Freie Universität Berlin

Grothendieck-Ogg-Shafarevich formula for ℓ-adic sheaves

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29.04.2016

Abstract

In this master thesis, we revisit the Grothendieck-Ogg-Shafarevich formula. In order to do this, we recall constructions and results from arithmetic (more concretely, about the ramification groups of a Galois extension), representation theory (specifically the tools needed to measure the wild ramification of an ℓ-adic Galois representation) and arithmetic geometry (mainly constructions and results of the cohomology of constructible sheaves). In the last part, we give a detailed proof of the formula.

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1 Introduction

The aim of this master thesis is to understand the Grothendieck-Ogg-Shafarevich formula, which measures the *Euler characteristic* of a *lisse sheaf* over a smooth curve and puts it in terms of its rank, the geometry of the curve and its wild ramification. Let's explain this a little bit more:

Let U be a curve defined over an algebraically closed field k, and let $\mathcal{F}$ be a lisse sheaf (one can think, for example, in a locally constant étale sheaf) defined over U. Let C be a compactification of U.

If k has characteristic zero, then we know that $\chi_c(U, \mathcal{F}) = \mathrm{rk}(\mathcal{F})\chi_c(U, \overline{\mathbb{Q}_\ell})$, where χ_c denotes the *Euler characteristic with compact support.*

If k has positive characteristic and $U = C$, then this formula remains true, but if U is non proper then the formula becomes false. The reason for this is that it may appear some *wild ramification* of $\mathcal{F}$ at $C \backslash U$. What is this wild ramification? We will define it later, but let's try to explain it a little bit: for a given lisse sheaf $\mathcal{F}$ on U, we want to define the wild ramification of $\mathcal{F}$ at a point $x \in C \setminus U$, denoted $\mathrm{Swan}_x(\mathcal{F})$; in order to achieve this, we will construct from $\mathcal{F}$ a continuous representation of the absolute Galois group G_x of a complete discretely valued field determined by x. This kind of representations are called *ℓ-adic Galois representation*, and one can measure the wild ramification of these representations.

In order to measure this wild ramification, one has to construct the so called *Artin* and *Swan representations.* We construct these representations by defining their characters directly. One important fact that we use is the Hasse-Arf theorem, which is very deep although at first it looks mild, and that tells us that the *breaks* of a filtration of certain Galois groups are integer.

All in all, we have that using geometry, representation theory and arithmetic we are able to define the wild ramification of our sheaf $\mathcal{F}$ at a point $x \in C \setminus U$, denoted $\mathrm{Swan}_x(\mathcal{F})$. With this notion, we obtain the Grothendieck-Ogg-Shafarevich formula, which tells us that

$$\chi_c(U, \mathcal{F}) = \mathrm{rk}(\mathcal{F})\chi_c(U, \overline{\mathbb{Q}_\ell}) - \sum_{x \in C \backslash U} \mathrm{Swan}_x(\mathcal{F}).$$

For a more precise statement, see theorem 4.63 below.

The structure of this thesis, following the presentation of [KR15], is the other way around: we first define and study the ramification groups from arithmetic (section 2); after this we recall some facts from representation theory and construct the Artin and Swan representations, which will allow us to measure the wild ramification of an ℓ-adic Galois representation (section 3); in the last section we go to the geometric setting, and we study how to associate from a lisse sheaf $\mathcal{F}$ a representation of the fundamental group of U, and from this we will obtain an ℓ-adic representation; finally, after recalling some facts from ℓ-adic cohomology, we define precisely the wild ramification of an ℓ-adic sheaf and we finish the thesis proving the Grothendieck-Ogg-Shafarevich formula.

Having a way of computing the Euler characteristic of a sheaf is very useful, because combined with vanishing theorems it gives us information about the dimension of the cohomology groups. This formula has a lot of applications: for example, it is used (among other things) in the proof of the Weil conjectures by Laumon.

Lately these notions have been generalized to higher dimension. There are two approaches: the first one, initiated by Wiesend and developed by Kerz, Schmidt, Drinfeld and Deligne, consists in reducing the study to dimension 1 by considering the family of all curves in our algebraic variety; the second approach, followed by Kato and Saito, develops a ramification theory directly in higher dimension and they prove an analogue of the Grothendieck-Ogg-Shafarevich formula. Kindler and Rülling survey both approaches in the last sections of [KR15], and this can be the first thing one can read after this master thesis.

Finally, there are essentially no new ideas in this thesis. Its presentation follows the one by Kindler and Rülling in [KR15], and the only merit that I can expect is having clarified some explanations, having enlightened some paragraphs that were not so detailed, having motivated a little bit more some parts, presenting sometimes examples where they don't and having restructured some of their proofs: this was challenging because their material is really nice. When I found that I couldn't improve their approach, I have quoted their notes. Hopefully this thesis serves as a first step towards their notes and gives the reader the opportunity to understand this nice formula and its proof.

Acknowledgements

First of all, I would like to express my special appreciation and thanks to my advisor Hélène Esnault for suggesting me such an interesting topic: one has to learn really nice things in order to understand the Grothendieck-Ogg-Shafarevich formula, and the proof itself is wonderful; I feel really lucky to work in your group, this is a privilege and I will do my best in order to continue and to return to the group as much as possible. Then, of course, my gratitude goes to all the people that have been around the office A3 111: the discussions with all of you allowed me to understand hundreds of things better, and also to enjoy the way by sharing our knowledge (our little branch of knowledge is quite difficult, so we must unite and help each other in order to create a friendly and productive atmosphere). I also want to thank all the people in the research group of Hélène, specially Lars Kindler and Kay Rülling: your notes were a necessary condition for this thesis, and moreover you also helped me a lot when I was having problems with the material. Hopefully this thesis complements your excellent work. I also want to thank the financial support of Fundación Mutua Madrileña.

Finally, but more important than all before, I want to celebrate the support of my family at all the levels: you always make me feel happy, take care of me and help me with my decission of living abroad. I know that you miss me a lot, and so do I. This experience is a great opportunity for me and it is very important that you understand this and that you have confidence in me, that's priceless.

Pedro A. Castillejo
Berlin, April 2016

2 Arithmetic

A general reference for this part is [Ser79]. In this section, L/K will be a finite Galois extension of complete discretely valued fields with separable residue field extension, and $G := \mathrm{Gal}(L/K)$. We denote by v the valuation in K, v_L the extension to L and p will be the characteristic $\mathrm{char}(k(v))$, and let

$$\begin{tikzcd} K \arrow[r, hook] & L \\ A \arrow[u] \arrow[r, hook] & B \arrow[u] \\ \mathfrak{m}_K \arrow[u] \arrow[r, hook] & \mathfrak{m}_L \arrow[u] \end{tikzcd}$$

be the diagram where the arrows are inclusions, A and B the ring of integers of K and L, and $\mathfrak{m}_K$ and $\mathfrak{m}_L$ their maximal ideals.

2.1 Ramification filtration - lower numbering

In the above setting, we define the ramification groups with lower numbering:

Definition 2.1. For $i \geq -1$, we know that G acts on $B/\mathfrak{m}_L^{i+1}$. The *i-th ramification subgroup of G* is the following subgroup:

$$G_i := \{\sigma \in G | \sigma \text{ acts trivially on } B/\mathfrak{m}_L^{i+1}\}.$$

These ramification subgroups form a filtration of G called the *ramification filtration of G in the lower numbering.*

One sees immediately that $G_{-1} = G$ and that $G_0 = I$ (the inertia group, i.e. the kernel of the surjective map $G \twoheadrightarrow \mathrm{Gal}(k(v_L)/k(v))$.

Remark 2.2. Note that G_i is just

$$G_i := \{\sigma \in G | \forall b \in B,\ v_L(\sigma(b) - b) \geq i + 1\},$$

and since we know [Ser79, III, Prop. 12] that there exists $x \in B$ such that $B = A[x]$, we can also write

$$G_i = \{\sigma \in G | v_L(\sigma(x) - x) \geq i + 1\},$$

which gives us an easy way to compute these subgroups. For example, we see immediately that G_i is trivial for $i \gg 0$.

