Introduction to crystalline cohomology

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Abstract

This is the second talk of the Research Seminar "Supersingular K3 surfaces are unirational" of the Arithmetic Geometry group (FU Berlin), which is on the Winter Semester 2015-2016. In the seminar we study the result of Liedtke in [Lie15b]:

Theorem 1. *Supersingular K3 surfaces are unirational.*

One of the ingredients of the proof is crystalline cohomology, and this talk is devoted to give an introduction to it. In these notes for the talk you can find the following:

- We first give a motivation, explaining why crystalline cohomology is like a "p-adic" cohomology, and first we recall the construction and some properties of ℓ-adic cohommology.
- After this, we recall the construction and m ain properties of the Witt vector, since crystalline cohomology will have these coefficients. After this, we sheafify this construction as in [Ser58], but the first idea that one may have, which is to take sheaf cohomology with this coefficients, doesn't work.
- Hence, we move on to crystalline cohomology, explaining some properties and giving the definition. We also do a couple of examples.
- Finally, we look at the de Rham-Witt complex, and state the canonical decomposition of the crystalline cohomology given by the Frobenius action.

1 Motivation

What is brown outside, white inside and is very delicate?
Crystalline cocohomology.

Crystalline cohomology was at first motivated by the search of a cohomology theory analogous to the ℓ-adic cohomology for a scheme over a field of characteristic p, with $p \neq \ell$. In fact, under the assumption $\ell \neq p$, ℓ-adic cohomology has a lot of nice properties which become false if we allow $\ell = p$.

For example, let's consider a smooth and proper scheme $\mathcal{X}$ over $\mathbb{Z}_p$, with special fibre X over $\mathbb{F}_p$, and let $\overline{X} := X \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$, where $\overline{\mathbb{F}_p}$ is an algebraic closure. On the other hand, consider $\mathcal{X}^{an}$ the complex analytic variety defined by X and an inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{C}$.

$$\begin{array}{ccccccc} \overline{X} & \longrightarrow & X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}^{an} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(\overline{\mathbb{F}_p}) & \longrightarrow & \operatorname{Spec}(\mathbb{F}_p) & \longrightarrow & \operatorname{Spec}(\mathbb{Z}_p) & \longleftarrow & \operatorname{Spec}(\mathbb{C}) \end{array}$$

Then, for $\ell \neq p$, we have isomorphisms

$$H^i_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell) \cong H^i(\mathcal{X}^{an}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

Hence, knowing the ℓ-adic cohomology of $\overline{X}$ is the same as knowing the rank and the ℓ-torsion of the singular cohomology of $\mathcal{X}^{an}$. But this isomorphism is no longer true for $\ell = p$, as we will see in an example, so we want to define a different cohomology theory that has the properties of the ℓ-adic cohomology, but with $\ell = p$. Once we have this, we will be able to study also the p-torsion phenomena. This was done in Berthelot's thesis [Ber74].

2 ℓ-adic cohomology

Recall that for a prime number ℓ, we define the ℓ-adic cohomology as

$$H^n_{\acute{e}t}(X, \mathbb{Z}_\ell) := \lim_{\leftarrow} H^n_{\acute{e}t}(X, \mathbb{Z}/\ell^m\mathbb{Z})$$

and we also define $H^n_{\acute{e}t}(X, \mathbb{Q}_\ell) := H^n_{\acute{e}t}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. This cohomology theory has a lot of convenient properties. For instance, if we assume that X is smooth, irreducible and proper over an algebraically closed field (of any characteristic), we have:

1. $H^n_{\acute{e}t}(X, \mathbb{Q}_\ell)$ is a contravariant functor in X. The cohomology groups are finite dimensional $\mathbb{Q}_\ell$-vector spaces, zero if $n \notin [0, \dots, 2\dim(X)]$ and we also have that $H^{2\dim(X)}_{\acute{e}t}(X, \mathbb{Q}_\ell)$ is 1-dimensional.

