Introduction to stacks

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Abstract

This is the second talk of the Research Seminar of the Arithmetic Geometry group (FU Berlin), which is on the Summer term of 2015. In the seminar we study the paper [JHS11]. In this talk I'll try to give an introduction to the language of stacks by following [Góm01]: first I give some motivation and historical remarks, then I define the notion of algebraic spaces and finally we go for the algebraic stacks. I'll try to give as many examples as possible, although in the references one can find more.

1 Motivation

O sol che sani ogne vista turbata,
tu mi contenti sì quando tu solvi,
che, non men che saver, dubbiar m'aggrata.
Dante Alighieri

1.1 From Apollonius of Perga to Grothendieck

Apollonius of Perga (3rd century BC) was a Greek geometer and astronomer, and he is very well known because of his study of conic sections. He gave the hyperbola, the ellipse and the parabola their names and he studied several properties.

On the other hand, Diophantus of Alexandria (3rd century AD) was an Alexandrian Greek mathematician who was interested in algebraic problems, mainly finding integer and rational solutions to a given polynomial equation. For example, the worlwide famous equation $x^n + y^n = z^n$ was contained in his *Arithmetica*, and Fermat's claim was written in a margin of his copy of *Arithmetica*.

In the 19th century, the geometry of the polynomial equations was developed, and number theorists were looking for analogue methods, since a lot of problems were very similar: those involving polynomial equations. If we work over real or complex numbers, we have geometry, but if we work over integer or rational numbers, we have arithmetic. But the main point is that the equation is the same in both sides!

In order to achieve this, in the 20th century Algebraic geometry was first founded in commutative algebra (van der Waerden, Zariski, Weil) in order to give a rigorous framework for the Italian school of Algebraic geometry (the typical structure of their results was Theorem - Proof - Counterexample); and in the 50's and 60's, Serre and

Grothendieck introduced in Algebraic geometry the sheaf theory, and finally in the 60's Grothendieck introduced the notion of scheme.

In what sense did the notion of scheme unify geometry and arithmetic? One way to see this is via the functor of points: we start with an abstract polynomial equation, say $p(x, y) = y^2 - x^3 + x = 0$. Note that $p(x, y)$ can be regarded as a polynomial in $R[x, y]$ for every ring R different from zero. We may be interested in its real or complex solutions, or maybe we are just wondering if it has any solution in the integers. But to give a solution of $p(x, y)$ in a ring R, is the same as giving a ring homomorphism $\varphi : \mathbb{Z}[x, y]/p(x, y) \to R$, or equivalently, a morphism of (affine) schemes $\text{Spec}(R) \to \text{Spec}(\mathbb{Z}[x, y]/p(x, y))$: indeed, in order to have a well defined homomorphism, $\varphi(x)$ and $\varphi(y)$ must satisfy the equation $p(\varphi(x), \varphi(y)) = 0$, so this defines a solution of $p(x, y)$ in the ring R. This leads to the following definition:

Definition 1 (Functor of points)**.** Given a scheme X, its functor of points h_X is, for every other scheme Y, the set $\text{Hom}(Y, X)$. If $Y = \text{Spec}(R)$ is affine, we will denote the set as $\text{Hom}(R, X)$.

Hence, the integer solutions of $y^2 - x^3 + x = 0$ may be identified with the set $\text{Hom}(\mathbb{Z}, (\text{Spec}(\mathbb{Z}[x, y]/p(x, y)))$, and if we are interested in the equation itself, then we have to focus in the functor of points $h_{\text{Spec}(\mathbb{Z}[x,y]/p(x,y))}$.

Therefore, if we study the functor of points, we are studying at the same time arithmetic and geometry.

1.2 Moduli spaces

In some situations, it is easier to define the functor of points of a scheme rather than the scheme itself. This situation happens, for example, when we study the moduli space that parametrizes some objects: vector bundles, curves, subspaces... In these situations, the moduli space ma be very complicated, but to give a morphism with target this moduli space is the same as choosing some elements that are being parametrized.

For example, we may be interested in the closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$. One desirable condition is that we have an object that parametrizes flat families of these objects. By the following proposition (c.f. [EH77][p. 126]), we should restrict our attention to families of objects that have the same Hilbert polynomial:

Proposition 1. *A family $\mathcal{X} \subset \mathbb{P}^r_B$ of closed subschemes of a projective space over a reduced connected base B is flat if and only if all fibers have the same Hilbert polynomial.*

Assume that we have a nice object $\mathcal{H}_P$ that parametrizes all the closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ with a given Hilbert polynomial. Then a flat family $\mathcal{X}$ of subschemes of $\mathbb{P}^n_{\mathbb{C}}$ with Hilbert polynomial P parametrized by B is just a closed scheme $\mathcal{X} \subset \mathbb{P}^n_{\mathbb{C}} \times B$. This gives us a map $B \to \mathcal{H}_P$ between the points $b \in B$ and the fibers X_b, which are our closed subspaces. We can ask that for every scheme B over $\mathbb{C}$, the set of flat families of closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ with Hilbert polynomial P and the set of maps from B to $\mathcal{H}_P$ are identified.