Let's compute an example:

Example 2.3. Let k be an algebraically closed field of $\mathrm{char}(k) = p > 0$, and let $K := k((x))$ be the Laurent series with coefficients in k. Consider the polynomial $u^p + xu^{p-1} - x \in K[u]$, which by the Eisenstein criterion is seen to be irreducible and therefore defines the field extension $L := K[u]/(u^p + xu^{p-1} - x)$, which is called an *Artin-Schreier extension*[1]. This extension L/K is Galois with group $G = \mathbb{Z}/p\mathbb{Z}$, and $a \in G$ acts via $u \mapsto u/(1+au)$. We get this diagram:

$$\begin{tikzcd}
K \arrow[r, hook] & K[u]/(u^p + xu^{p-1} - x) \\
k[[x]] \arrow[u] \arrow[r, hook] & k[[x]][u]/(u^p + xu^{p-1} - x) \arrow[u] \\
(x) \arrow[u] \arrow[r, hook] & (u) \arrow[u]
\end{tikzcd}$$

and since $u^p = x(1 - u^{p-1})$, we see that the ramification index of this extension is $e(L/K) = p$ (in particular it is totally ramified, so $G_{-1} = G_0 = G$). Let's compute the other ramification groups. For this, we just need to know how does $a \in G$ act on u. Since, for $a \neq 0$,

$$v_L(u - u/(1+au)) = v_L(u(1 + au - 1)/(1+au)) = 2,$$

we have that $G_1 = G$ and that $G_2 = 0$.

Similarly (c.f. [Lau81]), one proves that $L = K[t]/(t^{p^n} - t - x^{-m})/K$, with $(m, p) = 1$, has the following ramification filtration:

$$\mathbb{Z}/p^n\mathbb{Z} = G_0 = \ldots = G_m \supsetneq G_{m+1} = 0.$$

This finishes the example.

Since $B = A[x]$, the generator x will also be a generator over intermediate valuation rings, so we have the following compatibility when taking subgroups:

Proposition 2.4. *If $H < G$ is a subgroup and L^H its fixed field, then L/L^H is Galois with group H, and for all i,*

$$H_i = G_i \cap H.$$

Remark 2.5.

1. Applying the proposition to $H = I$, we see that we may assume L/K to be totally ramified.

2. Quotients will not respect this filtration in general. In order to fix this, we will define a different ramification filtration, which will have the same subgroups but in a different numbering (c.f. [KR15, Cor. 3.43]). In order to distinguish the filtrations, we denote the second one as G^i, and we call it *ramification filtration with upper numbering.* But before we define this second filtration, let's study a little bit more the structure of our lower numbering filtration.

Once we assume that L/K is totally ramified, we know by [Ser79, III, Lem. 4] that x, the generator of $B = A[x]$, may be chosen so that it is an uniformizer for L. Hence, for $\sigma \in G_0$, we see that $v_L(\sigma(x)) = v_L(x) = 1$ and therefore $\sigma(x)/x \in B^\times =: U_L^0$. Since $G_i = \{\sigma \in G | \sigma(x) - x \in (x)^{i+1}\}$, we see that

$$G_i = \{\sigma \in G | \sigma(x)/x \equiv 1 \pmod{\mathfrak{m}_L^i}\}.$$

This motivates a filtration of the group of units of L:

[1] Changing coordinates $t = 1/u$, we see that L is isomorphic to $K[t]/(t^p - t - 1/x)$.

Definition 2.6. For $i > 0$, we define $U_L^i := 1 + \mathfrak{m}_L^i \subset B^\times$ to be the group of *i-th units*. For $i = 0$, we will just define $U_L^0 := U_L := B^\times$.

These subgroups will give us information about the ramification subgroups because they are easier to handle and we can do the following[2]:

Proposition 2.7. *The assigment $\sigma \mapsto \sigma(x)/x$ induces an injective homomorphism of groups*

$$G_i/G_{i+1} \hookrightarrow U_L^i/U_L^{i+1},$$

and this homomorphism is independent of the choice of x.

The right hand side of this homomorphism is very explicit. Indeed, let's look closer at both $i = 0$ and $i > 0$. Since we may assume that L/K is totally ramified, the residue fields of L and K are the same. Let's denote them by k.

- The case $i = 0$: $U_L^0/U_L^1 \cong (k^\times, \cdot)$, where the map $U_L^0 \to (k^\times, \cdot)$ is given by fixing a local parameter x and mapping $u_0 + u_1 x + \ldots + u_n x^n \mapsto [u_0]$. We can also map $U_L^0 \to U_L^0/\mathfrak{m}_L = k^\times$, so we don't need to make any choice.
- The case $i > 0$: $U_L^n/U_L^{n+1} \cong (k, +)$. For this, we can show that $U_L^n/U_L^{n+1} \to \mathfrak{m}_L^n/\mathfrak{m}_L^{n+1} : [1 + x] \mapsto [x]$ is an isomorphism, and that the latter is a 1-dimensional k-vector space (for this just need to choose a local parameter x^n).

Using this, one shows the following:

Corollary 2.8.
- *G_0/G_1 is cyclic of order prime to $p = \mathrm{char}(k)$.*
- *If $p = 0$, then $G_i = 0$ for $i > 0$.*
- *If $p > 0$, then for $i \geq 1$ the groups G_i are p-groups, and the quotients G_i/G_{i+1} are abelian p-groups.*
- *G_0 is a semi-direct product of a cyclic group of order prime to p and a p-group. In particular, G_0 is solvable and G_1 is its unique p-Sylow group.*

2.2 Ramification filtration - upper numbering

Now we want to introduce the upper numbering filtration, that will respect the quotients. We said that we have to change the numbering, and in order to do it we first need to fix some notation. For $u \in \mathbb{R}_{\geq -1}$, we denote $G_u := G_{\lceil u \rceil}$, where $\lceil u \rceil$ is the smallest integer greater or equal to u.

Definition 2.9 (Herbrand's function)**.** We define the function $\varphi_{L/K} : [-1, \infty) \to [-1, \infty)$ as follows:

$$\varphi_{L/K}(u) = \int_0^u \frac{dt}{(G_0 : G_t)},$$

where $(G_0 : G_t)$ is defined, for $t \in [-1, 0)$, as $(G_0 : G_t) := (G_t : G_0)^{-1}$. In other words, $(G_0 : G_{-1}) = 1/f$ and $(G_0 : G_t) = 1$ for $t \in (-1, 0)$, where f is the degree of the extension of the residue fields.

[2] The two lines computation can be checked in [KR15, Prop. 3.37].

Remark 2.10. Note that if $u \in \mathbb{Z}_{\geq 0}$, then

$$\varphi_{L/K}(u) + 1 = \frac{1}{|G_0|} \sum_{i=0}^{u} |G_i|.$$

This definition arises naturally in the computation of the image of G_i in the quotient G/H, where $H \lhd G$ is a normal subgroup (c.f. proof of [KR15, Cor. 3.43]). Indeed, we obtain the following compatibility when taking quotients:

Proposition 2.11. *Let $H \lhd G$ be a normal subgroup. Then, for $u \in \mathbb{R}_{\geq -1}$, we have*

$$G_u H/H = (G/H)_{\varphi_{L/L^H}(u)}.$$

Hence, if $\psi_{L/K}$ denotes the inverse to $\varphi_{L/K}$, we define the upper numbering filtration as follows:

Definition 2.12. For $v \in \mathbb{R}_{\geq -1}$, define $G^v := G_{\psi_{L/K}(v)}$.

In order to be able to write the filtration, we will only keep track of the *jumps*, i.e. the $v \in \mathbb{R}_{\geq -1}$ such that $G^v \supsetneq G^{v+\varepsilon}$ for all $\varepsilon > 0$. In other words, the jumps are just the $\varphi_{L/K}(u)$, where u are the integers such that $G_u \supsetneq G_{u+1}$.

Example 2.13.

