Periods and the conjectures of Grothendieck and of Kontsevich-Zagier

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Abstract

This is the tenth talk of the Research Seminar of the Arithmetic Geometry group (FU Berlin), which is on the Summer term of 2016. In the seminar we study the papers [Ayo14a] and [Ayo14b], where the reader will find the details ommited here. In this talk I'll try to give a brief overview of the classical theory of periods, the motivic Galois group defined by Ayoub, Grothendieck's period conjecture, its relation with Kontsevich-Zagier's conjecture and at the end, I make a small digression on a Galois theory for periods following [And09]. Special acknowledgments go for Javier Fresán, Simon Pepin Lehalleur and Sinan Ünver for helping me enjoying the preparation of this talk.

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"Parmi toutes les choses mathématiques que j'avais eu le privilège de découvrir et d'amener au jour, cette réalité des motifs m'apparaît encore comme la plus fascinante, la plus chargée de mystère — au coeur même de l'identité profonde entre la "géométrie" et l' "arithmétique". Et le "yoga des motifs"... est peut-être le plus puissant instrument de découverte que j'aie dégagé dans cette première période de ma vie de mathématicien."

A. Grothendieck, Récoltes et Semailles, Introduction.

1 Classical periods

The set of periods is a subset of the complex numbers $\mathbb{C}$ that appear naturally in Grothendieck's period isomorphism, which relates singular cohomology (a purely topological invariant) with algebraic de Rham cohomology (defined algebraically).

Let's recall this isomorphism, as it is of central importance for this talk. First, let k be a subfield of $\mathbb{C}$ with a fixed embedding σ, and let X be a smooth variety (not necessarily proper) over k. Let X^{an} denote its complex analytification. Then, there exists a canonical isomorphism

$$\varpi_X : H^*_{dR}(X) \otimes_k \mathbb{C} \to H^*_{sing}(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

where we have on the left hand side the (algebraic) de Rham cohomology and singular cohomology on the right hand side. Let's unravel a little bit this isomorphism:

First of all, we have an isomorphism $H^*_{sing}(X^{an}, \mathbb{C}) \cong H^*_{dR}(X^{an})$. Here the right hand side means de analytic de Rham cohomology, which is defined analogously to the algebraic one. This isomorphism comes from the natural map of sheaves $\mathbb{C} \to \mathcal{O}^{hol}_{X^{an}}$, because singular cohomology can be computed as sheaf cohomology and the complex

$$0 \to \mathbb{C} \to \mathcal{O}^{hol}_{X^{an}} \to \Omega^1_{X^{an}} \to \Omega^2_{X^{an}} \to \dots$$

is exact. Indeed, we have to check this locally, and Poincaré lemma precisely says that on small balls (indeed on any contractible set) closed p-forms are exact. Hence we obtain the isomorphism.

Secondly, GAGA gives us an isomorphism $H^*_{dR}(X \otimes_k \mathbb{C}) \cong H^*_{dR}(X^{an})$, where on the left hand side we have algebraic de Rham cohomology and on the right hand side, analytic de Rham cohomology. In the case that X is proper this follows automatically from GAGA, and if X is not proper then we get it from considering a compactification with a simple divisor with normal crossings, and allowing log-poles on this divisor (cf. [HMS15, Prop. 4.1.7]).

Then, the isomorphism $\varpi_X : H^*_{dR}(X) \otimes_k \mathbb{C} \cong H^*_{sing}(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is just the composition of the following isomorphisms:

$$H^*_{dR}(X) \otimes_k \mathbb{C} \to H^*_{dR}(X \otimes_k \mathbb{C}) \to H^*_{dR}(X^{an}) \to H^*_{sing}(X^{an}, \mathbb{C}) \to H^*_{sing}(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Note that this isomorphism forgets all torsion classes present in $H^*_{sing}(X^{an}, \mathbb{Z})$, but we can still say something in the proper case: recently Bhatt, Scholze and Morrow showed in [BMS16] that if X is proper and smooth over $\mathbb{Z}[1/N]$, then for all $i \geq 0$ and $n > 1$ coprime to N, the order of $H^i_{dR}(X)[n]$ is at least the order of $H^i_{sing}(X^{an}, \mathbb{Z})[n]$, where the suffix $[n]$ means the kernel of the multiplication by n (cf. for example [Sch16, Thm. 2.3]).

But let's go back to our more classical isomorphism ϖ_X and look closer on it. It can be described explicitly via integration, which induces a pairing

$$\langle -, - \rangle : H^i_{dR}(X) \times H_{sing,i}(X^{an}, \mathbb{Q}) \to \mathbb{C}.$$

More precisely, if ω represents a closed form in $H^i_{dR}(X) \otimes_k \mathbb{C}$ and $\gamma = \sum_j a_j \gamma_j$ a singular i-cycle, with the $\gamma_j : \Delta_i \to X^{an}$ differentiable (note that we can always do this because X is smooth and we are only interested in the homology class), then

$$\langle [\omega], [\gamma] \rangle := \int_\gamma \omega = \sum_j a_j \int_{\Delta_i} \gamma_j^* \omega.$$

Note that Stokes' formula tells us that the pairing is well defined. Moreover, this pairing is compatible with products and long exact sequences for cohomology.