But maybe we are even more ambitious and ask that not only for $\mathbb{C}$, but for any field K, we have this, since the problem of parametrizing subspaces of $\mathbb{P}^n_K$ should be similar for different fields K, so so we would like to do this over $\mathrm{Spec}(\mathbb{Z})$ (which is the final object in (Sch)), and then when we choose a field, we want that $\mathcal{H}_P \times \mathrm{Spec}(K)$ parametrizes the closed subschemes of $\mathbb{P}^n_K$.

What we indeed are looking for is for the following functor:

Definition 2 (Hilbert functor)**.** The Hilbert functor h_P, also called the "functor of flat families of schemes in $\mathbb{P}^n_{\mathbb{Z}}$ with Hilbert polynomial P", is the functor that associates to any scheme B the set of subschemes $\mathcal{X} \subset \mathbb{P}^n_B$ flat over B whose fibers over points of B have Hilbert polynomial P.

We have a concrete description of this functor, and one may ask if there is a scheme $\mathcal{H}_P$ that represents this functor. Later we will solve the mystery, but just notice that if it exists, it has to very a very difficult object, since it must encode really a lot of information.

This is one of the advantages of using the functor of points point of view.

2 Stacks

La conclusion pratique à laquelle je suis arrivé dès maintenant, c'est que chaque fois que en vertu de mes critères, une variété de modules (ou plutôt, un schéma de modules) pour la classification des variations (globales, ou infinitésimales) de certaines structures (variétés complètes non singulières, fibrés vectoriels, etc.) ne peut exister, malgré de bonnes hypothèses de platitude, propreté, et non singularité éventuellement, la raison en est seulement l'existence d'automorphismes de la structure qui empêche la technique de descente de marcher.

Grothendieck's letter to Serre, 1959 Nov 5.

2.1 Algebraic spaces

First recall that a Grothendieck topology on an arbitrary category is basically a choice of class of morphisms that will play the analogue role of the open sets (for the details, see for example [Mum63, p. 38]): a morphism $f : V \to U$ is to be though of as an open set in the object U. The concept of intersection is replaced by the fiber product (cf [Mum63, pp. 35-36]): the 'intersection' of $f_1 : U_1 \to U$ and $f_2 : U_2 \to U$ is $f_{12} : U_1 \times_U U_2 \to U$.

A category with a Grothendieck topology is called a *site*. Here we will consider the following topologies on (Sch/S):

fppf topology. Let U be a scheme. Then a cover of U is a finite collection of morphisms $\{f_i : U_i \to U\}_{i \in I}$ such that $\coprod U_i \to U$ is '*fidèlement plat de présentation finie*', i.e., such that each f_i is a finitely presented flat morphism.

Étale topology. The same definition, replacing flat by étale.

Definition 3 (Presheaf of sets)**.** A presheaf of sets on (Sch/S) is a contravariant functor F from (Sch/S) to (Sets). We will use the following usual notation: if $X \in F(U)$ and $f_i : U_i \to U$ is a morphism, then $X|_i$ is the element of $F(U_i)$ given by $F(f_i)(X)$, and we call it the 'restriction of X to U_i' (even if f_i is not an inclusion!). If $X_i \in F(U_i)$, then $X_i|_{ij}$ is the element of $F(U_{ij})$ given by $F(f_{ij,i})_(X_i)$, where $f_{ij,i} : U_i \times_U U_j \to U_i$ is the pullback of f_j.

Example 1 (Functor of points)*.* Let X be an S-scheme. The *functor of points* h_X is a presheaf of sets on (Sch/S), given by assigning to every S-scheme Y the set of S-morphisms $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, X)$. We say that a presheaf of sets is *representable* if it is isomorphic to h_X for some S-scheme X.

Example 2 (Hilbert functor)*.* The Hilbert functor defined above is a presheaf of sets.

Example 3 (Curves)*.* The moduli functor $\mathcal{M}_g$ of smooth curves of genus g over a noetherian base S is the functor that sends each S-scheme B to the set $\mathcal{M}_g(B)$ of isomorphism classes of smooth and proper morphisms $C \to B$ (where C is also an S-scheme) whose fibers are geometrically connected curves of genus g. Each morphism $f : B' \to B$ is sent to the map of sets given by the pullback.

Example 4 (Fiber product)*.* Let $\alpha_1 : F_1 \to G$ and $\alpha_2 : F_2 \to G$ be two morphisms of presheaves (i.e. natural transformations of functors), then the *fiber product* $F_1 \times_G F_2$ is the presheaf defined, for every S-scheme B, by

$$(F_1 \times_G F_2)(B) = \{(u_1, u_2) \in F_1(B) \times F_2(B) | a_1(u_1) = a_2(u_2) \text{in } G(B)\}$$

and similarly for morphisms.