- Let's compute the upper numbering filtration of the Artin-Schreier extension. Since

  $$\mathbb{Z}/p\mathbb{Z} = G_0 = \ldots = G_m \supsetneq G_{m+1} = 0,$$

  we have that

  $$\varphi_{L/K}(u) = \begin{cases} \dfrac{up}{p} = u & \text{if } 0 \leq u \leq m, \\ m + \dfrac{u-m}{p} & \text{if } u > m. \end{cases}$$

  Hence, we get

  $$\psi_{L/K}(v) = \begin{cases} v & \text{if } 0 \leq u \leq m, \\ p(v-m) + m & \text{if } u > m. \end{cases}$$

  We see then that the only jump is on $v = m$, and therefore we write the upper numbering filtration as

  $$\mathbb{Z}/p\mathbb{Z} = G^0 = \ldots = G^m \supsetneq G^{m+\varepsilon} = 0.$$

- In the above example the only jump of the filtration is an integer, but this is not always the case. Indeed, Serre constructed in [Ser60, Sec. 4] a totally ramified Galois extension $L/\mathbb{Q}_2$ with Galois group G isomorphic to the quaternions group $\{\pm 1, \pm i, \pm j, \pm k\}$ with the usual relations. The center of the group is $Z(G) = \{\pm 1\}$, and $L/\mathbb{Q}_2$ has the following lower numbering filtration: $G = G_0 = G_1$, $G_2 = G_3 = Z(G)$, and $G_4 = \{1\}$. Hence, the jumps are $\varphi_{L/\mathbb{Q}_2}(1) = 1$ and $\varphi_{L/\mathbb{Q}_2}(3) = 3/2$.

  In the next subsection, we will see the theorem of Hasse-Arf, which asserts that when G is abelian, the jumps are integers. This result turns out to be very important, as we will see later.

The upper numbering filtration respects the quotients, as we were looking for (c.f. [KR15, Prop. 3.53]):

Proposition 2.14. *If $H \lhd G$ is a normal subgroup, we have*

$$G^v/(H \cap G^v) = (G/H)^v.$$

2.3 Theorem of Hasse-Arf

Recall that the *jumps* or *breaks* of the upper numbering filtration of G are the $v \in \mathbb{R}_{\geq -1}$ such that $G^v \supsetneq G^{v+\varepsilon}$ for all $\varepsilon > 0$. Then the following is true:

Theorem 2.15 (Hasse-Arf)**.** *If G is abelian, the jumps are integers.*

There are at least two ways of proving this result.

The first one uses local class field theory, and we need the extra assumption that the residue field of K is finite. Under this assumption, we know that there exists the local reciprocity map $\rho : K^\times \to \mathrm{Gal}(K^{ab}/K)$ such that, for any finite abelian extension L/K, the composition

$$K^\times \to \mathrm{Gal}(K^{ab}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$$

maps $U_K^{\lceil v \rceil}$ onto $\mathrm{Gal}(L/K)^v$ for any $v \in \mathbb{R}_{\geq 0}$ (c.f. [CF67, Ch. VI.4, Thm. 1]), so the jumps will occur at integer v's.

The second proof (c.f. [Ser79]) doesn't need the development of the local class field theory and doesn't need the extra assumption on the residue field, but on the other hand is a little bit longer and intricate. The idea of it consists of reducing to the case of a cyclic extension (using the transitivity of the norm and of the functions φ) and the last jump (i.e. the last v where G^v is non-trivial). In this particular case, we consider $V := \{\text{kernel of the norm } N_{L/K} : L^\times \to K^\times\}$. By Hilbert's theorem 90 this is just $V = \{gy/y |\ y \in L^\times\}$, where g generates G, and we consider the subgroup $W := \{gy/y |\ y \in U_L\}$. Then, fixing a local parameter $x \in L$, the assigment

$$\begin{array}{rcl} \theta : G & \to & V/W \\ \sigma & \mapsto & \sigma(x)/x \end{array}$$

is an isomorphism of groups that respects the filtrations, i.e. $\theta|_{G_i} : G_i \hookrightarrow V_i/W_i$ for all $i \geq 0$. Using this, if we assume that the jump v is not an integer, then there is an integer w such that $w < v < w+1$, and then our problem is reduced to showing that if $G^{w+1} = 0$ and $V_{\psi_{L/K}(v)+1}/W_{\psi_{L/K}(v)+1} = 0$, then $V_{\psi_{L/K}(v)}/W_{\psi_{L/K}(v)} = 0$, because once he have this we conclude (using that the isomorphism θ respects filtrations) that $G_{\psi_{L/K}(v)} = 0$, which contradicts the definition of v. One proves that fact studying the norm map, as it is perfectly done in [Ser79, Ch. V] [KR15, Sections 3.8 and 3.9].

3 Representation theory

A general reference for this part is [Ser77]. Let E be a field and G a finite group. Recall that a *class function* $\varphi : G \to E$ is a function which is constant on conjugacy classes.

Example 3.1. Let $G := \mathrm{Gal}(L/K)$, with L/K a finite Galois extension of complete discrete valued fields as in the previous section, and $x \in L$ a local parameter. Then we can defined the ramification subgroups using the following class function:

$$\begin{array}{rcl} i_G : G & \to & \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ \sigma & \mapsto & v_L(\sigma(x) - x) \end{array}$$

Note that according to our definition this wouldn't be a class map because we don't go to a field, but this is not so important. They key point is that it is constant on conjugacy classes.

One nice property of i_G is that it allows us to write the ramification groups as follows: $G_i = i_G^{-1}([i+1, \infty])$. Using this we will construct the Artin character (this one will be a class function to a field E), which will be very important in this thesis. This function will also appear in the proof of the Grothendieck-Ogg-Shafarevich formula.

One important example of a class function of special interest for us is the *character* of a representation. Recall that if $\rho : G \to \mathrm{GL}(V)$ is a representation of G on a finite dimensional E-vector space V, the character of ρ, $\chi_\rho : G \to E$, is defined as

$$\chi_\rho(g) := \chi_V(g) := \mathrm{Tr}(\rho(g)).$$

Note that if V is 1-dimensional, the character is the representation itself. The following facts can be easily shown:

Remark 3.2. Let V_1, V_2 be two representations of G.

1. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$.
2. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$.
3. $\chi_{V_1^\vee}(g) = \chi_{V_1}(g^{-1})$. In particular, if $E = \mathbb{C}$, we have $\chi_{V_1^\vee}(g) = \overline{\chi_{V_1}(g)}$.

Example 3.3. Given G, recall that the *regular representation* is the representation associated to $E[G]$ seen as an $E[G]$-module. In other words, if a basis of V is given by $\{e_g\}_{g \in G}$, then G acts by moving these elements[3]. Hence, if r_G is the character of the regular representation, $r_G(1) = |G|$ and $r_G(g) = 0$ for $g \neq 1$ (because in the diagonals of the matrices with respect to the above basis, we will have only zeroes).

Recall also the *augmentation representation*, which is just the kernel of the quotient from the regular representation to the trivial representation (of rank 1). Let u_G denote its character. If $|G|$ is invertible in E, then the regular representation is the direct sum of the trivial representation and the augmentation representation, so we have $r_G = u_G + 1_G$. In particular, $u_G(1) = |G| - 1$ and $u_G(g) = -1$ for the $g \neq 1$.

[3] For all $h \in G$, $he_g = e_{hg}$.

The set of class functions from G to a field E, $\mathbf{C}_{E,G}$, has a natural structure of an E-vector space, and if the characteristic of E doesn't divide $|G|$, we can define the following bilinear product:

$$\langle \varphi, \psi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \varphi(g)\psi(g^{-1}).$$

This is a symmetric bilinear form on $\mathbf{C}_{G,E}$.

If $E = \mathbb{C}$, we have the following nice theorem:

Theorem 3.4. *Let G be a finite group. Then, its irreducible characters $\chi_1, \ldots, \chi_r$ form a basis of $\mathbf{C}_{\mathbb{C},G}$. Moreover, this basis is orthonormal with respect to $\langle -, - \rangle_G$.*

Therefore, we have the following:

Corollary 3.5. *Over $\mathbb{C}$, a class function φ is the character of a representation of G if and only if it is of the form*

$$\varphi = a_1\chi_1 + \ldots + a_r\chi_r,$$

with the $a_i \in \mathbb{Z}_{\geq 0}$.

We will need two more things in the rest of the section: the Frobenius reciprocity and Brauer's theorem. First we need a couple of definitions:

Definition 3.6. Let $\alpha : H \to G$ be a group homomorphism (one can think on the inclusion of a subgroup), and let E be a field of characteristic 0.

1. If $\varphi \in \mathbf{C}_{E,G}$ is a class function on G, then $\alpha^*\varphi := \varphi \circ \alpha$ is a class function on H. We call it the *restriction of* φ.