2. There is a cup-product structure

$$\cup_{i,j} : H^i_{\acute{e}t}(X, \mathbb{Q}_\ell) \times H^j_{\acute{e}t}(X, \mathbb{Q}_\ell) \to H^{i+j}_{\acute{e}t}(X, \mathbb{Q}_\ell)$$

   which is a perfect pairing for $\cup_{n, 2\dim(X)-n}$ (and n in the above interval), called *Poincaré duality*.

3. $H^n_{\acute{e}t}(X, \mathbb{Z}_\ell)$ defines an integral structure on $H^n_{\acute{e}t}(X, \mathbb{Q}_\ell)$.

4. If $\operatorname{char}(k) = p > 0$, and $\ell \neq p$, then the dimensions $\dim_{\mathbb{Q}_\ell} H^n_{\acute{e}t}(X, \mathbb{Q}_\ell)$ is independent of ℓ. Hence, the Betti number

$$b_n(X) := \dim_{\mathbb{Q}_\ell} H^n_{\acute{e}t}(X, \mathbb{Q}_\ell)$$

   is well defined for $\ell \neq p$.

5. If $k = \mathbb{C}$, we can choose an inclusion $\mathbb{Q}_\ell \subset \mathbb{C}$ (note that this will not be a continuous inclusion), and then, there exist isomorphisms

$$H^n_{\acute{e}t}(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong H^n(X, \underline{\mathbb{C}})$$

where X is considered as a differentiable manifold on the right hand side and $\underline{\mathbb{C}}$ is locally constant with respect to the analytic topology.

6. There is a Lefschetz fixed point formula, there exist base change formulas, cycle classes in $H^{2q}_{\acute{e}t}(X, \mathbb{Q}_\ell)$ of codimension q subvarieties, ...

Remark 1. The assumption in 4. of $\ell \neq p$ is crucial. For example, if A is a g-dimensional abelian variety over an algebraically closed field of characteristic p, $\dim_{\mathbb{Q}_\ell} H^1_{\acute{e}t}(A, \mathbb{Q}_\ell) = 2g$ if $\ell \neq p$, and $\dim_{\mathbb{Q}_p} H^1_{\acute{e}t}(A, \mathbb{Q}_p) = r$ for some $r \leq g$.

The crystalline cohomology that we will later define will have similar properties. One of the differences is that instead of vector spaces, we will have modules over the ring of Witt vectors. Let's recall the construction of this ring.

3 Witt vectors

Serre observed in [Ser58] that there can't be a Weil cohomology with coefficients in $\mathbb{Q}_p$ for schemes X over a field of characteristic p. That's why we introduce the ring of Witt vectors, because there we will be able to obtain our desired cohomology theory.

We know (c.f. [Ser79]) the following theorem:

Theorem 2. *For every perfect field k of characteristic p, there exists a complete discrete valuation ring and only one (up to unique isomorphism) which is absolutely unramified and has k as its residue field.*

We define this ring as the ring of Witt vectors $W(k)$. The name is after the nazi mathematician (c.f. [Sch96]) Ernst Witt, who was able to put a ring structure and define the operations in a computable way. Note that $W(k)$ is unique up to unique isomorphism. More precisely, we know that $W(k)$ satisfies the following:

1. $W(k)$ is a complete discrete valuation ring of characteristic zero.

2. The unique maximal ideal $\mathfrak{m}$ of $W(k)$ is generated by p, and the residue field $W(k)/\mathfrak{m}$ is isomorphic to k.

3. Every $\mathfrak{m}$-adically complete discrete valuation ring of characteristic zero with residue field k contains $W(k)$ as a subring.

4. The Witt ring $W(k)$ is functorial in k, i.e. for every $\phi : k \to k'$, there exists a unique $f : W(k) \to W(k')$ making the obvious diagram commutative.