We define the set of periods of the variety X, denoted $Per(X)$ to be the extension of k generated by the image of this pairing.

Let's do an easy example: let $X = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[T, T^{-1}])$, so that $X^{an} = \mathbb{C}^*$. Its cohomology $H^1_{sing}(\mathbb{C}^*, \mathbb{Q}) = \mathbb{Q}$ is generated by the class α dual to the loop $\gamma : [0,1] \to \mathbb{C}^* : t \mapsto e^{2\pi i t}$. The algebraic de Rham complex is given by

$$\mathbb{Q}[T^{\pm 1}] \to \mathbb{Q}[T^{\pm 1}]dT : \sum_{n \in \mathbb{Z}} a_n T^n \mapsto \sum_{n \in \mathbb{Z}} n a_n T^{n-1} dT,$$

and we see that we can integrate 1-forms as long as $a_{-1} = 0$ via

$$\sum_{n \in \mathbb{Z} \setminus \{-1\}} a_n T^n dT \mapsto \sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{a_n}{n+1} T^{n+1}.$$

Hence, $H^1_{dR}(\mathbb{G}_m) = \mathbb{Q}$ and it is generated by $\omega = dT/T$. Then, the period pairing, which in this case defines also the period isomorphism, is given by

$$\langle [\omega], [\gamma] \rangle = \int_\gamma \omega = \int_0^1 e^{-2\pi i t} d(e^{2\pi i t}) = 2\pi i \int_0^1 dt = 2\pi i.$$

There is also a version of this isomorphism using relative cohomology: if X is a (now not necessarily smooth) variety over k and $D \subset X$ a closed subvariety, then there is a canonical isomorphism

$$\varpi_{X,D} : H^*_{dR}(X, D) \otimes_k \mathbb{C} \to H^*_{sing}(X^{an}, D^{an}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

We also obtain a well defined period pairing which is compatible with products and long exact sequences for relative cohomology. We can make this pairing explicit in the following case: if X is a smooth affine variety of dimension d and $D \subset X$ a simple divisor with normal crossings, then we pick $\omega \in \Omega^i_X(X)$ with associated cohomology class $[(\omega, \omega_D)] \in H^i_{dR}(X, D)$, and (γ, γ_D) a singular chain on X^{an} with boundary in D^{an}. Note that by the definition of cohomology and homology with support, they have to satisfy

$$\partial \gamma = -i_* \gamma_D, \quad i^* \omega = \omega_D, \quad d\omega = 0.$$

If we represent γ as $\sum_j a_j\gamma_j$, with $a_j \in \mathbb{Q}$ and $\gamma_j : \Delta_i \to X^{an}$, then

$$\langle[\omega],[\gamma]\rangle := \int_\gamma \omega + \int_{\gamma_D} \omega_D.$$

This definition allows us to write logarithms of rational numbers as periods. Indeed, if $X = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[T^{\pm}])$ and $D = \{1,\alpha\}$, with $\alpha > 1$ a rational number, then the singular homology of (X^{an}, D^{an}) is generated by the interval $[1,\alpha]$ and by a loop around 0. After a computation, we see that $H^1_{dR}(X,D)$ is generated by dT/T and $dt/(\alpha-1)$. Hence, if we write $\gamma :=$ the interval $[1,\alpha]$ and $\omega := dT/T$, the period pairing gives us

$$\langle[\omega],[\gamma]\rangle = \int_\gamma \omega = \int_1^\alpha \frac{dT}{T} = \log(\alpha).$$

We can also write easily algebraic numbers as periods. Indeed, if $\alpha \in \overline{\mathbb{Q}}$ and $f(T)$ is its minimal polynomial, let $X = \mathbb{A}^1_{\mathbb{Q}}$ and $D = \mathrm{Spec}(\mathbb{Q}[T]/(Tf(T)))$. Then, if we considering the singular 1-chain

$$\begin{array}{rcl} \gamma : [0,1] & \to & X^{an} = \mathbb{C} \\ s & \mapsto & s\alpha \end{array}$$

consisting on a path from 0 to α. Since D^{an} consists on the set of roots of $T \cdot f(T)$, both 0 and α belong to D^{an} and this 1-chain defines a class in $H_{1,sing}(X^{an}, D^{an}; \mathbb{Q})$, and taking the pairing together with the 1-form dT we get

$$\langle[dT],[\gamma]\rangle = \int_\gamma dT = \int_0^\alpha dT = \alpha.$$

There are more examples of trascendental numbers that appear in many contexts and that are periods, as for example the periods of elliptic curves, values of the ζ function, multiple-zeta values and so on. The interested reader can find more in [HMS15, Part IV] and in [BF a].

2 Voevodsky's definition of motive

In order to make these notes a little bit more self-contained, we briefly recall here Voevodsky's approach to the theory of motives. Ayoub's construction of the motivic Galois group of a field k with respect to $\sigma : k \hookrightarrow \mathbb{C}$ will be sketched in the next section.