Definition 4 (Space)**.** Fix a topology on (Sch/S). We say that F is an S-space (or a sheaf) if for every cover $\{f_i : U_i \to U\}_{i \in I}$ in the topology, the following two axioms are satisfied:

- *Mono*: let $X, Y \in F(U)$. If $X|_i = Y|_i$ for all i, then $X = Y$.
- *Glueing*: let $X_i \in F(U_i)$ for each i such that $X_i|_{ij} = X_j|_{ij}$; then there exists $X \in F(U)$ such that $X|_i = X_i$ for each i.

Example 5 (Functor of points)*.* The functor of points defined above is indeed a sheaf: first, note that we have the following sequence of finer Grothendieck topologies:

$$X_{Zar} \hookrightarrow X_{ét} \hookrightarrow X_{fppf} \hookrightarrow X_{fpqc}$$

Indeed, the two first arrows are easy to check, and a proof for the third can be found in [Fan+05, Prop. 2.35]. Hence, it is enough to show the sheaf axioms for the fpqc-topology, something that it is done, for example, in [Stacks, Tag 023Q].

Hence, S-schemes are a subcategory of S-spaces, and Yoneda's lemma states that it is indeed a full subcategory. We will sometimes abuse notation and write X for h_X when the context is clear.

Example 6 (Hilbert scheme). Now we solve the previous mystery: the Hilbert functor is representable by a scheme $\mathcal{H}_P$. For a proof, see [Kol96].

Example 7 (Curves). The functor $\mathcal{M}_g$ defined above is not a sheaf (and therefore is not representable by example 5) because of the presence of automorphisms. Indeed, given a curve C over S with nontrivial automorphisms, it is possible to construct a family $f : \mathcal{C} \to B$ such that every fiber of f is isomorphic to C, but $\mathcal{C}$ is not isomorphic to $B \times C$ (see [Edi97]).

Definition 5 (Equivalence relation and quotient space)**.** An equivalence relation in (Sch/S) consists of two S-spaces R and U and a monomorphism of S-spaces $\delta : R \to U \times_{h_S} U$ such that for all S-scheme B, the map $\delta(B) : R(B) \to (U \times_{h_S} U)(B)$ is the graph of an equivalence relation between sets. A quotient S-space for such an equivalence relation is by definition the sheaf cokernel of the diagram

$$R \underset{p_2 \circ \delta}{\overset{p_1 \circ \delta}{\rightrightarrows}} U.$$

Note that here the cokernel V exists, and can be built as follows: for every S-scheme B, let $V(B) = U(B)/\sim$. This gives you a presheaf, and after shefifying you obtain V.

Definition 6 (Algebraic space)**.** An S-space F is called an algebraic space if it is the quotient S-space for an equivalence relation such that R and U are S-schemes, $p_1 \circ \delta$ and $p_2 \circ \delta$ are étale (morphisms of S-schemes), and δ is a quasi-compact morphism (of S-schemes).

Roughly speaking, it is the quotient of a scheme by an étale equivalence relation. We have an alternative definition:

Definition 7 (Algebraic space (2))**.** An S-space F is called an algebraic space if there exists a scheme U (called atlas) and a morphism of S-spaces $u : U \to F$ such that

1. *The morphism u is étale.* For any S-scheme V and morphism (of S-spaces) $V \to F$, the (sheaf) fiber product $U \times_F V$ is representable, and the map $U \times_F V \to V$ is an étale morphism of schemes.

2. *Quasi-separatedness.* The morphism $U \times_F U \to U \times_S U$ is quasi-compact.

Remark 1. We recover the previous definition by taking $R = U \times_F U$.

Algebraic spaces and schemes

Now we follow Artin in [Art71], who first defined the notion of algebraic space, in order to compare the notions of algebraic spaces and schemes. The idea is to construct global objects by glueing affine schemes: if we glue them in one way or another, we get algebraic spaces or schemes.

He starts with an analytic space X, and the problem is to give an algebraic structure to it. First we give some definitions:

Definition 8 (Analytic variety (over $\mathbb{C}$))**.** We say that $(Z, \mathcal{O}_Z)$ is an analytic variety if Z is the zero locus of some analytic functions $f_1, \dots, f_k$ defined in some open set $U \subset \mathbb{C}^n$, and $\mathcal{O}_Z$ is the structure sheaf given by $\mathcal{O}_U / < f_1, \dots, f_k >$.

Remark 2. Note that analytic varieties are analogous to algebraic varieties: we just substitute polynomials by analytic functions.

Definition 9 (Analytic space (over $\mathbb{C}$))**.** An analytic space is a locally ringed space $(X, \mathcal{O}_X)$ such that for every point $x \in X$, there exists an open neighborhood U such that $(U, \mathcal{O}_U)$ is isomorphic to an analytic variety.

Remark 3. Closed analytic subspaces of $\mathbb{P}^n_{\mathbb{C}}$ are algebraic. This is Chow's theorem.