Explicit construction of $W(k)$

We now give an explicit construction of this ring: define the *Witt polynomials* (with respect to p) to be the following polynomials with coefficients in $\mathbb{Z}$:

$$\begin{array}{lcl} W_0(x_0) & := & x_0 \\ W_1(x_0, x_1) & := & x_0^p + px_1 \\ & \vdots & \\ W_n(x_0, \ldots, x_n) & := & x_0^{p^n} + px_1^{p^{n-1}} + \ldots + p^n x_n = \sum_{i=0}^n p^i x_i^{p^{n-i}} \end{array}$$

One can show [Ser79] that there exist unique polynomials S_n and P_n in $2n+2$ variables with coefficients in $\mathbb{Z}$ such that

$$\begin{array}{lclcl} W_n(x_0, \ldots, x_n) & + & W_n(y_0, \ldots, y_n) & = & W_n(S_0, \ldots, S_n(x_0, \ldots, x_n, y_0, \ldots, y_n)) \\ W_n(x_0, \ldots, x_n) & \cdot & W_n(y_0, \ldots, y_n) & = & W_n(P_0, \ldots, P_n(x_0, \ldots, x_n, y_0, \ldots, y_n)) \end{array}$$

For example, we have

$$\begin{array}{ll} S_0(a_0, b_0) = a_0 + b_0 & S_1(a_0, a_1, b_0, b_1) = a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0+b_0)^p}{p} \\ P_0(a_0, b_0) = a_0 b_0 & P_1(a_0, a_1, b_0, b_1) = b_0^p a_1 + b_1 a_0^p + p a_1 b_1 \end{array}$$

We now construct $W(R)$ for an arbitrary ring R (not necessarily of characteristic p). First, we define the *truncated Witt ring* $W_{n+1}(R)$ to be the set R^{n+1}, together with the operations:

$$\begin{array}{lcl} (x_0, \ldots, x_n) & \oplus & (y_0, \ldots, y_n) \\ & := & (S_0(x_0, y_0), \ldots, S_n(x_0, \ldots, x_n, y_0, \ldots, y_n)) \\ (x_0, \ldots, x_n) & \odot & (y_0, \ldots, y_n) \\ & := & (P_0(x_0, y_0), \ldots, P_n(x_0, \ldots, x_n, y_0, \ldots, y_n)) \end{array}$$

With this operations, $W_n(R)$ is a ring with zero $0 = (0, \ldots, 0)$ and unit $1 = (1, 0 \ldots, 0)$. For example, $W_1(R)$ is just R with the usual operations.

Now we assume that R is of characteristic p, and we introduce two important operations:

$$\begin{array}{rccc} \widetilde{V}: & W_n(R) & \to & W_{n+1}(R) \\ & (x_0, \ldots, x_{n-1}) & \mapsto & (0, x_0, \ldots, x_{n-1}) \end{array}$$

$$\begin{array}{rccc} \sigma: & W_n(R) & \to & W_{n-1}(R) \\ & (x_0, \ldots, x_{n-1}) & \mapsto & (x_0^p, \ldots, x_{n-2}^p) \end{array}$$

$\widetilde{V}$ is called the *shift* or *transfer* (in German this is *Verschiebung*), and it is additive. σ is called the *Frobenius* and it is a ring homomorphism. We have the following relation:

$$\sigma \circ \widetilde{V} = \widetilde{V} \circ \sigma = p \cdot id_{W_n(R)}$$

Here, obviously, multiplication by p is not multiplication by p coordinate-wise (otherwise, since A has characteristic p, this would be the same as multiplication by 0), but adding (using $\oplus$) p times any vector with itself.

Now the projection $W_n(R) \to W_{n-1}(R)$ onto the first $(n-1)$ components is a surjective ring homomorphism, and we can take the projective limit of this projective system in order to get $W(R)$:

$$W(R) := \lim_{\leftarrow} W_n(R)$$

For example, if $R = \mathbb{F}_p$, the Frobenius σ is trivial, so the Verschiebung V is just multiplying by p. Hence, $W_n(\mathbb{F}_p) \subset \mathbb{Z}/p^n\mathbb{Z}$. Moreover, by noting that there is an element of order p^n and comparing sizes, we conclude that this is an equality. Therefore,