First, let Sm/k denote the site of (not necessarily proper) smooth varieties over k equipped with the étale topology, and let $\mathbb{Q}$ be the coefficient ring (in general, we write Λ for this coefficient ring, but here we stick to the case $\Lambda = \mathbb{Q}$). The category $Shv((Sm/k)_{\acute{e}t}, \mathbb{Q})$ is a monoidal abelian category (actually a Grothendieck abelian category). Then, we derive the category in order to obtain

$$D(k) := D(Shv((Sm/k)_{\acute{e}t}, \mathbb{Q})),$$

which is a monoidal triangulated category $\mathbb{Q}$-linear (intuitively, this gives us cohomology theories satisfying étale descent, but not necessarily $\mathbb{A}^1$-invariance and $\mathbb{P}^1$-stability).

In order to obtain $\mathbb{A}^1$-invariance, we quotient the category by $I_{\mathbb{A}^1} :=$ smallest triangulated category stable by arbitrary direct sums and which contains, for all $X \in Sm/k$,

$$\cdots \to 0 \to \mathbb{Q}_{\acute{e}t}(X \times \mathbb{A}^1) \to \mathbb{Q}_{\acute{e}t}(X) \to 0 \to \cdots,$$

where $\mathbb{Q}_{ét}(X)$ is the sheaf associated to $U \rightsquigarrow \mathbb{Q}^{\mathrm{Hom}_k(U,X)}$. The idea is to identify the motives of X and $X \times \mathbb{A}^1$.

We call $DA^{eff}(k) := DA^{ét,eff}(k, \mathbb{Q}) := D(k)/I_{\mathbb{A}^1}$ the *category of effective motives.* We have a functor

$$\begin{aligned} Sm/k &\longrightarrow DA^{eff}(k) \\ X &\longmapsto M^{eff}(X), \end{aligned}$$

where $M^{eff}(X)$ is $\mathbb{Q}_{ét}(X)$ seen in the quotient $DA^{eff}(k)$.

In order to obtain the $\mathbb{P}^1$-stability, we have to localize $DA^{eff}(k)$. Let $T := \mathbb{Q}_{ét}(\mathbb{P}^1_k)/\mathbb{Q}_{ét}(\infty)$, and write $\mathbb{Q}(1)[2]$ for its image in $DA^{eff}(k)$. Denote, for every $n \in \mathbb{N}$, $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$, and for negative integers one takes the dual.

The first try is to consider $DA^{naive}(k)$, where the objects are pairs (M, m) with $M \in DA^{eff}(k)$ and $m \in \mathbb{Z}$, and the morphisms are

$$\mathrm{Hom}_{DA^{naive}(k)}((M,m),(N,n)) := \lim_{s \to +\infty} (M \otimes \mathbb{Q}(m+s), N \otimes \mathbb{Q}(n+s)).$$

The problem of this approach is that arbitrary direct sums don't work well. For example, $\mathbb{Q} \oplus \mathbb{Q}(-1) \oplus \mathbb{Q}(-2) \oplus \dots$ will be problematic.

In order to fix this, we need another approach. The idea, roughly speaking, is first to consider the category of complexes $\mathcal{P} := Compl(PSh(Sm/k, \mathbb{Q}))$, which turns out to be equivalent to $PSh(Sm/k, Ch_{\mathbb{Q}-mod})$, where $Ch_{\mathbb{Q}-mod}$ are $\mathbb{Z}$-graded $\mathbb{Q}$-complexes with chain maps between them, and $\mathcal{P}$ considered with more structure on it (the projective model structure on it, we don't give more details here). After doing a process called the Bousfield localisation on étale-local equivalences, we obtain a stable monoidal model category $\mathcal{P}_{ét-loc}$ such that its homotopy category $H_0\mathcal{P}_{ét-loc}$ is isomorphic to $D(Shv(Sm/k, \mathbb{Q}))$. If we now further localize with respect to $\mathbb{A}^1$, we get the category $\mathcal{P}_{\mathbb{A}^1-ét-loc}$. With this, we get an isomorphism $DA^{eff}(k, \mathbb{Q}) \cong H_0\mathcal{P}_{\mathbb{A}^1-ét-loc}$, where $H_0(-)$ denotes the homotopy category constructed from a model category. Hence, with this approach we recover the previous definition of effective motives.

