The topology of conjugate Berkovich spaces
(in 10 minutes)

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$X/\mathbb{C}$ projective algebraic variety
$\sigma \in \mathrm{Aut}(\mathbb{C})$

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Definition: Let X, Y be varieties over a field F. We say that they are *conjugate* if $\exists\ \sigma \in \mathrm{Aut}(F)$ such that

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, F & \xrightarrow{\sigma} & \mathrm{Spec}\, F \end{array}$$

is Cartesian. We will write $Y = X_\sigma$

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C^{an} is a Riemann surface.
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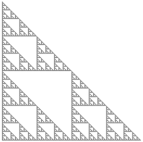
Question: What happens in the *non-archimedean* setting?
Are *non-archimedean* conjugate varieties homeomorphic?

K: complete non-archimedean field
In non-archimedean
geometry there are different analytification functors.
First approach: using rigid analytic spaces à la Tate.
Problem: the
underlying topological space is totally disconnected.



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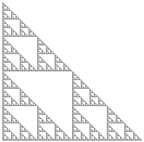

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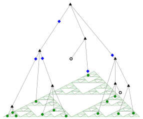
Second approach: using Berkovich spaces.

Remark: this approach will lead to locally path-connected and locally contractible topological spaces.



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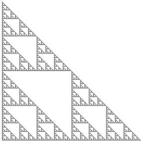

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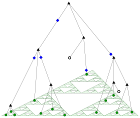

Example: The Berkovich analytification of smooth curves of genus 0 ($\mathbb{P}^1, \mathbb{A}^1, \mathbb{G}_m$, etc.) is contractible.

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- C/K a smooth curve of genus $g > 0$

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Example (C., idea from Chambert-Loir)

Assume for simplicity $K = \mathbb{C}_p$, with $p > 3$.
Let $E : y^2 = x^3 + px + a$, with $a \in \mathbb{C}_p$ of valuation equal to 1
$\Rightarrow$ E^{an} *is contractible*.

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If moreover a is transcendental over $\mathbb{Q}$, we can construct an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}_p)$ such that the analytification of $E_\sigma : y^2 = x^3 + px + \sigma(a)$ *is* **not** *contractible*!

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(Trick: play with the j-invariant)
$\Rightarrow E^{an}$ and E_σ^{an} are non-homeomorphic Berkovich elliptic curves.

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Given any non-continuous $\sigma \in \mathrm{Aut}(\mathbb{C}_p)$, we can construct an elliptic curve $E/\mathbb{C}_p$ such that E^{an} and E_σ^{an} are non-homeomorphic.

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Given any non-continuous $\sigma \in \mathrm{Aut}(\mathbb{C}_p)$, we can construct an elliptic curve $E/\mathbb{C}_p$ such that E^{an} and E_σ^{an} are non-homeomorphic.

The construction of the elliptic curve is explicit, and uses again the trick of the j-invariant.

Let $K = \mathbb{C}\{t\} := \left(\overline{\bigcup \mathbb{C}(t^{1/n})}\right)^{\wedge}$ be the field of Puiseux series.

Theorem (Nicaise, '20)

Every connected smooth and proper variety X over $\mathbb{C}\{t\}$ is conjugate to a smooth and proper variety Y whose Berkovich analytification is contractible.

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Every connected smooth and proper variety X over $\mathbb{C}\{t\}$ is conjugate to a smooth and proper variety Y whose Berkovich analytification is contractible.

We don't know an explicit construction of the conjugate variety, this is an existence statement.

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$\iota : X \hookrightarrow \mathbb{A}^n_K$
We can define the (extended) tropicalization map

$$\begin{array}{ccc} \pi_\iota : X^{\mathrm{an}} & \longrightarrow & (\mathbb{R} \cup \{\infty\})^n \\ & & \big\uparrow \\ & & (\mathrm{val}(K))^n \end{array}$$

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Theorem (Payne, '09)

Let X/K be an affine variety, K alg. closed non-archimedean field with nontrivial valuation.
X^{an} is homeomorphic to the inverse limit of all extended tropicalizations.

We fix now the field of Puiseux series $\mathbb{C}\{t\}$.

Construction (C.)

For any degree $d \geq 3$, we construct explicit smooth planar curves $C \hookrightarrow \mathbb{G}_m^2$ of degree d, and find an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}\{t\})$ such that:

- C is Mumford
- C_σ has good reduction

In particular, their (Berkovich) analytifications are non-homeomorphic conjugate curves.

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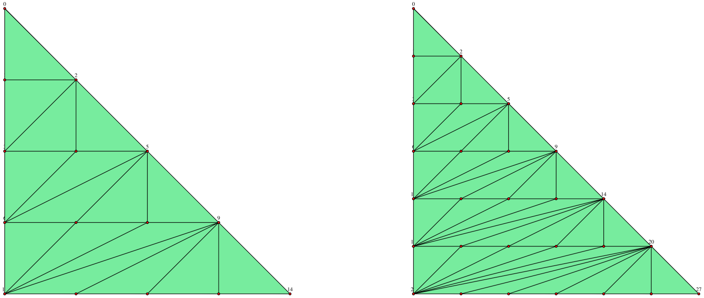
In particular, their (Berkovich) analytifications are non-homeomorphic conjugate curves. For example, for $d = 4$ the curve is given by the following polynomial $f \in \mathbb{C}\{t\}[x, y]$:

$$\begin{aligned} f(x,y) = \; & y^4 \\ & + ty^3 + txy^3 \\ & + t^3y^2 + t^4xy^2 + t^7x^2y^2 \\ & + t^{14}y + t^{24}xy + t^{46}x^2y + t^{127}x^3y \\ & + a_{704} + t^{2072}x + t^{4141}x^2y + t^{12353}x^3 + b_{49229}x^4. \end{aligned}$$

The automorphism σ fixes t and moves a_{704} and b_{49229}.



(a) Degree $d = 4$ (b) Degree $d = 6$

Figure: Newton polytopes of Mumford curves with their regular subdivisions.

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- (Future project? Question posed by Bernd Sturmfels): take 15 integer numbers $a_1, \ldots, a_{15} \in \mathbb{Z}$ defining a smooth planar quartic. Can we construct an algorithm describing the semi-stable reduction at all primes p?

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Thank you for your attention!