$$W(\mathbb{F}_p) = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$$

Sheafification of $W_n(R)$

Given a scheme X over an algebraically closed field of characteristic p, we can sheafify the construction of $W_n(R)$ to obtain sheaves of rings $W_n\mathcal{O}_X$ and $W\mathcal{O}_X$. This was done by Serre in [Ser58] as follows: for any closed $x \in X$, the ring $\mathcal{O}_{X,x}$ is a ring of characteristic p. If we fix n, we form $W_n(\mathcal{O}_{X,x})$, and when we vary x we get a sheaf of rings $W_n\mathcal{O}_X$. We construct similarly $W\mathcal{O}_X$, and we get the cohomology groups $H^i(X, W_n\mathcal{O}_X)$ and $H^i(X, W\mathcal{O}_X)$.

It is easy to prove that if X is a projective variety (as, for example, a K3 surface) over a field of characteristic p, the cohomology groups $H^i(X, W_n\mathcal{O}_X)$ are $W\mathcal{O}_X$-modules of finite length (c.f. [Ser58]). But we have to be careful, because this may not be true for $H^i(X, W\mathcal{O}_X)$: indeed, if X is a supersingular K3 surface, the torsion part of the $W(k)$-module $H^i(X, W\mathcal{O}_X)$ is not finitely generated. In particular, we don't want this to be the p-adic analogous of ℓ-adic cohomology, so we will have to do something else.

4 Crystalline cohomology

Properties of crystalline cohomology

> "Un cristal possède deux propriétés caractéristíques: la rigidité, et la faculté de croitre, dans un voisinage approprié. Il y a des cristaux de toute espèce de substance: des cristaux de soude, de souffre, de modules, d'anneaux, de schémas relatifs etc."[1]
>
> Grothendieck, in a letter to Tate in 1966

Let X be a scheme over k. In the next section we will define the cohomology groups $H^i_{crys}(X/W_n)$, which are finitely generated $W_n(k)$-modules. For $n = 1$, we have $W_1(k) = k$ and we obtain the de Rham cohomology:

$$H^i_{crys}(X/k) \cong H^i_{dR}(X/k)$$

[1] A crystal has two characteristic properties: the rigidity, and the faculty to grow in an adequate neighbourhood. There are crystals of all kind of substances: crystals of soda, of sulphur, of modules, of rings, of relative schemes etc.

The limit of these groups will be the crystalline cohomology:

$$H^i_{crys}(X/W) := \lim_{\leftarrow} H^i_{crys}(X/W_n)$$

Note that X may not have a lift $\mathcal{X}$ over $W(k)$. For example, recall the following theorem:

Theorem 3 (Delligne-Illusie '87)**.** *Let X be a smooth and proper variety over a perfect field of characteristic p of $\dim(X) \leq p$, and assume that X admits a lift to $W_2(k)$. Then, the Fröhlicher spectral sequence of X degenerates at E_1.*

But, after Mumford's work in [Mum61], we know examples of projective and smooth surfaces in positive characteristic p whose Fröhlicher spectral sequence does not degenerate at E_1, so in particular they don't admit a lift to $W_2(k)$.

However, we can still construct crystalline cohomology, we don't need a lift of X: we make the cohomology groups "grow" locally from k, and this grow is kind of "rigid", because we can glue these groups in order to get a group over $W(k)$. This explains the origin of the terminology.

Before we give the construction of crystalline cohomology, let's see which properties will this cohomology satisfy. All the demonstrations can be found in [Ber74].

If X is a proper and smooth scheme over k a perfect field of characteristic $p > 0$, then we can define a cohomology satisfying the following properties:

1. $H^n_{crys}(X/W)$ is a contravariant functor in X. These groups are finitely generated W-modules, and zero if $n \notin [0, 2\dim(X)]$.

2. There is a cup-product $\cup_{i,j}$ structure module torsion that induces a perfect pairing at $\cup_{n,2\dim(X)-n}$ when $n \in [0, 2\dim(X)]$.

3. $H^n_{crys}(X/W)$ defines an integral structure on $H^n_{crys}(X/W) \otimes_W K$.