Recall that the aim is to get $\mathbb{P}^1$-stability in a category well behaved with arbitrary direct sums, and we will do this using spectra. If we denote $T := \mathbb{P}^1_k \otimes \mathbb{Q}/\infty \otimes \mathbb{Q}$, then the functor $T \otimes \bullet : \mathcal{P}_{\mathbb{A}^1-ét-loc} \to \mathcal{P}_{\mathbb{A}^1-ét-loc}$ defines a Quillen adjunction, and from it we construct the category of spectra $Sp_T(\mathcal{P}_{\mathbb{A}^1-ét-loc}) := Sp_{T\otimes\bullet}(\mathcal{P}_{\mathbb{A}^1-ét-loc})$, with objects consisting on complexes in $\mathcal{P}_{\mathbb{A}^1-ét-loc}$, where transfer maps given by $T \otimes \bullet$, together with arrows between the complexes which are compatible with the transfer maps. This construction has suspension (resp. evaluation) functors, from $\mathcal{P}_{\mathbb{A}^1-ét-loc}$ to $Sp_T(\mathcal{P}_{\mathbb{A}^1-ét-loc})$ (resp. from $Sp_T(\mathcal{P}_{\mathbb{A}^1-ét-loc})$ to $\mathcal{P}_{\mathbb{A}^1-ét-loc}$). Levelwise, as a model category is fibrant with respect to $\mathcal{P}_{\mathbb{A}^1-ét-loc}$. Localizing $Sp_T(\mathcal{P}_{\mathbb{A}^1-ét-loc})$ with respect to the suspensions Sus^i of cofibrant replacements of objects, we get the category $Sp_T(\mathcal{P}_{\mathbb{A}^1-ét-loc})_{stable}$, from which we redefine $DA(k, \mathbb{Q}) := H_0(Sp_T(\mathcal{P}_{\mathbb{A}^1-ét-loc})_{stable})$.

With this, we obtain compatible functors

$$\begin{array}{ccccc} Sm/k & \longrightarrow & \mathcal{P}_{\mathbb{A}^1-ét-loc} & \xrightarrow{Sus^0} & Sp_T(\mathcal{P}_{\mathbb{A}^1-ét-loc}) \\ & \searrow^{M^{eff}} & \downarrow & & \downarrow \\ & & DA^{eff}(k,\mathbb{Q}) & \xrightarrow{\mathbb{L}Sus^0} & DA(k,\mathbb{Q}) \end{array}$$

where we denote M the composition $M := \mathbb{L}Sus^0 \circ M^{eff}$. This fixes the problem of the direct sums and we obtain $\mathbb{P}^1$-stability, which is precisely what we were looking for.

Hence, the motive associated to a smooth variety over k X is $M(X) = Sus^0(M^{eff}(X))$, where $M^{eff}(X)$ means the previous M^{eff} composed with the isomorphism $DA^{eff}(k,\mathbb{Q}) \cong H_0\mathcal{P}_{\mathbb{A}^1-\acute{e}t-loc}$. We continue our abuse of notation and write $\mathbb{Q}(1) := Sus^0(\mathbb{Q}(1))$.

Since this category of motives may be too big, we look at the smallest triangulated category containing $M(X)$ for all $X \in Sm/k$, denoted $DA_c(k)$, and called the category of constructible (or compact) motives.

2.1 Betti realization

The category of motives specializes in all cohomology theories. In particular, there is a Betti realization that gives the singular cohomology attached to the motive. One constructs this functor by going from Sm/k to the analytic varieties over a point An/pt. One constructs an analytic analogue of $DA(k)$, written $AnDA$, which comes with a functor $An* : DA(k) \to AnDA$. As before, we have an evaluation functor $Ev_{0,an} : AnDA \to AnDA^{eff}$, and this last category is equivalent to $D(\mathbb{Q}-mod)$ via a functor $i_* : AnDA^{eff} \to D(\mathbb{Q}-mod)$. Composing both functors, we obtain the Betti realization

$$Bti^* := i_* An^* : DA(k) \to D(\mathbb{Q}-mod),$$

which has a right adjoint Bti_*.

With $Bti_*\mathbb{Q}$, we recover the Betti cohomology of $X \in Sm/k$ via

$$H^n_B(X) \cong \mathrm{Hom}(M(X), Bti_*\mathbb{Q}[n]).$$

3 Ayoub's motivic Galois group

Here we define the motivic Galois group from the Betti realization $Bti^* : DA(k) \to D(\mathbb{Q}-mod)$ constructed by Ayoub. Its right adjoint Bti_* satisfies the following projection formula:

$$Bti_*(K) \otimes M \xrightarrow{\cong} Bti_*(K \otimes Bti^* M),$$

for every $K \in D(\mathbb{Q}-mod)$ and $M \in DA(k)$. This, together with the fact that Bti^* is monoidal, allows Ayoub to use a weak Tannakian formalism in order to obtain the structure of a Hopf algebra object on $Bti^*Bti_*\mathbb{Q}$, with $\mathbb{Q}$ regarded in $D(\mathbb{Q}-mod)$.

Hence we define the following motivic Hopf algebra:

$$\mathcal{H}_{mot}(k,\sigma) := Bti^*Bti_*\mathbb{Q}.$$

Here we write explicitly σ because this construction depends on the chosen embedding $\sigma : k \hookrightarrow \mathbb{C}$. Before we were hiding σ in order to make the notation easier to read.