So let's assume that X is closed. Then we can cover X by a finite set of affines schemes U_i. Since the finite[1] disjoint union of affine schemes is again affine, we get a surjective map $f : \coprod U_i = U \to X$, where f is a map between analytical spaces. We write $R = U \times_X U \subset U \times_{\mathbb{C}} U$.

Now, we put some requirements to $U \to X$: if the U_i's are mapped isomorphically onto their images in X, we get the structure of a scheme; if instead we just ask to $U \to X$ to be étale, we get an algebraic space.

Putting everything together, we recover our first definition: an algebraic space X consists on an affine scheme U and a closed subscheme $R \subset U \times U$ such that

1. R is an equivalence relation.

2. The projection maps are étale.

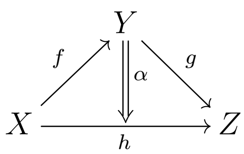
The algebraic space is a scheme if moreover

3 The restriction of R to every connected component of U is the trivial diagonal equivalence relation.

Remark 4. The underlying point set of X is precisely $|U|/|R|$.

2.2 Interlude on 2-categories

Definition 10 (2-category)**.** A 2-category C consists of the following data: a class of objects $\mathrm{ob}(C)$, and for each pair of objects X, Y, a category $\mathrm{Hom}(X, Y)$. We call the objects of $\mathrm{Hom}(X, Y)$ *1-morphisms*, and given two 1-morphisms $f, g \in \mathrm{Hom}(X, Y)$, an element $\alpha \in \mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f, g)$ is called a *2-morphism*. We add the expected axioms (identity elements, associativity, composition, etc.; c.f. [Góm01] for the details), and we will say that a diagram of 1-morphisms commutes in a 2-category if is of the shape



[1]Note that an infinite disjoint union of affine schemes is not affine, since $\coprod U_i$ is not quasi-compact and affine schemes are.

and α defines a 2-isomorphism between $g \circ f$ and h. Warning: note that in general, $g \circ f \neq h$. Some authors say that the diagram 2-commutes, because it commutes up to a 2-morphism.

Example 8. The example to keep in mind is (Cat), the 2-category of all categories (1-categories). The objects of (Cat) are categories, and for each pair of categories X, Y, $\mathrm{Hom}(X, Y)$ is just the category of functors between X and Y: 1-morphisms are functors and 2-morphisms are natural transformations.

Example 9. If the only 2-morphisms in our category C are the identities (i.e. $\mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f, g)$ is empty if $f \neq g$, and else equal to the identity functor $\{f \to f\}$), and each of the categories $\mathrm{Hom}(X, Y)$ is small (i.e. the class of objects is indeed a set), then we can regard C as a 1-category.

2-functors

A covariant 2-functor F between two 2-categories C and C' is a law that for each object X in C, it gives an object $F(X)$ in C', and similarly with 1-morphisms and 2-morphisms. Moreover, this law must satisfy the usual commutative rules. Note that in particular, F respects composition of 1-morphisms up to a 2-morphism.

2.3 Stacks

Here we will follow [Fan01] as a complement for [Góm01]. We begin with an example in order to have something down to earth in mind while developing the abstract definitions.

Example 10 (The category V_r). Let B be an S-scheme, and let X be a vector bundle of rank r over it, i.e. a S-scheme X and a morphism $p : X \to B$ with additional data consisting of an open covering $\{U_i\}$ of B, and isomorphisms $\phi_i : p^{-1}(U_i) \to \mathbb{A}^r_{U_i}$, such that for any i, j, and for any open affine subset $V = \mathrm{Spec}(A) \subset U_i \cap U_j$, the automorphism $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ of $\mathbb{A}^r_V = \mathrm{Spec}(A[x_1, \dots, x_r])$ is given by a linear automorphism θ of $A[x_1, \dots, x_r]$, i.e., $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij} x_j$ for suitable $a_{ij} \in A$. Alternatively, one can just take a locally free sheaf of rank r, and we will also denote it by X (c.f. [Har77, Ex. II.5.18]).

Now let $f : B' \to B$ be a morphism of S-schemes. We call a diagram

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & X \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

pullback diagram if X' is a vector bundle over B', and the diagram makes X' into the pullback of X via f (hence, the diagram is cartesian and $\overline{f}$ induces a linear isomorphism on fibers). We will also say that $(X', \overline{f})$ is a *pullback of* X *via* f. Pullback is esentially unique (given another pullback X'', there is a unique isomorphism from X' to X'').

We define the category V_r as follows: its objects are rank r vector bundles over S-schemes, and its morphisms are pullback diagrams (i.e. the above diagram defines a morphism from X' to X). There is a natural forgetful functor $p_{V_r} : V_r \to (\mathrm{Sch}/S)$, which sends $X \to B$ to the base B and every pullback diagram to f.