2. If $\varphi \in \mathbf{C}_{E,H}$ is a class function on H, then we define the *induced class function* on G, $\alpha_*\varphi$, as follows:

   - If α is injective,

   $$\alpha_*\varphi(g) := \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \varphi(xgx^{-1}).$$

   - If α is surjective,

   $$\alpha_*\varphi(g) := \frac{1}{|\ker(\alpha)|} \sum_{h \mapsto g} \varphi(h).$$

   - In general, factor α onto its image and an inclusion.

Remark 3.7. Both the restricted and the induces class functions respect characters, i.e. if χ is a character on H (resp. G), then $\alpha_*\chi$ (resp. $\alpha^*\chi$) is again a character on G (resp. on H).

The Frobenius reciprocity gives us an adjuction relation between induced and restricted representations:

Proposition 3.8 (Frobenius recoprocity). *Let $\alpha : H \to G$ be a group homomorphism, $\psi \in \mathbf{C}_{\mathbb{C},H}$, and $\varphi \in \mathbf{C}_{\mathbb{C},G}$. Then*

$$\langle \psi, \alpha^* \varphi \rangle_H = \langle \alpha_* \psi, \varphi \rangle_G .$$

Finally, Brauer's theorem allows us to write any character of G as a combination of 1-dimensional characters of subgroups $H_i < G$:

Theorem 3.9 (Brauer). *Let G be a finite group and χ a character corresponding to a finite dimensional complex representation of G. Then, χ is a $\mathbb{Z}$-linear combination of characters of the form $\alpha_{i*}\chi_i$, where $\alpha_i : H_i \hookrightarrow G$ is an inclusion of a subgroup and χ_i is a 1-dimensional representation of H_i.*

3.1 Artin and Swan representations

In this section we define the Artin and the Swan representations. Given a complete discrete valued field K with perfect residue field, we will use the Swan representation to define a measure of "wildness" of the pro-p-subgroup $P_K < G_K$.

Let L/K be a finite Galois extension of complete discretely valued fields with separable residue extension of degree f and Galois group G. Then, we define the *Artin character*, which is the class function given by

$$a_G(g) := \begin{cases} -f i_G(g) & \text{if } g \neq 1, \\ f \sum_{g' \neq 1} i_G(g') & \text{if } g = 1, \end{cases}$$

where i_G is the class function defined in Example 3.1. Of course, we call it a character because it is a character:

Theorem 3.10 (Artin). *The Artin character a_G is indeed a character (of a representation of G over $\mathbb{C}$).*

Proof. (Sketch) First of all, we reduce the theorem to the totally ramified case, because if $\iota : G_0 \hookrightarrow G$ is the inclusion of the inertia subgroup, we have (c.f. [KR15, Lem. 4.47]) $\iota_* a_{G_0} = a_G$, and we know that the induced class function of a character is again a character. By corollary 3.5, it is enough to show that for any character χ, then $\langle a_G, \chi \rangle =: f(\chi) \in \mathbb{Z}_{\geq 0}$. For this, one proves first that $f(\chi)$ is a *non-negative* rational number[4] (c.f. [KR15, Lem. 4.48]). Once we have this, then by Brauer's theorem we have that $\chi = \sum a_i \chi_i'$, where $a_i \in \mathbb{Z}$ and $\chi_i' := \alpha_{i*}\chi_i$ is the induced representation of a 1-dimensional character χ_i on the subgroup H_i. Hence, we just need to show that $f(\chi_i') \in \mathbb{Z}$.

But by Frobenius reciprocity, we have that $f(\chi_i') = \langle a_G, \alpha_{i*}\chi_i' \rangle_G = \langle \alpha_i^* a_G, \chi_i' \rangle_{H_i}$. Now we can write $\alpha_i^* a_G$ in terms of the regular representation of H_i and the Artin character of H_i (c.f. [KR15, Lem. 4.51]):

$$\alpha_i^* a_G = \lambda r_{H_i} + a_{H_i},$$

[4] Indeed, we have that $f(\chi) = \sum_{i \geq 0} \frac{1}{|G : G_i|} (\dim V - \dim V^{G_i})$.

where $\lambda = v_{K'}(\mathfrak{D}_{K'/K})$ is a non-negative integer and $K' := L^{H_i}$. Therefore we have

$$f(\chi_i') = \langle \alpha_i^* a_G, \chi_i \rangle = \lambda \langle r_{H_i}, \chi_i \rangle + \langle a_{H_i}, \chi_i \rangle = \lambda \chi_i(1) + \langle a_{H_i}, \chi_i \rangle ,$$

where χ_i is 1-dimensional, so $\chi_i(1) = 1$. Hence, we reduced our problem to the 1-dimensional case, because if we prove that $\langle a_{H_i}, \chi_i \rangle$ is an integer, we are done.

We have the group homomorphism $\chi_i : H_i \to \mathbb{C}^\times$. Let $H' := \ker(\chi_i)$. Then given the chain of subgroups $\{1\} < H' < H_i < G$, we denote the corresponding chain of field extensions as $L/L_i/K'/K$. If c_i' denotes the largest integer such that $\mathrm{Gal}(L_i/K')_{c_i'} = (H_i/H')_{c_i'} \neq \{1\}$, then we have (c.f. [KR15, Lem. 4.50])

$$\langle a_{H_i}, \chi_i \rangle = \varphi_{K'/K}(c_i') + 1,$$

and since H_i/H' is a subgroup of $\mathbb{C}^\times$, K'/K is an abelian extension. Finally Hasse-Arf theorem tells us that $\varphi_{K'/K}(c_i')$ is an integer, so we are done.

□

Definition 3.11. The *Swan character* is the following function:

$$\mathrm{sw}_G := a_G - (r_G - r_{G/G_0}).$$

Note that if L/K is totally ramified, then $\mathrm{sw}_G = a_G - (r_G - 1_G) = a_G - u_G$.

Remark 3.12. Again, we have that the Swan character is a character. To see this, we can assume as before that L/K is totally ramified, and then, for any character χ of a representation V,

$$\begin{aligned}
\langle \mathrm{sw}_G, \chi \rangle &= \langle a_G, \chi \rangle - \langle r_G - 1_G, \chi \rangle \\
&= \langle a_G, \chi \rangle - \dim V/V^G \\
&= \sum_{i \geq 0} \frac{1}{|G : G_i|}(\dim V - \dim V^{G_i}) - (\dim V - \dim V^G) \\
&= \sum_{i \geq 1} \frac{1}{|G : G_i|}(\dim V - \dim V^{G_i}),
\end{aligned}$$

which is greater or equal to 0, and since $\langle a_G, \chi \rangle$ is an integer, we conclude that sw_G is the character of a representation.

So far, we have seen that a_G and sw_G are the characters of *complex* representations. Since $\mathbb{C}$ has characteristic 0 and G is finite, we know that these representations are realizable over $\overline{\mathbb{Q}}$ (c.f. [KR15, Prop. 4.19]). We could ask if we can go further, i.e. if these representations are realizable over a smaller field, for example over $\mathbb{Q}$, but this turns out to be false (c.f. [Ser60]). Nonetheless, we can still do something: since both representations are realizable over $\overline{\mathbb{Q}}$, they are also realizable over $\overline{\mathbb{Q}_\ell}$, and we have the following theorem:

Theorem 3.13. *Let ℓ be a prime number different from the residue characteristic of K. Then,*

1. *The Artin and the Swan representations are realizable over $\mathbb{Q}_\ell$.*

2. *There exists a finitely generated projective left-$\mathbb{Z}_\ell[G]$-module Sw_G, unique up to isomorphism, such that $\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is isomorphic to the Swan representation (i.e. it has character sw_G).*

Remark 3.14. There is no direct construction of the $\mathbb{Z}_\ell[G]$-module known, since all the proofs that we have give just the existence of the module.

The proof of the existence of Sw_G lies on the study of representations over a field whose characteristic may divide the order of $|G|$ and over discrete valuation rings of mixed characteristic. This makes things complicated, and since we don't obtain a direct construction we omit the proof. More explanations can be found on [KR15, Section 4.3], and a complete proof in [Ser77].

In the next section we will use Sw_G to study the group G_K.