The complex $\mathcal{H}_{mot}(k,\sigma)$ is concentrated in negative degree, and it is posible to describe it explicitly. Its zero homology group $H_0(\mathcal{H}_{mot}(k,\sigma))$ defines a Hopf algebra, and taking it's spectrum we obtain the motivic Galois group

$$G_{mot}(k,\sigma) := \mathrm{Spec}(H_0(\mathcal{H}_{mot}(k,\sigma))).$$

There is a functor

$$DA_c(k) \longrightarrow D^b(Rep^{fd}(G_{mot}(k,\sigma)))$$

that it is expected to be an equivalence. This is the so called motivic t-structure conjecture, and it implies Grothendieck's standard conjectures, Bloch-Beilinson conjecture on Chow groups, etc.

If this conjecture is true, then $MM(k) := Rep^{fd}(G_{mot}(k,\sigma))$ would satisfy all the properties expected for the abelian category of mixed motives over k.

In any case, without assuming the conjecture, we can define the motivic Galois group of a smooth k-variety X as follows: let M denote the image of $M(X)$ in $D^b(Rep^{fd}(G_{mot}(k,\sigma)))$. Then, we consider the set of cohomology groups of the complex M together with the action of $G_{mot}(k,\sigma)$, which satisfy by construction that $H^i(M) = H^i_{sing}(X^{an},\mathbb{Q})$ (not just as vector spaces, but as representations of $G_{mot}(k,\sigma)$), and we further consider the Tannakian subcategory of $Rep^{fd}(G_{mot}(k,\sigma))$ generated by (direct sums, tensor products and duals of) these cohomology groups, denoted $\langle M\rangle$. This category is again a neutral Tannakian category, and therefore we can consider its fundamental group. This group, denoted $G_{mot}(X)$, is the motivic Galois group of X.

Remark 3.1. One can try to define the motivic Galois group of X directly on $DA(k)$, without going to the category of representations, by doing the following: consider the subcategory $\langle M(X)\rangle$ of $DA(k)$ generated by (tensor products and duals of) $M(X)$. Then, we can restrict the Betti realization to $\langle M(X)\rangle$,

$$Bti^*|_{\langle M(X)\rangle} : \langle M(X)\rangle \to D(\mathbb{Q}-mod),$$

which has also a right adjoint $Bti_*|_{\langle M(X)\rangle}$, and see if this also induces a Hopf algebra structure on $\mathcal{H}_{mot}(X) := Bti^*|_{\langle M(X)\rangle} \circ Bti_*|_{\langle M(X)\rangle}\mathbb{Q}$. If this is the case, we could define the motivic Galois group of X via

$$G_{mot}(X) := \mathrm{Spec}(H_0(\mathcal{H}_{mot}(X))).$$

One of course should check if this construction gives us the same group. In any case Ayoub doesn't do this in his paper, so we stick to the more classical setting of $MM(k)$.

With this definition we get that for a finite Galois extension L/k, the motivic Galois group coincides with the usual Galois group (considered as a group scheme). The motivic Galois group of $\mathbb{P}^1$ is $\mathbb{G}_m$.

Remark 3.2. In $MM(k) = Rep^{fd}(G_{mot}(k,\sigma))$ there is a second fiber functor that neutralizes he category. From this construction it is not clear where is it coming from, and one has to consider a different approach developed by Nori. Indeed, Nori constructs in a different way a Tannakian category of mixed motives. From his construction, it is not clear what is the fundamental group of this Tannakian category, but it has been recently proved in [CG14] that it is isomorphic to the motivic Galois group defined by Ayoub, which is our approach here. Hence, a posteriori, we have that both constructions of $MM(k)$ are isomorphic, and the advantage of Nori's construction is that it comes with a fiber functor

$$MM(k) \to Vec_k$$

that realizes the de Rham cohomology. We will use this later.

4 Grothendieck's period conjecture

Through this section, we assume that k is an algebraic extension of $\mathbb{Q}$. Recall that every smooth k-variety X induces a canonical isomorphism ϖ_X between $H^*_{dR}(X)\otimes_k\mathbb{C}$ and $H^*_{sing}(X^{an},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$. We have already seen how does the entries of this matrix look like: for example, if $X=\mathbb{G}_m$, ϖ_X is given by multiplication by $2\pi i$. Note that the field generated by this period, $\mathbb{Q}(2\pi i)$, has trascendence degree over $\mathbb{Q}$ equal to 1. Covering $\mathbb{P}^1$ and using the Mayer-Vietoris' sequence, we get also that $Per(\mathbb{P}^1)=\mathbb{Q}(2\pi i)$.

On the other hand, we have also seen that the motivic Galois group of $\mathbb{P}^1$ is $\mathbb{G}_m$.

Grothendieck's conjecture states that the trascendence degree of $Per(X)$ is equal to the dimension (as a group scheme) of the motivic Galois group of X, i.e.

$$tr.deg(Per(X)/\mathbb{Q})=\dim G_{mot}(X).$$

In the case of $\mathbb{P}^1$, we see that this is true because of the above discussion.

There is a more geometrical way to describe this conjecture, and for this we need the notion of the motivic period torsor.