Category fibered on groupoids

Definition 11. A *category over* (Sch/S) is a category $\mathcal{F}$ and a covariant functor $p_{\mathcal{F}} : \mathcal{F} \to (\mathrm{Sch}/S)$ (called the structure functor). If X is an object (resp. ϕ is a morphism) of $\mathcal{F}$, and $p_{\mathcal{F}}(X) = B$ (resp. $p_{\mathcal{F}}(\phi) = f$), then we say that X lies over B (resp. ϕ lies over f). If B is an S-scheme, the *fiber* of $\mathcal{F}$ over B is the subcategory of objects over B, and morphisms over the identity of B. We denote it $\mathcal{F}(B)$.

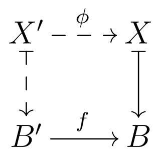
Example 11. V_r is a category over (Sch/S); the fiber over B is the category whose objects are vector bundles over B, and whose morphisms are the isomorphisms among them.

Recall that a *groupoid* is a category where every morphism is invertible. Note that if we start with a group G and consider the category $\mathcal{G}$, given by an object $\heartsuit$ and the set of morphisms $\mathrm{Mor}(\heartsuit, \heartsuit) = G$, we get a groupoid.

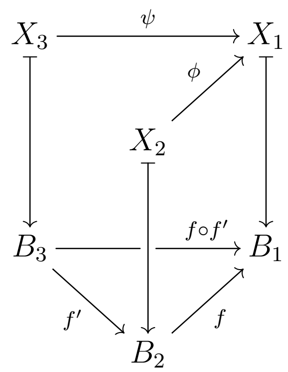
Hence, the fiber of V_r over B, $V_r(B)$, is a groupoid.

Definition 12. A category $\mathcal{F}$ over (Sch/S) is called a *category fibered on groupoids* if

1. For every $f : B' \to B$ in (Sch/S) and every object X lying over B, there exists at least one object X' and a morphism $\phi : X' \to X$ over f:



2. For every commutative diagram



where the vertical arrows are given by $p_{\mathcal{F}}$, there exists a unique $\varphi : X_3 \to X_2$ that fits in the diagram and makes everything commutative.

Remark 5. Condition 2 implies that the object X', whose existence is asserted by the first condition, is unique up to unique isomorphism. Indeed, if we take in the big diagram $B_2 = B_3$ and $f' = id_{B_2}$, we get a unique morphism $\varphi : X_3 \to X_2$. Since we can do the same construction with $(id_{B_2})^{-1}$, φ is an isomorphism. Hence, every fiber $\mathcal{F}(B)$ is indeed a groupoid.

Remark 6. For each X and f, we choose once and for all such a X' and we denote it as f^*X, and if the morphism is clear from he context, we write $X|_{B'}$ instead of f^*B (for example, when f is an inclusion).

Descent data

Now we want to define the descent data. This will allow us to glue objects.

Example 12 (Descent data on V_r). For example, if we have an open covering $\{B_i \to B\}$ of an S-scheme, and a vector bundle X over B, we may not be able to recover X from the $X_i := X|_{B_i}$, as X may not be trivial and at the same time, trivial in the restrictions. But the fact that X_i is the pullback of X means that we have induced isomorphisms $\phi_{ij} : X_i|_{B_{ij}} \to X_j|_{ij}$ which satisfy the cocycle condition on B_{ijk}, i.e. $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on B_{ijk}, and now we are able to recover X from the X_i, since now we know how to glue them. In general, we have the following definition:

Definition 13 (Descent data)**.** Let $\mathcal{F}$ be a category fibered in groupoids over (Sch/S). A *descent datum* for $\mathcal{F}$ over an S-scheme B is the following: an open covering $\{B_i \to B\}$; for every i, a lifting X_i of B_i to $\mathcal{F}$; for every i, j an isomorphism $\phi_{ij} : X_i|_{B_{ij}} \to X_j|_{B_{ij}}$ in the fiber which satisfies the cocycle condition $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ over B_{ijk}.

We say that the descent datum is *effective* if there exists a lifting X of B to $\mathcal{F}$ together with isomorphisms $\phi_i : X|_{B_i} \to X_i$ in the fiber such that $\phi_{ij} = \phi_j|_{B_{ij}} \circ (\phi_i|_{B_{ij}})^{-1}$.

Example 13. The descent data of V_r is effective.

Stacks as categories

First, we fix a topology on (Sch/S). Here we will be mainly interested in the étale and the fppf topologies. Until the definition of algebraic stacks, the distinction will not matter at all.

Definition 14 (Stack)**.** A stack is a category $\mathcal{F}$ fibered in groupoids over (Sch/S) such that

1. For every S-scheme B and pair of objects X, Y of $\mathcal{F}$ over B, the contravariant functor

$$\begin{array}{rccc} \mathrm{Hom}_B(X,Y): & (\mathrm{Sch}/B) & \longrightarrow & (\mathrm{Sets}) \\ & (f : B' \to B) & \longmapsto & \mathrm{Hom}(f^*X, f^*Y) \end{array}$$

   is a sheaf on the site (Sch/B).