4. If ℓ is a prime different from p, then

$$\dim_{\mathbb{Q}_\ell} H^n_{\acute{e}t}(X, \mathbb{Q}_\ell) = \text{rank}_W H^n_{crys}(X/W)$$

   so crystalline cohomology computes ℓ-adic Betti numbers.

5. Crystalline cohomology computes the de Rham cohomology:

$$0 \to H^n_{crys}(X/W) \otimes_W k \to H^n_{dR}(X/k) \to Tor_1^W(H^{n+1}_{crys}(X/W), k) \to 0$$

6. There is a Lefschetz fixed point formula, there exist base change formulas, cycle classes in $H^{2q}_{crys}(X/W)$ of codimension q subvarieties, ...

As an application of these facts, we can show that the following are equivalent:

1. For all $n \geq 0$, the W-module $H^n_{crys}(X/W)$ is torsion-free.

2. We have

$$\dim_{\mathbb{Q}_\ell} H^n_{ét}(X, \mathbb{Q}_\ell) = \dim_k H^n_{dR}(X/k)$$

for all $n \geq 0$ and all primes $\ell \neq p$.

Indeed, if $H^n_{crys}(X/W)$ is torsion-free, the exact sequence of 5. induces an isomorphism $H^n_{crys}(X/W) \otimes_W k \cong H^n_{dR}(X/k)$. But by 4., the rank of $H^n_{crys}(X/W)$ is precisely the n-th Betti number, so 2. holds.

Conversely, if we have that equality, looking at

$$\dim_{\mathbb{Q}_\ell} H^n_{ét}(X, \mathbb{Q}_\ell) = \mathrm{rank}_W H^n_{crys}(X/W) \leq \dim_k (H^n_{crys}(X/W) \otimes_W k) \leq H^n_{dR}(X/k)$$

we deduce that the inequality must be an equality. Therefore, the term on the right of the exact sequence appearing in 5. must be zero: in other words, $H^n_{crys}(X/W)$ is torsion-free.

Definition of the crystalline cohomology

Divided powers: let A be a ring and $I \subset A$ an ideal. A PD[2] structure over I is a collection of maps $\gamma_n : I \to A$ for $n \geq 0$ s.t., morally, "$\gamma_n(x) = x^n/n!$". We impose the following conditions:

1. $\gamma_0(x) = 1$ and $\gamma_1(x) = x$ for every $x \in I$.
2. $\gamma_n(x) \in I$ if $n \geq 1$.
3. $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$.
4. $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$ for all $\lambda \in A$.
5. $\gamma_n(x)\gamma_m(x) = \binom{n+m}{n} \gamma_{m+n}(x)$.
6. $\gamma_m(\gamma_n(x)) = \dfrac{(mn)!}{m!(n!)^m} \gamma_{mn}(x)$.

Using the fact that $\gamma_1(x) = x$, condition 5. and induction one easily sees that $n!\gamma_n(x) = x^n$. With this trick, we can talk about divided powers even if the characteristic of A is positive.

Example 1. Consider the ideal $(p) \subset W(k)$, where k is a perfect field of characteristic p (you can think on $\mathbb{F}_p$, so that the Witt ring becomes $\mathbb{Z}_p$). Then, since we are in characteristic zero, we must define $\gamma_n(p) = p^n/n!$ and extend to the whole ideal (p) by the above properties. This is well defined because $p^n/n!$ lies on $W(k)$: for instance, its p-adic valuation is positive for $n \geq 0$, and non-zero for $n \geq 1$, so this definition satisfies the first two conditions. By a boring computation, you can check the other conditions.

Crystalline category: Let k be a perfect field of characteristic p, and we denote $W := W(k)$ and $W_n := W_n(k) = W/p^n$. Let X be a scheme over k. We define the category $\mathrm{Crys}(X/W_n)$ as follows:

[2]From French, *puissances divisées*.