First, we describe now more precisely the motivic Hopf algebra $\mathcal{H}_{mot}(k,\sigma)$ of the previous section. We have that $\mathcal{H}_{mot}(k,\sigma)\cong\mathcal{P}(k,\sigma)\otimes\mathbb{C}$, where $\mathcal{P}(k,\sigma):=\widetilde{\Omega}^{\infty-\bullet}_{alg}(\overline{\mathbb{D}}^{\infty})[\overline{\pi}^{-1}]$ is the complex of Valentina's talk inverting what we need to invert.

Ayoub defines the ring of abstrac periods as follows: we look at $P_{Ay}(k,\sigma):=H_0(\mathcal{P}(k,\sigma))$. If we don't invert $\overline{\pi}$, then we obtain the ring of abstract effective periods, which is canonically isomorphic, as a k-vector space, to

$$\mathcal{O}_{alg}(\overline{\mathbb{D}}^{\infty})/\left(\frac{\partial f}{\partial t_i}-(f|_{t_i=1}-f|_{t_i=0})\right).$$

Here $\mathcal{O}_{alg}(\overline{\mathbb{D}}^{\infty}):=colim(\mathcal{O}_{alg}(\overline{\mathbb{D}}^{n}))$ is defined by

$$\mathcal{O}_{alg}(\overline{\mathbb{D}}^{n}):=\left\{f\in\mathbb{C}[[t_1,\dots,t_n]]\;\middle|\;\begin{array}{l}f \text{ has radius of convergence } >1\\ \text{and } f \text{ is algebraic over } k(t_1,\dots,t_n)\end{array}\right\}.$$

With this description, we see easily that we have an evaluation morphism

$$\begin{array}{rccc}Ev: & P_{Ay}(k,\sigma) & \to & \mathbb{C}\\ & f & \mapsto & \displaystyle\int_{[0,1]^n} f,\end{array}$$

where n is just an integer such that $f\in\mathcal{O}_{alg}(\overline{\mathbb{D}}^{n})$. The image of this evaluation morphism gives us classical periods, we come to this later.

We can interpret $\mathrm{Spec}(P_{Ay}(k,\sigma))$ as follows:

In $MM(k)$, we have two realizations (two fibre functors) specializing in Betti and de Rham cohomology:

$$\begin{array}{rl}H_{dR}: & MM(k)\to Vec_k\\ H_{sing}: & MM(k)\to Vec_{\mathbb{Q}}\end{array}$$

where the first one is the de Rham realization coming from Nori's construction and the second one is the Betti realization discussed above. Note that H_{sing} is the fibre functor giving the

Tannakian fundamental group $G_{mot}(k,\sigma)$. In other words, we can describe for example the motivic Galois group of X as

$$G_{mot}(M) = Aut^{\otimes} H_{sing}|_{\langle M\rangle}.$$

Then, if we consider the torsor of isomorphisms between the functors $H_{dR}|_{\langle M\rangle}$ and $H_{sing}|_{\langle M\rangle}$, denoted $Isom^{\otimes}(H_{dR}|_{\langle M\rangle}, H_{sing}|_{\langle M\rangle})$, we obtain a variety that has a canonical complex point given by Grothendieck's isomorphism ϖ_X. Hence, this is a torsor under $G_{mot}(X)$, which is acting on the right.

Moreover, the residue field $k(\varpi_X)$ of this complex point ϖ_X is exactly $Per(X)$ (cf. [Ayo14c, Lem. 27]). Now, we can relate the previous statement of the Grothendieck's conjecture with this torsor. Indeed, since $Isom^{\otimes}(H_{dR}|_{\langle M\rangle}, H_{sing}|_{\langle M\rangle})$ is a torsor over $G_{mot}(X)$, it is equidimensional of dimension

$$\dim(Isom^{\otimes}(H_{dR}|_{\langle M\rangle}, H_{sing}|_{\langle M\rangle})) = \dim(G_{mot}(X)).$$

Hence, since $k(\varpi_X) = Per(X)$, Grothendieck's period conjecture

$$tr.deg(Per(X)/\mathbb{Q}) = \dim G_{mot}(X)$$

holds for X if and only if ϖ_X is a generic point of $Isom^{\otimes}(H_{dR}|_{\langle M\rangle}, H_{sing}|_{\langle M\rangle})$. Note that we are saying that ϖ_X is a generic point, and this doesn't imply that it is a dense point because we don't know if $Isom^{\otimes}(H_{dR}|_{\langle M\rangle}, H_{sing}|_{\langle M\rangle})$ is connected (although we expect it). We come to this later.

In particular, we get the inequality

$$tr.deg(Per(X)/\mathbb{Q}) \leq \dim G_{mot}(X),$$

which gives us non-trivial information about multizeta values.

Passing to the limit of all the smooth k-varieties X, we get a pro-k-algebraic variety $Isom^{\otimes}(H_{dR}, H_{sing})$, which is a torsor for the pro-$\mathbb{Q}$-algebraic group $G_{mot}(k,\sigma)$ and that it is called the torsor of periods. This coincides with $\mathrm{Spec}(P_{Ay}(k,\sigma))$, and Grothendieck conjecture is equivalent to the fact that, if $k=\mathbb{Q}$, ϖ is a generic point of $Isom^{\otimes}(H_{dR}, H_{sing})$.