2. Descent data is effective.

Remark 7. The notion of stack is just a categorical notion. Now we will define morphisms of stacks and the fiber product of stacks, and then we will give define the notion of algebraic stack. The advantage of this notion will be that it has an analogy with the notion of algebraic space, so we will be able to see some geometry behind these categorical concepts. If we choose the étale topology on (Sch/S), we obtain the definition of a Deligne-Mumford stack (c.f. [DM69]), and with the fppf we get an Artin stack.

Stacks as functors

There is another way to define a stack, and this point of view is to regard a stack as a sheaf of groupoids.

Definition 15 (Stack as a functor)**.** A stack is a sheaf of groupoids $\mathcal{F}$, i.e. a contravariant 2-functor (a presheaf) from (Sch/S) to (Groupoids) that satisfies the following sheaf axioms. Let $\{f_i : U_i \to U\}_{i\in I}$ be a covering of U in the site (Sch/S). Then

1. *Glueing of morphisms.* If X and Y are two objects of $\mathcal{F}(U)$, and $\phi_i : X|_i \to Y|_i$ are morphisms such that $\phi_i|_{ij} = \phi_j|_{ij}$, then there exists a morphism $\phi : X \to Y$ such that $\phi|_i = \phi_i$.

2. *Monopresheaf.* If X and Y are two objects of $\mathcal{F}(U)$, and $\phi : X \to Y$, $\psi : X \to Y$ are morphisms such that $\phi|_i = \psi|_i$, then $\phi = \psi$.

3. *Glueing of objects.* If X_i are objects of $\mathcal{F}(U_i)$ and $\phi_{ij} : X_i|_{ij} \to X_j|_{ij}$ are morphisms satisfying the cocycle condition, then there exists an object X of $\mathcal{F}(U)$ and isomorphisms $\phi_i : X|_i \to X_i$ such that $\phi_{ij} \circ \phi_i|_{ij} = \phi_j|_{ij}$.

Note that the third condition is equivalent to saying that the descent data is effective.

We now see how to go from one definition to the other, first comparing a category fibered on groupoids and a presheaf on groupoids: start with an object B of (Sch/S). Recall that we defined $\mathcal{F}(B)$ to be the fiber of $\mathcal{F}$ over B. Since $\mathcal{F}(B)$ is a groupoid, this association defines a presheaf of groupoids (everything is well defined because of the conditions of the definition 12 of category fibered on groupoids). Conversely, given a presheaf (a 2-contravariant functor) from (Sch/S) to the category of groupoids $\mathcal{G}$, we can define the category $\mathcal{F}$ whose objecs are pairs (B, X) where B is an object of (Sch/S) and X is an object of $\mathcal{G}$, and whose morphisms $(B', X') \to (B, X)$ are pairs (f, α), where $f : B' \to B$ is a morphism in (Sch/S) and $\alpha : f^*X \to X'$ is an isomorphism, where $f^* = \mathcal{G}(f)$. This gives us the relationship netween category fibered on groupoids and presheaves of groupoids, and one sees that the conditions of the definitions of stack are just a translation from one side to the other.

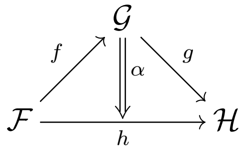
Hence, we have the following picture:

$$\begin{array}{ccccccc} & & & & \text{Stacks} & \longrightarrow & \text{Presheaves of groupoids} \\ & & & & \uparrow & & \uparrow \\ (\text{Sch}/S) & \longrightarrow & \text{Algebraic spaces} & \longrightarrow & \text{Spaces} & \longrightarrow & \text{Presheaves of sets} \end{array}$$

where each arrow means that we have a full subcategory. We will complete the picture by defining algebraic stacks, but first we will define morphisms of stacks.

Morphisms of stacks

Definition 16. A morphism of stacks (over (Sch/S)) $f : \mathcal{F} \to \mathcal{G}$ is a 2-functor between the categories such that $p_{\mathcal{G}} \circ f = p_{\mathcal{F}}$. A commutative diagram of stacks is a diagram



such that $\alpha : g \circ f \to h$ is an isomorphism of functors. If f is an equivalence of categories, then we say that the stacks $\mathcal{F}$ and $\mathcal{G}$ are isomorphic. We denote by $\mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$ the category whose objects (1-morphisms) are morphisms of stacks and whose morphisms (2-morphisms) are natural transformations. With this structure, (Stacks) forms a 2-category.

Remark 8. Note that Grothendieck's difficulties to define a moduli space were the existance of automorphisms (c.f. remark 9), and this is solved by conidering 2-categories as we will see.

Before we study different properties of morphisms, we need the notions of representability and of fiber product.

