Hans Böckler Stiftung

Mitbestimmung · Forschung · Stipendien

Exposé

Abstract

In 1964, Serre (the youngest mathematician to be awarded with the Fields medal, which is the highest recognition in Mathematics) found explicit examples of non homeomorphic conjugate complex varieties. The aim of my PhD "The topology of conjugate Berkovich spaces" will be to study, under the supervision of Prof. Hélène Esnault, the analogue problem for non-archimedean varieties. More precisely, I will understand the behaviour of topological invariants under conjugation of Berkovich spaces. This PhD will allow to understand better the topology of Berkovich spaces as well as the interaction between field automorphisms of $\mathbb{C}_p$ and the variation of the topology of the Berkovich spaces defined over it.

Keywords— Mathematics, algebraic geometry, p-adic geometry, Berkovich spaces, topology.

Introduction

The structure of this Exposé is the following: first I briefly explain the main mathematical theories that will be referred, which are algebraic geometry, p-adic geometry and topology. After this, I explain the research topic and the status of research, and I finally explain the preparatory work that has been already done and I write a list of references.

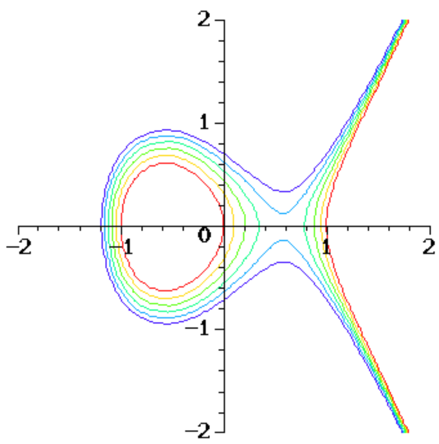
The topic of my PhD can be stated in one sentence: "the study the behaviour of topological invariants of Berkovich spaces under conjugation, in analogy with the study of conjugate complex analytic spaces". In order unravel this sentence, I will try to explain a little bit each of the elements appearing in the statement, so the reader can understand the topic.

The first step consists on explaining what do we mean by a topological invariant. Secondly, I'll explain the complex setting, i.e. what complex analytic spaces are. After this, I will introduce the p-adic world, which is a framework analogous to the complex setting that turns out to be very useful; after this, I will explain what an algebraic variety is, and I will give an overview on the process of analytification that leads us to the concept of Berkovich spaces. Finally, I will explain what do I mean by a conjugate variety, and then I will put everything together in order to get back the previous sentence.

Topology of analytic spaces

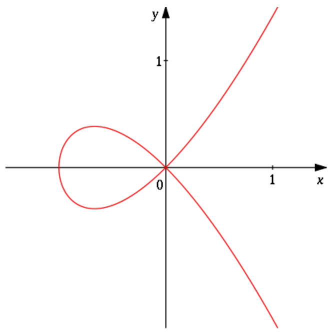
An analytic space is what we obtain if we allow singularities in the definition of a manifold. But first, what is a manifold? And what is a singularity?

A manifold, roughly speaking, is a geometric object that locally looks like the "usual" space. For example, a one-dimensional manifold looks like a line if we look close enough. In the picture, we see a family of 1-dimensional manifolds, usually called curves. A nice example of a manifold of two dimensions is the sphere: locally, if we are over a sphere and we are small enough, it seems that the sphere is flat, or in other words, that it looks locally like the plane, which is the 2-dimensional space. This is why people thought during so many years that the Earth was flat: it is a sphere, but locally it looks like a flat surface. Of course, we generalize this notion to an arbitrary dimension in an abstract process, so that a 5-dimensional manifold is an object that locally looks like the 5-dimensional space, whatever this is.



And what is a singularity? A singularity is a point that is different from the rest, and in a manifold this means that it doesn't matter how close do we look to it, it will never look like the usual space. In the second picture, we see that there is one point which is different from the others, which is the origin. If we look close enough, it will look like a cross, which is not like the usual 1-dimensional space (a line); this doesn't happen in the rest of the points, and that's why we call it singular. Hence, in this case, we don't have a manifold, but an analytic space.