5 Abstract periods à la Kontsevich-Zagier

The ring of abstract efective periods of Kontsevich-Zagier, denoted P_{KZ}^{eff}, is a free $\mathbb{Q}$-vector space generated by some symbols modulo some relations.

Here the symbols are 5-tuples $(X, D, i, \gamma, \omega)$, where

- X is a $\mathbb{Q}$-variety (possibly singular!).
- $D \subset X$ is a closed subvariety.
- The symbol i is just a natural number.
- The symbol γ is a class in $H_{sing,i}(X^{an}, D^{an})$.
- The symbol ω is a class in $H_{dR}^{i}(X, D)$.

And the relation are the following:

- Additivity: the map $(\gamma,\omega)\mapsto(X,D,i,\gamma,\omega)$ is bilinear.
- Base change: given $f:X'\to X$ with $f(D')\subset D$, then $(X,D,i,f_*\gamma',\omega)=(X',D',i,\omega',F^*)$.
- Stokes' formula: if $E\subset D\subset X$ are closed subvarieties, and $\gamma\in H_{sing,i}(X^{an},D^{an}),\omega\in H^{i-1}_{dR}(D,E)$, then $(X,D,i,\gamma,d\omega)=(D,E,i-1,\partial\omega,\omega)$.

Let $\underline{2\pi i}$ denote the class $(\mathbb{G},\emptyset,1,\gamma,dT/T)$, where γ is the loop, then the ring of abstract periods (of Kontsevich-Zagier) is obtained by inverting $\underline{2\pi i}$: $P_{KZ}:=P^{eff}[\underline{2\pi i}^{-1}]$.

As in the case of Ayoub's periods, we have an evaluation morphism

$$\begin{array}{rccc} Ev: & P_{KZ}(k,\sigma) & \to & \mathbb{C} \\ & (X,D,i,\gamma,\omega) & \mapsto & \displaystyle\int_\gamma \omega, \end{array}$$

which is well defined because of the additivity, the base-change and fulfill of the Stoke's formula by integration. We see that its image is precissely the set of classical periods defined via Grothendieck's comparisson isomorphism. In other words, Ev factors surjectively through the set of periods.

Here there is a remark to make: we are considering here possibly singular varieties, but at the end of the day we get the same thing as if we only consider smooth varieties.

6 Comparing the different definitions of periods: the Kontsevich-Zagier conjecture

So far, we have three different definitions of periods: the classical one as entries of the comparison isomorphism, the one by Ayoub and the one by Kontsevich-Zagier. What is the relation between them?

First of all, we have an isomorphism

$$P_{Ay}\cong P_{KZ},$$

and this is compatible with the evaluation morphisms (cf. [Ayo14c, Prop. 11]). In particular, this tells us that the evaluation morphism from Ayoub's definition also gives us all the classical periods. Let's construct the morphism:

Given $f\in P_{Ay}$, we know that there exists an $n\in\mathbb{N}$ such that $f\in\mathcal{O}_{alg}(\overline{\mathbb{D}}^n)$, we know from Valentina's talk that the exists an étale $\mathbb{Q}[t_1,\dots,t_n]$-algebra A contained in $\mathcal{O}_{alg}(\overline{\mathbb{D}}^n)$ containing f. Setting $X:=\mathrm{Spec}(A)$, we already have two elements of the 5-tuple.

Now we set $D\subset X$ to be the divisor given by $\prod_{i=1}^n t_i(t_i-1)=0$. Here we are using that A is contained in $\mathcal{O}_{alg}(\overline{\mathbb{D}}^n)$. Let τ_n be the composition $[0,1]^n\hookrightarrow\overline{\mathbb{D}}^n\to X^{an}$. Then, the morphism $P_{Ay}\to P_{KZ}$ is given by

$$f\mapsto(X,D,n,\tau_n,f\cdot dt_1\wedge\cdots\wedge dt_n).$$

Proving that this morphism is actually an isomorphism is quite difficult, and requires the isomorphism between of Nori's motivic Galois group and the one described in these notes, due to Ayoub.

Finally, one could ask how big P_{KZ} is, compared with the algebra of classical periods. The conjecture, by Kontsevich and Zagier, is that it is as big as the classical periods. In other words, the Kontsevich-Zagier's conjecture states that the surjective evaluation morphism

$$Ev : P_{KZ} \to \{\text{Classical periods}\}$$

is injective.

Another way to think on this conjecture is by saying that all the algebraic relations between (classical) periods are of motivic origin.