3.2 Measuring the wild ramification of an ℓ-adic Galois representation

Let K be a complete discretely valued field with perfect residue field of characteristic $p > 0$. Fix a separable closure K^{sep} of K, and let $G_K := \mathrm{Gal}(K^{sep}/K)$ be the absolute Galois group of K. Let $\ell \neq p$ be a second prime, and $E/\mathbb{Q}_\ell$ a finite field extension. In this section we consider continuous representations of the shape $\rho : G_K \to \mathrm{GL}(V)$, where V is a finite dimensional vector space over E. Such a representation is called an *ℓ-adic Galois representation.*

We are interested in the restriction of this action to the wild ramification subgroup of G_K (we define it right now). Recall that for any finite Galois extension L/K, $\mathrm{Gal}(L/K)_1$ is the unique p-Sylow subgroup of $\mathrm{Gal}(L/K)$ (c.f. Cor. 2.8), and that the quotient $\mathrm{Gal}(L/K)/\mathrm{Gal}(L/K)_1$ is cyclic of order prime to p. Given two finite extensions $L/L'/K$, we know that $\mathrm{Gal}(L/K)_1$ maps to $\mathrm{Gal}(L'/K)_1$, since the image of a p-group is a p-group. Hence we can take the inverse limit of $\mathrm{Gal}(L/K)_1$ over all the finite Galois L/K, and we obtain a closed normal pro-p-group $P_K \trianglelefteq G_K$. We call this group P_K the *wild ramification subgroup of G_K*. Note that G_K/P_K is pro-cyclic with every finite quotient of order prime to p. In particular, for any $H \trianglelefteq G$ open normal subgroup, the image of P_K in G_K/H is precisely $(G_K/H)_1$.

We use this group P_K for the following definition:

Definition 3.15. Let R be a commutative ring and $\rho : G_K \to \mathrm{GL}_n(R)$ be a group homomorphism.

1. ρ is called *unramified* if $G_K^0 \subset \ker(\rho)$.

2. ρ is called *tame* or *tamely ramified* if $P_K \subset \ker(\rho)$. Otherwise ρ is called *wild* or *wildly ramified.*

We are mainly interested in the cases $R = E, \mathcal{O}_E$ and $\mathbb{F}_\lambda := \mathcal{O}_E/\mathfrak{m}_E$, and ρ continuous.

Given an ℓ-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$, we can factor it via a model $\mathrm{GL}(\mathcal{V})$ over $\mathcal{O}_E$. Indeed, we can do it for any profinite group G:

Lemma 3.16. *If G is a compact topological group (in particular a profinite group as G_K) and $\rho : G \to \mathrm{GL}(V)$ a continuous representation, then there exists a free $\mathcal{O}_E$-submodule $\mathcal{V} \subset V$ such that $V = \mathcal{V}_E$ and ρ factors*

$$\rho : G \to \mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(V),$$

where $\mathrm{GL}(\mathcal{V}) := \mathrm{Aut}_{\mathcal{O}_E}(\mathcal{V})$.

Proof. Choose a basis $e_1, \ldots, e_r$ of V. This gives us an inclusion $\mathrm{GL}_r(\mathcal{O}_E) \subset \mathrm{GL}_r(E)$, and makes $\mathrm{GL}_r(\mathcal{O}_E)$ into a topological group. Then

$$G = \bigcup_{M \in \mathrm{GL}_r(E)} \rho^{-1}(M\mathrm{GL}_r(\mathcal{O}_E))$$

is an open covering, so we can find a minimal $n \geq 1$ and matrices $M_1, \ldots, M_n \in \mathrm{GL}_r(E)$ such that $\mathrm{im}(\rho) \subset \bigcup M_i \mathrm{GL}_r(\mathcal{O}_E)$.

Taking $\mathcal{V}' := \sum_j^r e_j \mathcal{O}_E \subset V$, for any $g \in G$ we have that $\rho(g)\mathcal{V}' = M_i \mathcal{V}'$ for some i, so $\mathcal{V} := \sum_i M_i \mathcal{V}'$ is a G-stable free $\mathcal{O}_E$-submodule of V satisfying our conditions. □

Now, if λ is a local parameter of $\mathcal{O}_E$, we specialize the representation as follows:

Definition 3.17. Given a continuous representation $\rho : G_K \to \mathrm{GL}(\mathcal{V})$ over $\mathcal{O}_E$, then the composition $\overline{\rho} : G_K \to \mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(\overline{\mathcal{V}})$, with $\overline{\mathcal{V}} = \mathcal{V}/\lambda\mathcal{V}$ is called the *reduction modulo* λ of ρ. Note that $\overline{\rho}$ is a representation over $\mathbb{F}_\lambda$.

We now want to see that P_K acts on $\mathcal{V}$ and $\overline{\mathcal{V}}$ through the same group, and this group will be finite. In particular, given an ℓ-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$, $\rho|_{P_K}$ factors through a finite quotient of P_K (i.e. the action is, in some sense, almost trivial). It is important to emphasize here that we are assuming all the time that $\ell \neq p$.

Indeed, we prove something more general:

Lemma 3.18. *Let $E/\mathbb{Q}_\ell$ be a finite field extension, $\ell \neq p$. If P is a pro-p-group and $\rho : P \to \mathrm{GL}_r(\mathcal{O}_E)$ a continuous representation, then the image of ρ is finite and $\rho(P) \cap \ker(\mathrm{GL}_r(\mathcal{O}_E) \twoheadrightarrow \mathrm{GL}_r(\mathbb{F}_\lambda)) = \{1\}$.*

Proof. If $M_r(\mathcal{O}_E)$ denotes the ring of $r \times r$ matrices with coefficients in $\mathcal{O}_E$, then $H := \ker(\mathrm{GL}_r(\mathcal{O}_E) \twoheadrightarrow \mathrm{GL}_r(\mathbb{F}_\lambda)) = id + \lambda M_r(\mathcal{O}_E)$. Since ρ is continuous, $\rho^{-1}(H)$ is an open subgroup of P. Since $\mathbb{F}_\lambda$ is a finite extension of $\mathbb{F}_\ell$, H is a pro-ℓ-group, and since $\ell \neq p$ there are no non-trivial maps between pro-p- and pro-ℓ-groups. Hence $\rho^{-1}(H) \subset \ker(\rho)$, so $\rho(P) \cap H = \{1\}$. Finally H has finite index in $\mathrm{GL}_r(\mathcal{O}_E)$, so $\rho^{-1}(H)$ has finite index in P which implies that $\ker(\rho)$ also has finite index, and we are done. □

Corollary 3.19. *In the above situation, ρ is tame if and only if $\overline{\rho}$ is tame.*

Now we want to define an invariant of a given ℓ-adic Galois representation that measures its wild ramification. There are two ways of defining this invariant: one using the Swan representation defined over $\mathbb{Z}_\ell$ (last chapter of [Ser77]), and another one using the break decomposition of the representation, which allows us to define the Swan conductor (beginning of [Kat88]). At the end of the section we show that both definitions coincide.

3.2.1 First approach: the invariant $b(V)$

Let $\rho : G_K \to \mathrm{GL}(\mathcal{V})$ be, as before, a continuous representation, where $\mathcal{V}$ is a free $\mathcal{O}_E$-module.

Definition 3.20. Let $G := G_K/\ker(\overline{\rho})$, which is a finite group (since $\mathbb{F}_\lambda$ is a finite field) corresponding to a finite Galois extension L/K. Hence, we are in the situation of section 3.1, so we can consider the Swan representation over $\mathbb{Z}_\ell$ of G, Sw_G. Then, we define

$$b(\rho) := b(\mathcal{V}) := \dim_{\mathbb{F}_\lambda} \mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \overline{\rho}).$$

Remark 3.21.

1. Note that the number $b(\mathcal{V})$ only depends on the class of $\overline{\mathcal{V}}$ (i.e. the reduction $\overline{\rho}$ of the representation ρ) in the Grothendieck ring $R_{\mathbb{F}_\lambda}(G)$, which is the abelian group generated by the isomorphism classes $[\overline{\mathcal{W}}]$ of finite dimensional representations of G with the extra relation $[\overline{\mathcal{W}}] = [\overline{\mathcal{W}_1}] + [\overline{\mathcal{W}_2}]$ if there exists an exact sequence of representations $0 \to \overline{\mathcal{W}_1} \to \overline{\mathcal{W}} \to \overline{\mathcal{W}_2} \to 0$. $R_{\mathbb{F}_\lambda}(G)$ becomes a ring with the tensor product.

