarrow \perp$.

Atomic Formulas. Any propositional variable or $\perp$ is an atomic formula.

Well-formed Formulas. The (well-formed) formulas of $\mathbf{S}_4$ are defined inductively as follows:

- Each atomic formula is a well-formed formula.
- If φ and ψ are well-formed formulas, so are $\Box\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \rightarrow \psi$.
- Nothing else is a well-formed formula.

4.4.2 Proof System

McKinsey and Tarski (1948) present the following collection of axiom schemes for a Hilbert-style proof system for $\mathbf{S}_4$:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \mu)) \to ((\varphi \to \psi) \to (\varphi \to \mu))$
3. $(\varphi \wedge \psi) \to \varphi$
4. $(\varphi \wedge \psi) \to \psi$
5. $\varphi \to (\psi \to (\varphi \wedge \psi))$
6. $\varphi \to (\varphi \vee \psi)$
7. $\psi \to (\varphi \vee \psi)$
8. $(\varphi \to \mu) \to ((\psi \to \mu) \to ((\varphi \vee \psi) \to \mu))$
9. $\bot \to \varphi$
10. $\varphi \vee (\varphi \to \bot)$
11. $\Box\varphi \to \varphi$
12. $\Box\varphi \to (\Box(\varphi \to \psi) \to \Box\psi)$
13. $\Box\varphi \to \Box\Box\varphi$

where φ, ψ, μ are some arbitrary formulas in $\mathbf{S}_4$.

Additionally, this collection of inference rules is presented:

1. From φ and $\varphi \to \psi$, conclude ψ.
2. From φ, conclude $\Box\varphi$.

where φ, ψ are some arbitrary formulas in $\mathbf{S}_4$.

The second rule is called the Necessity Rule.

4.5 Relation of $\mathcal{G}$ and $\mathbf{S}_4$

Comparing the definitions given in sections 4.1 and 4.4 it becomes clear, that if Bφ is understood as 'φ is necessary', then the expanded system $\mathcal{G}$ results as the Lewis modal system $\mathbf{S}_4$ with B written for the necessity operator $\Box$. Hence, Gödel's results show that there is an interpretation of the intuitionistic propositional logic **IPC** in the modal logic $\mathbf{S}_4$. If φ can be derived from **IPC**, then φ's translation, $g(\varphi)$, can be derived from $\mathbf{S}_4$ and vice versa:

$$\vdash_{\mathbf{IPC}} \varphi \text{ if, and only if } \vdash_{\mathbf{S}_4} g(\varphi).$$

Since the system $\mathbf{S}_4$ is a very popular and famous system in the broad field of logics, this is a very outstanding and important contribution. In fact, Gödel was one of the first who proposed an embedding of $\mathbf{IPC}$ into classical logic. And this conjecture above is what McKinsey and Tarski finally prove years later in their paper from 1948.

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