Periods and the conjectures of Grothendieck and of Kontsevich-Zagier

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In the memory of Bridget Bishop (1632 – 10 June 1692),

first woman executed for witchcraft during the Salem witch trials 328 years ago

Pope Clement VII commissioned Michelangelo to decorate the altar wall of Sistine Chapel

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Less known: in 1533, Clement VII was told about the Copernican system...
And he was actually very happy about it!

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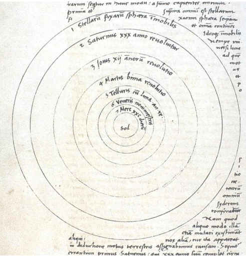
Heliocentric model of Copernicus

- Planetary orbit is a circle



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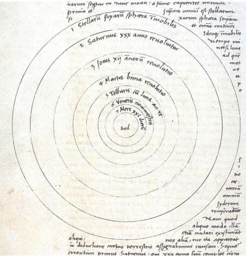

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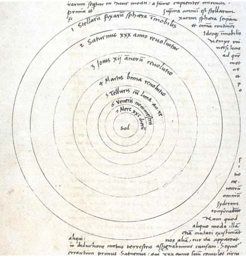

Figure: Heliocentric model

4 Martis bima revolutio
5 Telluris cū Luna . an . re .
6 Veneris nono mense
7 Merc
Sol

Figure: Inquisition asking for Galileo in 1615. Meanwhile, in the Holy Roman Empire...

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- The orbital period of a planet is algebraically related to the semi-axis of its orbit

(For the experts: the square of the period is directly proportional to the cube of the semi-axis)

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Reformulation in modern language:
Theorem (Newton, 1687; reformulation by Arnold, 1987, Newton's proof essentially works):
There exists no algebraically integrable convex non-singular algebraic curve.

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Definition (period as a complex number)

A *period* is a complex number $z \in P_C \subset \mathbb{C}$ whose real and imaginary parts can be written as absolutely convergent integrals of the form

$$\int_S f \, dx_1 \ldots dx_n,$$

where $f \in \mathbb{Q}(x_1, \ldots, x_n)$ and S is a $\mathbb{Q}$-semi-algebraic set. In other words,

$$z = \int_{S_1} f_1 + \left(\int_{S_2} f_2 \right) i$$

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Still open questions

- Is $1/\pi$ a period?
- Is $e = 2.71828\ldots$ a period?

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Linear relations between (representations of) periods (a.k.a. rules of integration):

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Kontsevich-Zagier period conjecture

The linear relations between representations of periods come all from the above rules.

Ayoub constructs periods in a different way:

Fix $\sigma : K \hookrightarrow \mathbb{C}$

- $\mathcal{O}_{K-alg}(\overline{D}^n) :=$ K-vec. sp. of formal power series on $(z_1, \ldots, z_n)$, convergent on a polydisk D of radius $r > 1$, algebraic over $K(z_1, \ldots, z_n)$

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We integrate elements of $\mathcal{O}_{K-alg}(\overline{D}^\infty)$ on the unit hypercube:

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The element $g := 2z - 1 \in \mathcal{O}_{\mathbb{Q}-alg}(\overline{D}^1)$ maps to zero:
$\int_{[0,1]}(2z-1)dz = (z^2 - z)|_1 - (z^2 - z)|_0 = 0.$

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$\int_{[0,1]}(2z-1)dz = (z^2 - z)|_1 - (z^2 - z)|_0 = 0.$
Note that $g = \frac{df}{dz} - f(1) + f(0)$ for $f(z) = z^2$.

In general, for any $f \in \mathcal{O}_{K-alg}(\overline{D}^{\infty})$, the element

$$\frac{\partial f}{\partial z_i} - f|_{z_i=1} + f|_{z_i=0}$$

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Kontsevich-Zagier period conjecture (Ayoub's formulation)

The kernel of

$$\int_{\square} : \mathcal{O}_{\mathbb{Q}-alg}(\overline{D}^{\infty}) \to \mathbb{C}$$

is spanned by the elements of the form

$$\frac{\partial f}{\partial z_i} - f|_{z_i=1} + f|_{z_i=0}.$$

Grothendieck's point of view

$K \hookrightarrow \mathbb{C}$, X/K smooth variety, X^{an} its analytification.
Grothendieck's period isomorphism:

$$H^*_{dR}(X) \otimes_K \mathbb{C} \to H^*_{sing}(X^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

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Bhatt-Morrow-Scholze (2018) obtained information of the torsion part (for X proper and smooth, but this is unrelated with today's topic)

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We can describe it via the period pairing:

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$$[\omega] \otimes [\gamma] \mapsto \int_\gamma \omega$$

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Definition: The *period field* of a smooth variety X is the subfield $Per(X)$ of $\mathbb{C}$ generated by the image of the period pairing:

$$Per(X) := K\left(\mathrm{im}\left(\int\right)\right)$$

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Example: $X = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[T, T^{-1}])$, $X^{\mathrm{an}} = \mathbb{C}^*$.