2.4 Algebraic stacks

Representability of stacks

Given an S-scheme U, consider the category (Sch/U). If we define the functor p_U : (Sch/U) $\to$ (Sch/S) that sends B to B regarded as an S-scheme, and U-morphisms $f : B' \to B$ to the same morphism, but regarded as a morphism of S-schemes, then (Sch/U) becomes a stack. By abuse of notation, we call this stack U. If we think in the 2-functor point of view, then the stack associated to U is the 2-functor that associates, for each S-scheme B, the category $\mathrm{Hom}_S(B, U)$ (where the 2-morphisms are just the identities).

Definition 17. We say that a stack is represented by a scheme (resp. algebraic space) U if it is isomorphic to the stack associated to U.

Remark 9. Note that if a stack has an object with an automorphism other that the identity, then the stack can't be represented by a scheme, since we would have a 2-morphism different from the identity.

Remark 10. Let $\mathcal{F}$ be an stack and U an S-scheme. The functor

$$u : \mathrm{Hom}_S(U, \mathcal{F}) \to \mathcal{F}(U)$$

that sends f to $f(id_U)$ is an equivalence of categories. This follows from Yoneda's lemma.

Fiber product of stacks

Given two morphisms of stacks $f_1 : \mathcal{F}_1 \to \mathcal{G}$, $f_2 : \mathcal{F}_2 \to \mathcal{G}$, we define a new stack $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ as follows: the objects are triples (X_1, X_2, α) where X_i belongs to $\mathcal{F}_i$ and they lie over the same S-scheme U, and $\alpha : f_1(X_1) \to f_2(X_2)$ is an isomorphism in $\mathcal{G}$ (or equivalently, $p_{\mathcal{G}}(\alpha) = id_U$). A morphism from (X_1, X_2, α) to (Y_1, Y_2, β) is a pair (ϕ_1, ϕ_2) of morphisms $\phi_i : X_i \to Y_i$ that lie over the same morphism of schemes $f : U \to V$, and such that $\beta \circ f_1(\phi_1) = f_2(\phi_2) \circ \alpha$. One can check that the fiber product satisfies the usual universal property.

Representability of morphisms and properties

A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is representable if for all objects U in (Sch/S) and morphisms $U \to \mathcal{G}$, the fiber product stack $U \times_{\mathcal{G}} \mathcal{F}$ is representable by an algebraic space. Let P be a property of morphisms of schemes that is local in nature on the target for the topology chosen on (Sch/S) (étale or fppf), and it is stable under arbitrary base change (e.g. separated, quasi-compact, unramified, flat, smooth, étale, surjective, finite type, locally of finite type,...). Then, for a representable morphism f, we say that f has property P if for every $U \to \mathcal{G}$, the pullback $U \times_{\mathcal{G}} \mathcal{F} \to U$ has property P.

A very important notion is the diagonal of a stack:

Definition 18. Given a stack $\mathcal{F}$, the diagonal is the obvious $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$, i.e. the fiber product of the structure functor $p_{\mathcal{F}} : \mathcal{F} \to (\mathrm{Sch}/S)$ with itself.

Remark 11. Note that a morphism from a S-scheme U to $\mathcal{F} \times_S \mathcal{F}$ is, by remark 10, equivalent to two objects X_1, X_2 of $\mathcal{F}(U)$. Taking now the fiber product of U and $\mathcal{F}$, we get precisely

$$\begin{array}{ccc} \mathrm{Hom}_U(X_1, X_2) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \Delta_{\mathcal{F}} \\ U & \xrightarrow{(X_1, X_2)} & \mathcal{F} \times_S \mathcal{F} \end{array}$$

and recall that $\mathrm{Hom}_U(X_1, X_2)$ forms a groupoid because of condition 2 on definition 12. Because of this, sometimes people denote this as $\mathrm{Iso}_U(X_1, X_2)$. Hence, the information of the group of automorphisms of an object is encoded in the diagonal morphism.

Now, we state a proposition that will allow us to define the notion of an algebraic stack. A proof can be found, for example, in [Góm01, Prop. 2.23].

Proposition 2. *Let $\mathcal{F}$ be a stack. The following are equivalent:*

1. *The morphism $\Delta_{\mathcal{F}}$ is representable.*

2. *The stack $Iso_U(X_1, X_2)$ is representable for all U, X_1 and X_2.*

3. *For all scheme U, every morphism $U \to \mathcal{F}$ is representable.*

4. *For all schemes U, V and morphisms $U \to \mathcal{F}$ and $V \to \mathcal{F}$, the fiber product $U \times_{\mathcal{F}} V$ is representable.*

Algebraic stacks

Definition 19 (Deligne-Mumford stack)**.** Let (Sch/S) with the étale topology. Let $\mathcal{F}$ be a stack. We say that $\mathcal{F}$ is a Deligne-Mumford stack if

1. *Quasi-separatedness.* The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated.

2. There exists a scheme U (called atlas) and an étale surjective morphism $u : U \to \mathcal{F}$.

Definition 20 (Artin stack)**.** Let (Sch/S) with the fppf topology. Let $\mathcal{F}$ be a stack. We say that $\mathcal{F}$ is a Deligne-Mumford stack if

1. *Quasi-separatedness.* The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated.