- The objects are commutative diagrams

$$\begin{CD} U @>i>> V \\ @VVV @VVV \\ \mathrm{Spec}(k) @>>> \mathrm{Spec}(W_n) \end{CD}$$

  where $U \subset X$ is a Zariski open, and $i : U \to V$ is a PD thickening of U. More concretely, i is a closed immersion of W_n-schemes such that the ideal $\mathrm{Ker}(\mathcal{O}_V \to \mathcal{O}_U)$ is endowed with a PD structure δ compatible with the canonic PD structure on $pW_n \subset W_n$, i.e. $\delta(pa) = \gamma_n(p)a^n$ for any $pa \in \mathrm{Ker}(\mathcal{O}_V \to \mathcal{O}_U)$. Note that we always have that $(p)\mathcal{O}_V \subset \mathrm{Ker}(\mathcal{O}_V \to \mathcal{O}_U)$, but the latter may be bigger and we may not have a PD structure, so the condition is not empty.

- The morphisms from (U, V, δ) to (U', V', δ') are the commutative diagrams formed by an open immersion $U \hookrightarrow U'$ and a morphism $V \to V'$ compatible with the PD structure.

Crystalline site: We have to define the Grothendieck topology. This will be given by covers of the form $(U_i, V_i, \delta_i) \to (U, V, \delta)$, where $V_i \to V$ is an open immersion and $V = \bigcup_i U_i$.

Now that we have a site, so we can define sheaves, and we can relate sheaves for the crystalline topology over (U, V) with sheaves for the Zariski topology on V as follows:

- Let F be a sheaf and (U, V) an object of $\mathrm{Crys}(X/W_n)$. For a Zariski open $W \subset V$, we define $\widetilde{F}_{(U,V)}(W) := F(U \times_V W, W)$, and this way we get a sheaf on V. Moreover, for a morphism $g : (U, V) \to (U', V')$ in $\mathrm{Crys}(X/W_n)$, we obtain a morphism $g_F^* : g^{-1}\widetilde{F}_{(U',V')} \to \widetilde{F}_{(U,V)}$ that satisfies
  (i) Transitivity for $(U, V) \to (U', V') \to (U'', V'')$.
  (ii) g_F^* is an isomorphism if $V \to V'$ is an open immersion and $U = U' \times_V' V$.

- Conversely, if for every object (U, V) of $\mathrm{Crys}(X/W_n)$ we have a sheaf $\widetilde{F}_{(U,V)}$ for the Zariski topology of V s.t. for every morphism $g : (U, V) \to (U', V')$ in $\mathrm{Crys}(X/W_n)$ we have a morphism $g_F^* : g^{-1}\widetilde{F}_{(U',V')} \to \widetilde{F}_{(U,V)}$ satisfying the two conditions, we get a sheaf F on the crystalline site.

This way, we can define a structure sheaf on $\mathrm{Crys}(X/W_n)$, $\mathcal{O}_{X/W_n}$, that associates to every object (U, V, δ) the sheaf $\mathcal{O}_V$.

Finally, we are able to define the crystalline cohomology with the crystalline structure sheaf:

$$H^i_{crys}(X/W_n) := H^i(\mathrm{Crys}(X/W_n), \mathcal{O}_{X/W_n})$$

and, passing to the limit,

$$H^i_{crys}(X/W) = \lim_{\leftarrow} H^i_{crys}(X/W_n)$$

Examples

Let A be an abelian variety of dimension g. Then, the $H^n_{crys}(A/W)$ are torsion-free W-modules. More precisely, $H^1_{crys}(A/W)$ is free of rank $2g$ and for all $n \geq 2$ there are isomorphisms $H^n_{crys}(A/W) \cong \Lambda^n H^1_{crys}(A/W)$.

For a smooth and proper variety X, recall that its Albanese morphism is (once we fix a point $p \in X$) $\alpha : X \to Alb(X)$, where $Alb(X)$ is an abelian variety that satisfies the following universal property: for any morphism to an abelian variety A $(X,p) \to (A,e)$, this factors through $(Alb(X),e)$.