2. If we start with an ℓ-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$, where V is a vector space over E, then by lemma 3.16 it factors through $\rho : G_K \to \mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(V)$. Then, we can define $b(V) := b(\mathcal{V})$, and this number doesn't depend on the $\mathcal{O}_E$-lattice $\mathcal{V}$ that we choose. This is because the class of $\overline{\mathcal{V}}$ in the ring $R_{\mathbb{F}_\lambda}(G)$ only depends on $\rho : G_K \to \mathrm{GL}(V)$: in order to check this, one has to develop a little bit of representation theory in mixed characteristic, and we refer to [Ser77] or [KR15, Prop. 4.61] for details. Here we just need to know that $b(V)$ is well defined.

3. Here we use $G := G_k/\ker(\overline{\rho})$ to define $b(\rho)$, but we can use G_K/N, where N is an open normal subgroup of finite index contained in $\ker(\overline{\rho})$ without changing the result, as we will see at the end of the section.

4. If ρ factors through a finite quotient G of G_K, then

$$\begin{aligned} b(\rho) &= \dim_{\mathbb{F}_\lambda[G]} \mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \overline{\rho}) \\ &= \mathrm{rank}_{\mathcal{O}_E} \mathrm{Hom}_{\mathcal{O}_E[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathcal{O}_E, \rho) \\ &= \dim_E \mathrm{Hom}_{E[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Q}_\ell} E, \rho \otimes E), \end{aligned}$$

c.f. [KR15, Rem. 4.72].

With the next proposition, we see that $b(V)$ tells us if V has wild ramification or not.

Proposition 3.22. *In the above situation, with $G := G_K/\ker(\overline{\rho})$, we have*

$$b(\mathcal{V}) = \sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}/\overline{\mathcal{V}}^{G_i}).$$

The proof uses some facts about representation theory, and we refer again to [Ser77] or [KR15, Prop. 4.73] for the details. We summarize here what we know about $b(V)$ measuring the the wild ramification:

Proposition 3.23. *Let $\rho : G_K \to \mathrm{GL}_r(\mathcal{O}_E)$ be a continuous representation. Then the following are equivalent:*

1. *The composition $\rho \otimes E : G_K \to \mathrm{GL}_r(\mathcal{O}_E) \hookrightarrow \mathrm{GL}_r(E)$ is tame.*
2. *ρ is tame.*
3. *$\overline{\rho} : G_K \to \mathrm{GL}_r(\mathbb{F}_\lambda)$ is tame.*
4. *$b(\mathcal{V}) = 0$.*

3.2.2 Second approach: the Swan conductor $\mathrm{Swan}(V)$

In this section, given an ℓ-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$, we want to construct a decomposition of V, called the *break decomposition* $V = \bigoplus_{x \in \mathbb{R}_{\geq 0}} V(x)$, that will encode part of the ramification information of G_K. Once we have it, we will define the *Swan conductor of* V, which is the real number

$$\mathrm{Swan}(V) := \sum_{x \in \mathbb{R}_{\geq 0}} x \dim V(x),$$

and we will see that it coincides with $b(V)$.

In order to construct the break decomposition, we need some more facts about the ramification filtration on G_K:

Lemma 3.24. *Let $\lambda \in \mathbb{R}_{\geq 0}$, and denote $G_K^{\lambda+} := \overline{\bigcup_{\lambda' > \lambda} G_K^{\lambda'}}$ the closure of the union of the subgroups $G_K^{\lambda'}$ in the topological group G_K. Then, the upper numbering filtration G_K^λ satisfies the following:*

1. *$\bigcap_{\lambda > 0} G_K^\lambda = \{1\}$.*
2. *For $\lambda > 0$,*
$$G_K^\lambda = \bigcap_{0 < \lambda' < \lambda} G_K^{\lambda'}.$$
3. *$P_K = G_K^{0+}$.*

Proof. For the first part, note that if $g \in G_K^\lambda$ for all $\lambda > 0$, then for every finite Galois extension L/K, g maps to $1_{\mathrm{Gal}(L/K)}$, which is the only element of $\mathrm{Gal}(L/K)^\lambda$ for λ big enough. Hence g must be the identity element.

For the second part, first note that $G_K^\lambda \subset G_K^{\lambda'}$ for every $\lambda' < \lambda$, so $G_K^\lambda \subset \bigcap_{\lambda' < \lambda} G_K^{\lambda'}$. If we assume that there exists $g \in (\bigcap G_K^{\lambda'}) \setminus G_K^\lambda$, then there must be a finite Galois extension L/K such that $\overline{g}$, the image of g in $\mathrm{Gal}(L/K)$, lies in $(\bigcap \mathrm{Gal}(L/K)^{\lambda'}) \setminus \mathrm{Gal}(L/K)^\lambda$. In other words, $\overline{g} \in \mathrm{Gal}(L/K)^{\lambda'} \setminus \mathrm{Gal}(L/K)^\lambda$ for all $\lambda' < \lambda$, but this is a contradiction with the fact that the function $t \mapsto \#\mathrm{Gal}(L/K)^t$ is left continuous (this is because $\mathrm{Gal}(L/K)^t = \mathrm{Gal}(L/K)_{\psi(t)} = \mathrm{Gal}(L/K)_{\lceil \psi(t) \rceil}$, and the function $\lceil - \rceil$ is left continuous). Therefore we must have an equality.

Finally, for the third part, let L/K be a finite Galois extension. By definition, the image of P_K in $\mathrm{Gal}(L/K)$ is $\mathrm{Gal}(L/K)_1$. We know that for any $\varepsilon > 0$, $\psi_{L/K}(\varepsilon) > 0$, so $\mathrm{Gal}(L/K)^{\varepsilon} = \mathrm{Gal}(L/K)_{\lceil\psi(\varepsilon)\rceil} \subset \mathrm{Gal}(L/K)_1$. Hence, $G_K^{0+} \subset P_K$. Moreover, for L/K, there exists $\varepsilon_L > 0$ such that $\mathrm{Gal}(L/K)^{\varepsilon_L} = \mathrm{Gal}(L/K)_1$. Hence the image of G_K^{0+} in P_K is precisely $\mathrm{Gal}(L/K)_1$ for any finite Galois L/K, so we have an equality $G_K^{0+} = P_K$ because G_K^{0+} is closed and any closed subgroup H of a profinite group $P = \varprojlim_N P/N$ is isomorphic to $\varprojlim_N H/(H \cap N)$.

□

We will use this lemma to construct the break decomposition. This decomposition exists in a more general setting, i.e. we have it not just for any $E[G_K]$-module V, but for more general modules:

Definition 3.25. We say that a P_K-module is a $\mathbb{Z}[1/p]$-module M, together with a morphism $\rho : P_K \to \mathrm{Aut}_{\mathbb{Z}} M$ which factors through a finite discrete quotient. A morphism of P_K-modules is a morphism of $\mathbb{Z}[1/p]$-modules that respects the additional structure.

Note that by lemma 3.18, any ℓ-adic representation V of G_K is a P_K-module. Now we see that the break decomposition exists for P_K-modules, and we see also how does it look like:

Proposition 3.26. *For notational convenience, let's denote $G := G_K$ and $P := P_K$. Let M be a P-module.*

1. *There exists a unique decomposition $M = \bigoplus_{x \in \mathbb{R}_{\geq 0}} M(x)$ of P-modules such that*
   - *(a) $M(0) = M^P$.*
   - *(b) $M(x)^{G^x} = 0$ for $x > 0$.*
   - *(c) $M(x)^{G^y} = M(x)$ for $x > y$.*
2. *$M(x) = 0$ for all but finitely many $x \in \mathbb{R}_{\geq 0}$.*
3. *For every $x \in \mathbb{R}_{\geq 0}$, the assignment $M \mapsto M(x)$ is an exact endofunctor on the category of P-modules.*
4. *$\mathrm{Hom}_P(M(x), M(y)) = 0$ for $x \neq y$.*

The proof can be checked in [Kat88, Prop. I.1.1], and some more details of the proof in [KR15, Prop. 4.77]. Here we don't prove the whole statement, but we follow the second reference in order to define the P-modules $M(x)$.