2. There exists a scheme U (called atlas) and an smooth surjective morphism $u : U \to \mathcal{F}$.

Remark 12. This definition with an atlas is similar to the second definition of algebraic space.

Remark 13. The last proposition ensures us that the morphism u is representable, since the diagonal $\Delta_{\mathcal{F}}$ is assumed to be representable. Hence the definitions make sense.

This completes our previous picture:

$$\begin{array}{ccccccc}
 & \nearrow & \text{Algebraic stacks} & \longrightarrow & \text{Stacks} & \longrightarrow & \text{Presheaves of groupoids} \\
 & & \uparrow & & \uparrow & & \uparrow \\
(\mathrm{Sch}/S) & \longrightarrow & \text{Algebraic spaces} & \longrightarrow & \text{Spaces} & \longrightarrow & \text{Presheaves of sets}
\end{array}$$

We know follow [LMB00] in order to define properties of stacks and morphisms. As before, we are interested in the properties P that are local in nature and stable under base change.

Definition 21 ([LMB00][Def. 4.7.1])**.** An algebraic stack $\mathcal{F}$ has property P if for one (and therefore for all) presentation $u : U \to \mathcal{F}$, U has the property P.

Definition 22 ([LMB00][Def. 4.14])**.** We say that (1-)morphism $F : \mathcal{F}_1 \to \mathcal{F}_2$ of algebraic stacks has the property P if, for a presentation f of F, i.e. a commutative diagram of 1-morphisms

$$\begin{array}{ccccc}
U_1' & \xrightarrow{u_1'} & \mathcal{F}_1' & \longrightarrow & \mathcal{F}_1 \\
 & \searrow^{f} & \downarrow & & \downarrow F \\
 & & U_2 & \xrightarrow{u_2} & \mathcal{F}_2
\end{array}$$

where u_1' and u_2 are the presentations of $\mathcal{F}_1' = U_2 \times_{u_2, \mathcal{F}_2, F} \mathcal{F}_1$ and $\mathcal{F}_2$ respectively, the morphism $f : U_1' \to U_2$ has this property P.

2.5 Final remark

There is still another way to define a stack, which is analogue to the method used to define an algebraic space via a quotient.

If C is a category, we denote by U the class of objects and by R the class of morphisms. From the axioms of a category we get the 'maps'

$$R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{e} R \qquad\qquad R \times_{s,U,t} R \xrightarrow{m} R,$$

where s and t give the source and he target of a morphism respectively, e gives the identity morphism and m the composition. If C is a groupoid, we also have

$$R \xrightarrow{i} R$$

which assigns to a morphism its inverse. These maps satisfy

1. $s \circ e = t \circ e = id_U$, $s \circ i = t$, $t \circ i = s$, $s \circ m = s \circ p_2$ and $t \circ m = t \circ p_1$.
2. *Associativity.* $m \circ (m \times id_R) = m \circ (id_R \circ m)$.
3. *Identity.* Both compositions

$$R = R \times_{s,U} U \xrightarrow{id_R \times e} R \times_{s,U,t} R \xrightarrow{m} R$$

$$R = U \times_{U,t} R \xrightarrow{e \times id_R} R \times_{s,U,t} R \xrightarrow{m} R$$

   are the identity on R.

4. *Inverse.* In R, $m \circ (i \times id_R) = e \circ s$, $m \circ (id_R \times i) = e \circ t$.

Definition 23 (Groupoid space)**.** A groupoid space is a pair of spaces U, R with five morphisms s, t, e, m, i satisfying the above properties.

Definition 24. Given a groupoid space, define the groupoid over (Sch/S) as the category $[R, U]'$ over (Sch/S) whose objects over the S-scheme B are the elements of the set $U(B)$ and whose morphisms over B are elements of the set $R(B)$. Given $f : B' \to B$, we define a functor $f^* : [R, U]' \to [R, U]'(B')$ using the maps $U(B) \to U(B')$ and $R(B) \to R(B')$.

This defines just a prestack (i.e. there may be non effective descent data). We denote by $[R, U]$ the associated stack. This can be thought of as the sheaf associated to the presheaf of groupoids $B \mapsto [R, U]'(B)$ (c.f. [LMB00, Lemma 3.2]).

Example 14. Let R, U be a groupoid space such that R and U are algebraic spaces, locally of finite presentation. Assume that the morphisms s, t are flat, and that $\delta = (s, t) : R \to U \times_S U$ is separated and quasi-compact. Then $[R, U]$ is an Artin stack, locally of finite type (c.f. [LMB00, Cor. 10.4]).

Remark 14. Any Artin stack can be defined in this fashion. The algebraic space U will be the atlas of $\mathcal{F}$, and we set $R = U \times_{\mathcal{F}} U$. The morphisms s and t are the two projections, i exchanges the factors, e is the diagonal, and m is defined by the projection to the first and third factor.

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