Then, α induces an isomorphism $H^1_{crys}(X/W) \cong H^1_{crys}(Alb(X)/W)$, so by the above example we get that $H^1_{crys}(X/W)$ is always torsion-free.

5 The de Rham-Witt complex

Through this section X will be assumed to be smooth and projective over k a field of characteristic p. The sheaves $W_n\mathcal{O}_X$ and $W\mathcal{O}_X$ of the previous sections are just the zero part of the complexes $W_n\Omega^\bullet_X$ and $W\Omega^\bullet_X$. We can define (see [Ill79]) operators $V : W_n\Omega^\bullet_X \to W_{n+1}\Omega^\bullet_X$ and $F : W_n\Omega^\bullet_X \to W_{n-1}\Omega^\bullet_X$ between the complexes satisfying the following properties:

1. Both operators are additive.
2. Over $W_n\mathcal{O}_X$, $F : W_n\mathcal{O}_X \to W_{n-1}\mathcal{O}_X$ is our σ defined on W_n, and the previous $\widetilde{V}$ equals this $V : W_n\mathcal{O}_X \to W_{n+1}\mathcal{O}_X$.
3. $FV = VF = p$.
4. $FdV = d$. This property, combined with the previous, gives us $dF = pFd$.
5. $Fd[x] = [x^{p-1}]d[x]$, where $[x]$ is the Teichmüller representative of $x \in \mathcal{O}_X$, i.e. $[x] = (x, 0, \ldots)$.
6. $FxFy = F(xy)$, $xVy = V(F(xy))$ and $V(xdy) = V(x)dV(y)$.

Remark 2. Assume that $k = \mathbb{F}_p$. Then, we have the absolute Frobenius $F_X : X \to X$, which is the identity on the topological spaces and taking the p-th power on the ring structure. This absolute Frobenius induces a morphism F_X^* on the complex $W_n\Omega^\bullet_X$. Let's look at it at the first level, i.e. at $W_n\Omega^1_X \to W_n\Omega^1_X$. It is a fact that we can write every element of $W_n\Omega^1_X$ as a sum of $V^i([a])dV^j([b])$, with $[a],[b] \in \mathcal{O}_X$. Then, the induced Frobenius acts as follows:

$$F_X^* : \begin{array}{ccc} W_n\Omega^1_X & \to & W_n\Omega^1_X \\ V^i([a])dV^j([b]) & \mapsto & V^i([a^p])dV^j([b^p]) \end{array}$$

and then by properties 2 and 4 we get

$$F_X^*(V^i([a])dV^j([b])) = F(V^i([a]))dFV^j([b]) = pF(V^i([a])V^j([b]))$$

so we have that

$$F_X^*\alpha = pF\alpha$$

The technical part of divided powers in the construction of the crystalline cohomology is motivated by this, because we want a correct way to say $F =$ "F_X^*/p" in characteristic p.

Let's look now a little bit closer to the complex $W\Omega^\bullet$. We can consider the stupid filtration on it

$$\begin{array}{ccccccccc} W\Omega^\bullet: & W\mathcal{O} & \longrightarrow & W\Omega^1 & \longrightarrow & W\Omega^2 & \longrightarrow & \cdots \\ \cup & \uparrow & & \uparrow & & \uparrow & & \\ W\Omega^{\geq 1}: & 0 & \longrightarrow & W\Omega^1 & \longrightarrow & W\Omega^2 & \longrightarrow & \cdots \\ \cup & \uparrow & & \uparrow & & \uparrow & & \\ W\Omega^{\geq 2}: & 0 & \longrightarrow & 0 & \longrightarrow & W\Omega^2 & \longrightarrow & \cdots \\ \cup & & & & & & & \\ \vdots & & & & & & & \end{array}$$

Note that if we denote by $W\Omega^i$ the complex that is zero everywhere except on level 0, where it is precisely $W\Omega^i$, we get the equality of complexes

$$\frac{W\Omega^{\geq i}}{W\Omega^{\geq i+1}} = (W\Omega^i)[-i]$$

Now, as usual, this double complex induces a spectral sequence, and we have $E_1^{i,n-i} = H^n(W\Omega^{\geq i}/W\Omega^{\geq i+1}) = H^n(W\Omega^i[-i]) = H^{n-i}(W\Omega^i_X)$.