First, let $\rho : P \to \mathrm{Aut}_{\mathbb{Z}}(M)$ denote the representation that gives M the structure of a P-module, and let $H := \mathrm{im}(\rho)$, which is a finite discrete p-group by definition. Now, for $x \in \mathbb{R}_{\geq 0}$, let $H(x+) := \rho(G^{x+})$ and for $x > 0$, $H(x) := \rho(G^x)$. For example, $H = \rho(P) = \rho(G^{0+}) = H(0+)$. Note that $H(x)$ and $H(x+)$ are all normal subgroups of H.

Now, for the different x, we define the following elements of $\mathbb{Z}[1/p][H]$:

$$\pi(x) := \frac{1}{|H(x)|} \sum_{h \in H(x)} h \quad \text{and} \quad \pi(x+) := \frac{1}{|H(x+)|} \sum_{h \in H(x+)} h.$$

Since $G^{x+} \subset G^x$, we see that $H(x+) \subset H(x)$, and we see that they are equal if x is not a break in the upper numbering filtration of G. Since $\pi(x+)\pi(x) = \pi(x)$, we see that almost all the elements $\pi(x+)(1-\pi(x))$ are zero, and the non-zero elements correspond to the jumps of the filtration. Then one shows that these elements are orthogonal, idempotents and their sum is zero (c.f. [KR15, Lemma 4.78]). Now we can define the decomposition of M: $M(0) := \{m \in M |\, \pi(0+)m = m\}$, and $M(x) := \{m \in M |\, \pi(x+)(1-\pi(x))m = m\}$ for $x > 0$.

Corollary 3.27. *Let A be a $\mathbb{Z}$-algebra, and M an A-module on which $P = P_K$ acts A-linearly through a finite quotient (i.e. is a P-module and the representation factors also through $\mathrm{Aut}_A(M) \subset \mathrm{Aut}_{\mathbb{Z}}(M)$).*

1. *In the break decomposition $M = \bigoplus_{x \geq 0} M(x)$, every $M(x)$ is an A-submodule of M.*

2. *If B is an A-algebra, then the break decomposition of $B \otimes_A M$ is*

$$\bigoplus_{x \geq 0} B \otimes_A M(x).$$

3. *If A is local and noetherian, and M a free A-module of finite rank, then every $M(x)$ is free of finite rank.*

Proof.

1. If $a \in A$, multiplication by a is P-equivariant on M, so by the third part of the previous proposition, a maps $M(x)$ to $M(x)$.

2. This is because of the construction of $\pi(x)$ and $\pi(x+)$.

3. If M is a free A-module of finite rank, then $M(x)$ is a direct summand so it is projective. If A is noetherian, then $M(x)$ is also finitely generated. Finally, if A is local, then projective modules of finite rank are free modules of finite rank.

□

Now we define the Swan conductor of a P_K-module:

Definition 3.28. Let A be a local noetherian $\mathbb{Z}[1/p]$-algebra and M a free A-module of finite rank on which P_K acts A-linearly through a finite quotient. The *Swan conductor* of M is the real number

$$\mathrm{Swan}(M) := \sum_{x \geq 0} x \ \mathrm{rank}_A(M(x)).$$

Remark 3.29.

1. One sees immediately that $\mathrm{Swan}(M) = 0$ if and only if the action of P_K on M is trivial.

2. If B is an A-algebra, then $\mathrm{Swan}(M) = \mathrm{Swan}(M \otimes_A B)$.

3. $\mathrm{Swan}(M)$ is additive on short exact sequences.

Now we want to extend the notion of the Swan conductor for an ℓ-adic Galois representation V of G_K. We are almost there, we just need to see that the decomposition is not just a decomposition of P_K-modules, but of G_K-modules:

Lemma 3.30. *Let M be a $\mathbb{Z}[1/p]$-module on which G_K acts such that the restriction to P_K acts through a finite quotient on M. Then the break decomposition $M = \bigoplus_{x\geq 0} M(x)$ is a decomposition of G_K-modules.*

The proof of this lemma is easy, and we refer the reader to [KR15, Lem. 4.83]. Hence, by lemma 3.18, our ℓ-adic Galois representation V satisfies the conditions (recall that we are assuming all the time that $\ell \neq p$) and we obtain the break decomposition

$$V = \bigoplus_{x\in\mathbb{R}_{\geq 0}} V(x),$$

which is a decomposition of continuous E-representations of G_K (for the continuity, note that ρ factors through $\bigoplus_x \mathrm{GL}(V(x))$, and here we have the subspace topology).

Remark 3.31.

1. We see immediately that $\mathrm{Swan}(V)$ measures the wild ramification of V, because by the previous remark $\mathrm{Swan}(V) = 0$ if and only if V is tame.
2. If we have our ℓ-adic representation $G_K \to \mathrm{GL}(V)$, we know that it factors through a free $\mathcal{O}_E$-module $\mathcal{V}$ of the same rank as V. Then $V = \mathcal{V} \otimes E$ implies that $V(x) = \mathcal{V}(x) \otimes E$, and therefore
$$\mathrm{Swan}(V) = \mathrm{Swan}(\mathcal{V}).$$
3. Similarly, one gets that $\mathrm{Swan}(\mathcal{V}) = \mathrm{Swan}(\overline{V})$.

In the next section we prove that $\mathrm{Swan}(V)$ is an integer in the case of ℓ-adic representations. If V is just a representation of P_K, then $\mathrm{Swan}(V)$ may not be an integer, but it will still be a rational number (c.f. [Kat83, Cor. of p. 214]).

3.2.3 Both approaches give us the same number

Here we want to prove that $b(V) = \mathrm{Swan}(V)$:

Theorem 3.32. *If $\rho : G_K \to \mathrm{GL}(V)$ is an ℓ-adic Galois representation, then* $\mathrm{Swan}(V) = b(V)$.

Example 3.33. Let's compute the Swan conductors of the 1-dimensional non-trivial ℓ-adic Galois representations of the Artin-Schreier extension $L := K[t]/(t^p - t - x^{-m})$, where $K = k((x))$ and k is an algebraically closed field of characteristic p. Recall that if we assume that $(m,p) = 1$, then the lower numbering filtration of $G := \mathrm{Gal}(L/K)$ is $G = \mathbb{F}_p = G_0 = \ldots = G_m \supsetneq G_{m+1} = 0$ (c.f. example 2.3).

Now let $\ell \neq p$, and let $\chi : \mathbb{F}_p \to \mathbb{Q}_\ell$ be a 1-dimensional ℓ-adic Galois representation. By proposition 3.22, we know that $b(V) = \sum_i |G_i|/|G_0| \dim_{\mathbb{F}_\ell}(\overline{\chi}/\overline{\chi}^{G_i})$. The reduction $\overline{\chi} : \mathbb{F}_p \to \mathbb{F}_\ell^\times$ is non-trivial (since the image of a generator of $\mathbb{F}_p$

must be a root of unity different from 1, it belongs to $U_{\mathbb{Q}_\ell} \setminus U^1_{\mathbb{Q}_\ell}$), so we have that $\dim_{\mathbb{F}_\ell}(\overline{\chi}/\overline{\chi}^{G_i}) = 1$ for $i = 0, \ldots, m$.

Hence, applying the theorem we get that

$$\mathrm{Swan}(\chi) = \mathrm{Swan}(\overline{\chi}) = b(\chi) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\ell}(\overline{\chi}/\overline{\chi}^{G_i}) = \sum_{i=1}^{m} 1 = m.$$

Let's prove the theorem:

Proof. We saw that $\mathrm{Swan}(V) = \mathrm{Swan}(\overline{\mathcal{V}})$, where $\overline{\mathcal{V}}$ is the reduction of a lift of V, as usual. Let E denote the field of the coefficients of V and $\mathbb{F}_\lambda$ its residue field, which is finite. Since V is finite dimensional $\overline{\mathcal{V}}$ also is, so $\mathrm{GL}(\overline{\mathcal{V}})$ is a finite group, so G_K acts on $\overline{\mathcal{V}}$ through a finite quotient G. Let L/K be the subextension corresponding to this group G.