I haven't explained yet what do I mean by the "usual space", but we will come later to this question in the sections of complex analytic spaces and on the one about p-adic analysis.



When we study analytic varieties, their topology is one of the most important properties that concerns us, because it tells us how they look like if we allow continuous deformations of our objects and we identify them; a topological invariant is a property that is preserved under these continuous deformations.

For example, the size is not a topological invariant, because we can think on a balloon that we keep inflating: it is deformed continuously and getting bigger and bigger, so we can't expect that the size is a topological invariant. An example of a topological invariant in a 2-dimensional manifold is the "number of holes", something that the mathematicians call genus: for example, a mug has one hole, as well as a doughnut, and that's why a topologist can't notice any difference between a doughnut and a mug, as we can see in the picture.

Geometry over the complex numbers

We already know that an analytic space is a manifold that it is allowed to have singularities. We mentioned above that a manifold is a geometric object that locally looks like the usual space, but what do we mean by the usual space?

Here, it depends on what you are interested. A setting which turns out to be very fruitful and that it can be applied to physics and mechanics is the complex setting, and understanding complex geometry allows us to solve a big number of problems. The complex numbers $\mathbb{C}$ are a generalization of the usual real numbers, and they have been used for hundreds of years. One of the reasons is that the Fundamental Theorem of Algebra tells us that every polynomial can be factorised in complex linear terms. For example, if we consider the polynomial

$$p(X) = X^2 + 1,$$

we can't write it as a product of linear factors using only real numbers, because the square root of a negative number is not real. But, in the complex world, we can achieve this, and we have that

$$p(X) = (X + \sqrt{-1})(X - \sqrt{-1}).$$

A complex analytic space is then given locally by complex analytic functions: in other words, we have in some ambient usual space some functions that equal to zero in a set of points, and this vanishing locus is what we call an analytic space. There are ways to detect the singularities from these functions that define your complex analytic space.

Complex geometry is a very active and important subject of Mathematics, as we can see for example in the ongoing research on Calabi-Yau manifolds, which have a number of application in theoretical physics (cf. [YN10]).

Geometry over the p-adic numbers

The geometry of the p-adic numbers is the study of the geometric objects (the analytic spaces) that locally look like the p-adic numbers, that is, we take as usual spaces those given by the p-adic numbers, which we denote as $\mathbb{C}_p$ in analogy with the complex numbers. But what are the p-adic numbers?

Back in 1897, the German mathematician Hensel introduced in [Hen97] the p-adic numbers: he was looking for a tool in order to study the "local behaviour" around a rational point, and what he did is to consider the power expansions with respect to a prime number p (more precisely, this construction gives the set $\mathbb{Q}_p$, but if we consider the algebraic closure and we complete this field, then we obtain the set $\mathbb{C}_p$). This complicated construction was crucial in the solution of difficult problems in algebraic number theory, as for example the computation of the factors of the discriminant of a number field $K/\mathbb{Q}$.

This led to the study of p-adic analysis, which is a complete new branch of mathematics, and the whole theory of p-adic analysis was developed. In this way, the construction of geometric objects in this new branch became an important area of work, because it was a new tool that allowed the solution of old problems in algebraic number theory.

A major example of these geometric objects, of special interest for my PhD, is the definition of rigid analytic space, which plays the role of a complex analytic space in the p-adic world. We defined a complex analytic space as a space which locally looks like the zeroes of a finite set of complex analytic functions; in the p-adic world the same approach gives contradictory results because of the exoticism of the p-adic topology, so a different approach is needed. Nonetheless, Tate was able to solve this difficulty and introduced in 1962 the notion of rigid analytic space, see [Tat71], which plays the analogue role of a complex analytic space in the p-adic setting.