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so we see that $\mathbb{Q}(2\pi i) \subset Per(\mathbb{G}_m)$.

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so we see that $\mathbb{Q}(2\pi i) \subset Per(\mathbb{G}_m)$. Some extra work yields an equality.

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What is the motivic Galois group of X???
Two approaches: via Nori motives, via Voevodsky motives

Motivic Galois group of a variety X?

Recall

Absolute Galois group of a field $K \hookrightarrow \mathbb{C}$:

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) := \varprojlim_{L/K \text{ finite}} \mathrm{Aut}_K(L)$$

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Tannakian category $(\mathcal{C}, \omega)$ over $\mathbb{Q}$, where

- $\mathcal{C}$ ($\mathbb{Q}$-linear) abelian tensor category (+ some assumptions)
- $\omega : \mathcal{C} \to \mathtt{fVec}_{\mathbb{Q}}$ exact faithful tensor functor, called fiber functor

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Tannakian category $(\mathcal{C}, \omega)$ over $\mathbb{Q}$ induces an affine group scheme

$$G := \mathrm{Aut}^{\otimes}(\omega)$$

called the Tannakian fundamental group of $(\mathcal{C}, \omega)$.

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For any Tannakian category $(\mathcal{C}, \omega)$, we have an equivalence of categories:

$$(\mathcal{C}, \omega) \longleftrightarrow \mathtt{Rep}_G$$

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- Morphisms come from geometry

Motivic Galois group of a variety X?

Conjecture (Grothendieck): for $\sigma : K \hookrightarrow \mathbb{C}$, there *exists* a category of *mixed motives* $MM(K)$ such that:

- different cohomology theories should factor through $MM(K)$:

$$\begin{tikzcd}
\mathtt{Var}_K \arrow[r] & MM(K) & H^*_{sing}(X^{an}, \mathbb{Q}) \\
X \arrow[r] & M(X) \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & H^*_{dR}(X) \\
 & & H^*_\ell(X) \\
 & & \vdots
\end{tikzcd}$$

- Morphisms come from geometry
- Tannakian category, fiber functor should give singular cohomology (a.k.a. Betti realization)

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Hope: yield same theory.
Evidence (Choudhury, Gallauer 2017): if $K/\mathbb{Q}$ algebraic,

$$G^{\mathrm{abs}}_{N,\mathrm{mot}}(K) \cong G^{\mathrm{abs}}_{A,\mathrm{mot}}(K)$$

Motivic Galois group of a variety X!

For a variety X/K, consider

$$\langle M(X) \rangle \subset MM(K)$$

(smallest full Tannakian subcategory of $MM(K)$ containing $M(X)$ and stable under taking subobjects and quotiens)

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(smallest full Tannakian subcategory of $MM(K)$ containing $M(X)$ and stable under taking subobjects and quotiens)
$G_{\text{mot}}(X) :=$ Tannakian fundamental group of $\langle M(X) \rangle$

$$G_{mot}(X) := \mathrm{Aut}^{\otimes} H_{sing}|_{\langle M(X) \rangle}$$

Kontsevich-Zagier conjecture
$(\int_{\Box} : \mathcal{O}_{\mathbb{Q}-alg}(\overline{D}^{\infty}) \to \mathbb{C}$ has the expected kernel)

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Grothendieck period conjecture for $X/\mathbb{Q}$

$$tr.deg(Per(X)/\mathbb{Q}) = \dim G_{\mathrm{mot}}(X)$$

$+ \mathcal{O}_{\mathbb{Q}-alg}(\overline{D}^\infty)$ modulo the expected kernel of $\int_\Box$ is an integral domain

Geometric version

Power series of periods:

$$\sum_{i \geq 0} f_i \cdot t^i$$

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We have an integration map $\int_\square : \mathcal{O}^\dagger_{K-alg}(\overline{D}^\infty) \to \mathbb{C}((t))$:

$$\sum_{i >> -\infty} f_i \cdot t^i \mapsto \sum_{i >> -\infty} \left(\int_\square f_i \right) \cdot t_i$$

Geometric version

As in the previous case, $\int_{\square}$ maps elements of the form

$$\frac{\partial f}{\partial z_i} - f|_{z_i=1} + f|_{z_i=0}$$

to zero (here $f \in \mathcal{O}^{\dagger}_{K-alg}(\overline{D}^{\infty})$) (call them elements of the *first kind*).

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Ayoub (Annals '15 + Tohoku - to appear) answers this question:

Theorem (Geometric version of the KZ conjecture)

Assume K contains the number π in its algebraic closure. Then, the kernel of

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  - $h \in \mathcal{O}^{\dagger}_{K-alg}(\overline{D}^{\infty})$ *such that g and h don't depend simultaneously on the same variable (in other words:* $\frac{\partial g}{\partial z_i} \neq 0 \Rightarrow \frac{\partial h}{\partial z_i} = 0$*)*


Thank you for your attention!
Any questions?