There is still another (vague) way to think on this conjecture: if we start with a smooth projective $\mathbb{Q}$-variety and we consider an algebraic cycle Z of codimension c, we can look at their cohomology classes $[Z]_{dR}$ and $[Z]_{sing}$ in $H^{2c}_{dR}(X) \otimes \mathbb{C}$ and $H^{2c}_{sing}(X^{an}, \mathbb{Q}) \otimes \mathbb{C}$, respectively. Then, the period isomorphism ϖ_X sends $[Z]_{dR}$ to $(2\pi i)^c [Z]_{sing}$ (cf. [BC16]). This will induce some algebraic relations on the periods: if we look at a cycle Z on some power of X, Künneth formula allows us to write it as an element of the tensor product of $H^{2c}(X)$ (here we ommit the subindex because both hold), and this will give polynomial relations between the periods of X. Then, Kontsevich-Zagier conjecture says that these polynomial relations are all the relations between the periods of X.

7 Relation between Grothendieck's period conjecture and Kontsevich-Zagier's conjecture

Here we continue to assume that $k = \mathbb{Q}$. The conjecture of Grothendieck, which says that the complex point

$$\varpi : \text{Spec}(\mathbb{C}) \to Isom^{\otimes}(H_{dR}, H_{sing})$$

is a generic point of $Isom^{\otimes}(H_{dR}, H_{sing})$, is very related with Kontsevich-Zagier's conjecture, which says that $Ev : P_{Ay} \to \mathbb{C}$ is injective.

Recall that for $X \in Sm_{\mathbb{Q}}$, the residue field of ϖ_X is equal to the ring of periods of X. Hence, if we take $f \in \mathcal{O}(Iso^{\otimes}(H_{dR}, H_{sing}))$ and we look at its image on $k(\varpi) \subset \mathbb{C}$, denoted as usual $f(\varpi)$, we obtain something in the ring of periods.

This is not casual. Indeed, we have that $\mathcal{O}(Isom^{\otimes}(H_{dR}, H_{sing})) \cong P_{Ay}$, and that this isomorphism is compatible with both evaluation morphisms:

$$\begin{array}{ccc} \mathcal{O}(Isom^{\otimes}(H_{dR}, H_{sing})) & \xrightarrow{\simeq} & P_{Ay} \\ & \searrow & \downarrow{\scriptstyle Ev} \\ & & \mathbb{C} \end{array}$$

Therefore, Kontsevich-Zagier's conjecture holds if and only if Grothendieck's period conjecture holds and P_{Ay} is an integral domain. Indeed, Ev is injective if and only if ϖ is a generic point and $\mathcal{O}(Isom^{\otimes}(H_{dR}, H_{sing}))$ is an integral domain (note that this implies that $Isom^{\otimes}(H_{dR}, H_{sing})$ is connected).

8 Galois theory for periods

> *"Pour un nombre α appartenant à la "$\mathbb{Q}$-algèbre des périodes" (i.e. l'algèbre engendrée par les périodes de tous les motifs définis sur $\mathbb{Q}$, en inversant $2\pi i$), la théorie des motifs suggère que oui: le sens profond de la conjecture des périodes de Grothendieck est de pouvoir fair agir le groupe des points rationnels du Galois motivique absolu sur cette algèbre des périodes."*
>
> Y. André, Une introduction aux motifs.

André proposes in [And09] a Galois theory for periods. First, he takes some non algebraic numbers, say for example π. Since it is trascendental, $\mathbb{Q}(\pi)$ is infinite dimensional as a $\mathbb{Q}$-vector space, so there are no polynomials that vanish at π, but for example $\sin(x)$ vanishes at π. Since $\sin(x)$ also vanishes at 0, $\sin(x)$ would not be irreducible (if we think on $\sin(x)$ as a polynomial), so we consider $\sin(x)/x$, which vanishes at $n\pi$ for all $n \in \mathbb{Z} \setminus 0$.

If we write $\sin(x)/x$ as a power series, we have that

$$\prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 - \frac{x}{n\pi}\right) = \frac{\sin(x)}{x},$$

and if we write the product on the left hand side, we obtain indeed a formal power series with rational coefficients. Note that this is the closest thing to a polynomial that we can think on.

Here the semigroup $(\mathbb{Z} \setminus \{0\}, \cdot)$ acts on the set of roots of our power series. But in classical Galois theory, we have a group acting, not just a semigroup, so we consider the group generated by $(\mathbb{Z} \setminus \{0\}, \cdot)$, which is precisely $Gal(\mathbb{Q}(\pi)/\mathbb{Q}) := \mathbb{Q}^* = \mathbb{G}_m$. Here the set of conjugates of π is $\mathbb{Q}^* \cdot \pi$, and we have that $\mathbb{Q}(\pi)^{Gal(\mathbb{Q}(\pi),\mathbb{Q})} = \mathbb{Q}$.

If we compare this naive computation with the results that we obtain for $X = \mathbb{P}^1$, where $G_{mot}(X) = \mathbb{G}_m$, and $Per(X) = \mathbb{Q}(2\pi i)$, we see that essentially we obtain the same (we could have started with $2\pi i$ and the power series given by $\prod(1 - 1/(n2\pi i))$.

In [And09], André computes also the same for periods coming from elliptic curves. It seems plausible that there is a Galois theory for periods, and maybe in the future, when we understand better the motivic Galois group and the ring of periods, we can study it.

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