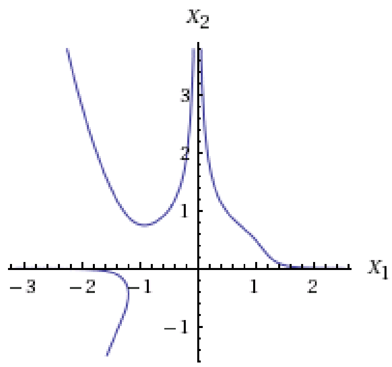
Analytification of an algebraic variety

Recall that a complex analytic variety was defined by the zeroes of complex analytic functions. If instead of considering analytic functions we look at polynomial equations, we obtain the notion of an algebraic variety. In this section we sketch how to go from an algebraic variety to an analytic space, both in the complex and the p-adic setting.

Recall from high-school that a polynomial equation is something of the form

$$p(X_1, X_2) = X_1^7 X_2 + 4X_1^2 X_2^3 - 1,$$

where we can of course have more terms, more variables, etc. If we look at the points of the coordinate plane (X_1, X_2) where this vanishes, we obtain what we call an algebraic curve:



An algebraic variety can be more complicated and be given my more equations, but essentially is this. There is a huge branch of Mathematics which is dedicated to the study of these varieties defined by polynomials, and which is called algebraic geometry and has a long history, as it is coming from the ancient Greeks.

There is a deep relation between algebraic geometry and complex analytic geometry, which we call nowadays GAGA-type theorems. These kind of theorems stablish equivalences between objects in the analytic world and objects in algebraic geometry, and its name comes from the foundational paper by the famous mathematician Serre [Ser55]. More concretely, for an algebraic variety X defined over $\mathbb{C}$, one considers X with the classical complex topology, and one sees that it has the structure of an analytic space. We denote such space with X^{an}.

If instead of starting with an algebraic variety X defined over $\mathbb{C}$, we start with an algebraic variety defined over the p-adic numbers $\mathbb{C}_p$, then there is a way to attach to it a p-adic analytification, which gives us a rigid analytic space X^{an} (is the one mentioned above and developed by Tate). Indeed, there is more than one way to do this: for example, one can use formal schemes following the construction by Raynaud, see [Ray74], or Berkovich spaces, introduced by Berkovich in [Ber90]. In this PhD we will follow this last approach.

Conjugation of a variety

The conjugation of an algebraic variety X defined over $\mathbb{C}$ or $\mathbb{C}_p$ consists, roughly speaking, in changing the coefficients of the equations that define the algebraic variety.

Let's look at this conjugation process in the complex analytic setting: given a variety X over the complex numbers $\mathbb{C}$, we can consider an automorphism σ of $\mathbb{C}$, and consider the *conjugate* of X under σ, denoted X_σ: if the variety X is given by some equations over the complex numbers $\mathbb{C}$, the resulting variety X_σ is given by the the same equations but changing the coefficients via the automorphism σ. Here we are mixing two different areas of study: arithmetic and topology. Nonetheless, we know that some of the topological properties of the analytifications of our varieties, X^{an} and X_σ^{an}, are preserved: for example, their Hodge numbers coincide, because Grothendieck's comparison isomorphism implies that they coincide with the Betti numbers of X and X_σ respectively, and in the algebraic side we know that these Betti numbers are equal (in other words: the conjugation σ doesn't change them). Similarly, we know that the profinite completions of the (topological) fundamental groups of X^{an} and X_σ^{an} coincide, because these groups are isomorphic to the algebraic fundamental groups of X and X_σ, and they are equal (see for example [Esn16, Prop. 6.1]). One reasonable question is whether more topological invariants are preserved, or even if X^{an} and X_σ^{an} are homeomorphic.

But the answer to this question is no! Serre found in [Ser64] examples of conjugate varieties with different (topological) fundamental groups, which means in particular that they are not homeomorphic. This result was surprising at the time, and now there are more examples of non homeomorphic conjugate varieties in the literature, see for example [Abe74], [BCG07], [EV07], [MS10], [ACC07] or [Cha09].