Assuming that X is smooth and proper over k, and that k is perfect, and after killing the torsion part (by tensoring with $K := \mathrm{Frac}(W(k))$), we have that the spectral sequence degenerates at E_1 (c.f. [Ill79, Thm. 3.2]), i.e.,

$$H^{n-i}(W\Omega^i_X)_K \Rightarrow H^n(X_{Zar}, W\Omega^\bullet)_K$$

Here the right hand side means the hypercohomology of the double complex obtained by considering an injective resolution of each $W\Omega^i_X$ in a compatible way[3]. One of the nice properties of crystalline cohomology is that we get the equality $H^n(X_{Zar}, W\Omega^\bullet)_K = H^i_{crys}(X/W)_K =: H^n_{crys}(X/K)$.

Remark 3. In general, if we consider the spectral sequence over W directly, it will not degenerate at E_1. But if the $H^j(W\Omega^i)$ are finitely generated, then the sequence will degenerate (c.f. [Ill79, Thm. 3.7]).

We have a decomposition of $H^n_{crys}(X/K)$ coming from the Frobenius. For simplicity, we will assume that k is a finite field, although these results hold in greater generality. The Frobenius $\varphi : k \to k$ induces a morphism on $X \to X$, and by the functoriality of the cohomology we get an automorphism on $H^n_{crys}(X/K)$. We want to look now to the eigenvalues of this automorphism, so we consider the vector space and the automorphism on the algebraic closure of K, i.e. we see the automorphism φ^* in $H^n_{crys}(X/K) \otimes_K \overline{K} =: H^n_{crys}(X/\overline{K})$. This automorphism will be linear (here we are using our assumption that k is finite: in general, we only have p-linearity),

[3]From the double complex $I^{\bullet\bullet}$ you form the complex $Tot(I^{\bullet\bullet})^\bullet$, where $Tot(I^{\bullet\bullet})^n = \bigoplus_{i+j=n} I^{i,j}$ and the differentials are the sum (up to a sign) of the two obvious ones.

and since we are working over an algebraically closed field we have a basis formed by eigenvectors.

Given $q \in \mathbb{Q}$, we define $H^{n,q}_{crys}(X/\overline{K})$ as the linear subspace generated by all the eigenvectors of φ^* whose eigenvalues have p-adic valuation equal to q (here the valuation is normed so that p^r, the size of the field, has valuation 1. Therefore we may have non-integer valuations). Note that we can see this subspaces already over K, because those linear subspaces will be invariant (there is a Galois action involved). Since the valuations of all the eigenvalues are rational (and positive), we get the decomposition

$$H^n_{crys}(X/K) = \bigoplus_{q \in \mathbb{Q}_{\geq 0}} H^{n,q}_{crys}(X/K)$$

One very nice fact about this decompositions is that is compatible with the graded structure coming from the spectral sequence. If we denote $H^{n,[q_1,q_2)}_{crys}(X/K) := \bigoplus_{q \in [q_1,q_2)} H^{n,q}_{crys}(X/K)$, we have the equality

$$H^{n,[i,i+1)}_{crys}(X/K) = H^n(W\Omega^i_X)_K$$

for all $i \in \mathbb{N}$. In this way, thanks to this decomposition associated to the Frobenius action we obtain a canonical decomposition of $H^n_{crys}(X/K)$, something which is not clear a priori.

Further information can be found in [Lie15a], where he discusses F-crystals and Ogus' crystalline theorem for supersingular K3 surfaces (among other things); in [CL98], where he discusses more on F-crystals, the link between the de Rham-Witt cohomology and the slopes of the Frobenius operator and the Hodge-Witt decomposition; in [Ill79], where he studies the de Rham-Witt complex; and in [Ber74], where he developes the whole theory of crystalline cohomology.

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