Research topic, content and pertinence

I can finally state in one sentence the aim of my PhD: I will study how the operation on conjugating a Berkovich space affects its topology, following the work initiated by Serre and bringing it to the p-adic setting. The plan is twofold: first, I want to understand how do the examples work in the complex setting, and try to imitate the constructions and the ideas behind in the p-adic setting; second, I want to study phenomena that occur in the p-adic setting but not in the complex setting.

This project will, on one hand, investigate the differences between the topological and the algebraic fundamental groups, and in the other hand, build one more analogy between classical and p-adic analysis by understanding the topology behind the Berkovich spaces, a relatively new topic of mathematics that nonetheless is rapidly growing due to the number of applications that are being discovered, not only in number theory but also in geometry and dynamics (see for example [DFN15]).

As I mentioned before, the topic originates from important work of Serre in [Ser64], where he studies the variation of topological invariants under conjugation in complex analytic varieties. The relevance of this project resides in the connection between different branches of mathematics that appear during the research, and the interaction among them: for instance, to state the problem, one has to understand tools from topology, from arithmetic geometry, from p-adic analysis and from Berkovich spaces. And probably, in order to solve the problem, one needs to understand elements of tropical geometry, as it is related with Berkovich spaces and allows explicit computations (cf [Pay09] and [HL16]). Finally, note that this research is mathematical, so there are no hypothesis assumed.

Status of the research

The big number of tools needed to attack this problem are available, and one just need to understand them all and combine them in a convenient way in order to solve it.

In order to start the study of the problem, I will look at the following base example, which combines a recently known fact with the topic that I want to study in such a way that has not been done before: it is known that the homotopy type of the Berkovich space associated to an elliptic curve $E/\mathbb{C}_p$ depends on its j-invariant only: if $|j(E)|_p \leq 1$, then E is contractible; otherwise, E is homotopic to a circle. One can try to take two elliptic curves E and E' with transcendental j-invariant $j(E)$, $j(E')$ in $\mathbb{C}_p$ such that $|j(E)| \leq 1 < |j(E')|$, and an automorphism of $\mathbb{C}_p$ which maps $j(E)$ to $j(E')$. Then we would get conjugate varieties with topologically different analytifications. Note that this example can't occur in the complex setting, as all the elliptic curves have genus 1, and therefore their analytifications will all be homeomorphic to the torus (which we drew before as a mug or a doughnut).

This would be the base example from which I could start the systematic study, including the construction of more conjugate p-adic varieties with different topological fundamental groups, the comparison between topological invariants that don't change under conjugation and those who do change, etc.

Methodology

In Mathematics, there are mainly two approaches to solve problems: one consists in solving particular cases which are easier to compute, and little by little generalize and extend the solutions, and the other one consists in generalizing the problem to a very abstract setting, so that only the necessary structure is involved, making the solution "natural" in an abstract sense.

In this case, we will follow in the beginning the first path, computing concrete examples where the fundamental group, or possibly other topological invariants, change under conjugation, starting with the example of elliptic curves.

Time plan

The time plan for my PhD is the following: the first year I will study the base example of elliptic curves, which involves understanding Berkovich curves, models of varieties defined over $\mathbb{C}_p$ and their special fibers, and non continuous automorphisms of $\mathbb{C}_p$. The second year I should be able to write down the first results and I will explore the question for higher genus curves. In the third year, depending on the progress of the study on curves, I will either continue the study for curves or move to higher dimensional varieties, starting with abelian varieties.

Meanwhile I plan to attend international conferences about Berkovich spaces and non-archimedian geometry, with special attention to those related with their topology and on curves. In particular, I will attend the summer school and workshop "Instruments of Algebraic Geometry" (IAG) that will take place in Bucharest in September 2017. This will allow me to establish contact with the experts in the areas involved in my research and to learn about of the ultimate tools that are being developed worldwide, so I can use them in my research.

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