Assume that $V \neq 0$ (else, both numbers are 0 and we are done). Then $\overline{\mathcal{V}} \neq 0$. Let $x \in \mathbb{R}_{\geq 0}$ such that $\overline{\mathcal{V}}(x) \neq 0$. Then x corresponds to a jump of the upper numbering filtration of G, as we saw in the discussion after proposition 3.26. In other words, $G^x \neq G^{x+\varepsilon}$ for all $\varepsilon > 0$, so $\psi_{L/K}(x) \in \mathbb{Z}_{\geq 0}$. Then by remark 2.10,

$$x = \varphi_{L/K}(\psi_{L/K}(x)) = \sum_{i=1}^{\psi(x)} \frac{|G_i|}{|G_0|}.$$

We also have that for any integer $i \leq \psi_{L/K}(x)$, $G^x = G_{\psi_{L/K}(x)} \subset G_i$, so $\overline{\mathcal{V}}(x)^{G_i} = \overline{\mathcal{V}}(x)^{G^x}$. But the last term is zero because of proposition 3.26, so both are zero.

Now, if $i > \psi_{L/K}(x)$, then $\varphi_{L/K}(i) > x$ and by the same proposition, $\overline{\mathcal{V}}(x)^{G_i} = \overline{\mathcal{V}}(x)^{G^{\varphi(i)}} = \overline{\mathcal{V}}(x)$. Hence, we have that

$$\dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}(x)/\overline{\mathcal{V}}(x)^{G_i}) = \begin{cases} \dim_{\mathbb{F}_\lambda} \overline{\mathcal{V}}(x) & \text{if } i \leq \psi_{L/K}(x), \\ 0 & \text{if } i > \psi_{L/K}(x). \end{cases}$$

With this, noting that $\left(\overline{\mathcal{V}}(x)\right)(y) = 0$ for $y \neq x$, we can compute

$$\mathrm{Swan}(\overline{\mathcal{V}}(x)) = x \dim_{\mathbb{F}_\lambda} \overline{\mathcal{V}}(x) = \sum_{i \geq 0} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}(x)/\overline{\mathcal{V}}(x)^{G_i}).$$

Since both sides are additive with respect to direct sums (c.f. remark 3.29) we conclude, using as in the previous example the proposition 3.22, that

$$\mathrm{Swan}(\overline{\mathcal{V}}) = \sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}/\overline{\mathcal{V}}^{G_i}) = b(V).$$

□

Note in particular that since $b(V)$ is independent of the choice of the finite quotient G, $\mathrm{Swan}(V)$ is also independent, as we mentioned in remark 3.21.

4 Geometry

General references for this part are [SGA1], [SGA4] and [SGA4.5].

4.1 Étale fundamental group

Here we recall the construction of the étale fundamental group and some of its properties. We assume that the reader is familiar with this topic, but we still give some definitions and state some propositions for the sake of unity.

4.1.1 Étale morphisms

In order to define the étale fundamental group, we proceed as in algebraic topology, where we can define the the fundamental group via the covering maps. Finite étale morphisms will play the role of finite covering maps, and this will be our starting point. First we give the local definition:

Definition 4.1. Let A be a ring (commutative and with 1, as always). An A-algebra B is said to be *étale* if B is finitely presented as an A-algebra, and one of the following equivalent conditions holds:

1. B is flat as an A-module and it is unramified, i.e. for each prime ideal $\mathfrak{q} \subset B$ over $\mathfrak{p} \subset A$, the natural map $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$ is a separable extension of fields.

2. If $B = A[x_1, \dots, x_n]/I$ is a presentation of B, then for all prime ideals $\mathfrak{p} \subset A[x_1, \dots, x_n]$ with $\mathfrak{p} \supset I$, there exist polynomials $f_1, \dots, f_n \in I$ such that
$$I_\mathfrak{p} = (f_1, \dots, f_n) \subset A[x_1, \dots, x_n]_\mathfrak{p}$$
and
$$\det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j} \notin \mathfrak{p}.$$

And now we give the global definition:

Definition 4.2. A morphism of schemes $f : X \to Y$ is *étale* if for any point $x \in X$ with image $y = f(x)$, there exist open neighborhoods $x \in V = \mathrm{Spec}(B)$ and $y \in U = \mathrm{Spec}(A)$ such that the induced restriction map $A \to B$ makes B into an étale A-algebra.

Example 4.3.
1. Isomorphisms are étale. More generally, open immersions are étale because they are local isomorphisms.

2. 0-dimensional case: let k be a field, and B a k-algebra. Then B is étale over k if and only if B is isomorphic to a finite product of finite separable field extensions L_i/k, i.e. $B = \prod L_i$.

3. 1-dimensional arithmetic example: let A be a Dedekind domain with fraction field K, and L/K a finite field extension with ring of integers B. Then the A-algebra B is étale if and only if B is flat over A and unramified. Every such extension is flat (because A is a Dedekind domain, so being flat is the same as being torsion-free), so B is étale over A if and only if L/K is unramified.

4. 1-dimensional geometric example: consider the $\mathbb{C}$-morphism $\operatorname{Spec}(\mathbb{C}[y, y^{-1}]) \to \operatorname{Spec}(\mathbb{C}[x, x^{-1}])$ associated to $x \mapsto y^2$. Then, with the notation of the definition, $A = \mathbb{C}[x, x^{-1}]$ and the A-module structure on B induced by the morphism is isomorphic to $\mathbb{C}[x, x^{-1}][z]/(f)$, where $f(z) = z^2 - x$. Here, of course, z plays the role of y, and we don't write y just to make explicit that the isomorphism is not canonical (z may be y or $-y$). Now, $\dfrac{\partial f}{\partial z} = 2z$, which doesn't lie in any $\mathfrak{p} \subset \mathbb{C}[x, x^{-1}][z]$ containing $(z^2 - x)$ because those prime ideals $\mathfrak{p}$ correspond to the prime ideals in the quotient, but in the quotient, $2z$ is a unit and hence can't be in any prime ideal. Therefore the morphism is étale.

5. One can similarly prove that the morphism

$$\operatorname{Spec}(\mathbb{Q}(i)[s, s^{-1}]) \to \operatorname{Spec}(\mathbb{Q}(i)[t, t^{-1}])$$

corresponding to $t \mapsto s^4$ is étale. We will develop this example later.

Étale morphisms satisfy the following nice properties:

Proposition 4.4. *Étale morphisms are stable under base change, composition and fibered products.*

For a proof, see for example [Liu02, Prop. 4.3.22]. We said that étale morphisms will play the role of finite covering maps. Recall that for covering spaces we have this proposition:

Proposition 4.5. *Consider the diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[dl, "p"] \\ S, & \end{tikzcd}$$

where S is a locally connected topological space, $p : Y \to S$ is a cover, X a connected topological space, and $f, g : X \to Y$ two continuous maps such that $p \circ f = p \circ g$. If there is a point $x \in X$ such that $f(x) = g(x)$, then $f = g$.

For a proof and the definitions, see for example [Sza09, Prop. 2.2.2]. Now, we have the analogous property for étale morphisms:

Proposition 4.6. *Consider the diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[dl, "p"] \\ S, & \end{tikzcd}$$

where S is connected, $p : Y \to S$ is separated and étale, and $f, g : X \to Y$ are two morphisms of schemes such that $p \circ f = p \circ g$. If there is a point $x \in X$ such that $f(x) = g(x)$ (not just topologically, but in the sense that the embeddings in the residue fields are the same), then $f = g$.

For a proof, check [Sza09, Cor. 5.3.3] or, if we further assume that everything is locally noetherian, [SGA1, Exp. I, Cor. 5.4].

4.1.2 Definition of $\pi_1^{ét}(X, \overline{x})$ and first properties

Now we want to define the étale fundamental group of a connected (not necessarily noetherian) scheme X. Let $\overline{x} : \mathrm{Spec}(\Omega) \to X$ be a *geometric point* (i.e. a morphism with Ω an algebraically closed field), and consider the *fibre functor* $\mathrm{Fib}_{\overline{x}}$ from the category of étale coverings of X (i.e. finite étale morphisms over X) to the category of finite sets given by

$$\mathrm{Fib}_{\overline{x}} : (Y \to X) \mapsto \mathrm{Hom}_X(\mathrm{Spec}(\Omega), Y).$$

Note that we can identify $\mathrm{Hom}_X(\mathrm{Spec}(\Omega), Y)$ with the finite set underlying the geometric fibre $Y_{\overline{x}} := Y \times_X \mathrm{Spec}(\Omega)$.

Definition 4.7. Let X be a connected scheme, and $\overline{x}$ a geometric point. The *(étale) fundamental group of X with respect to $\overline{x}$*, denoted $\pi_1^{ét}(X, \overline{x})$, is by definition $\pi_1^{ét}(X, \overline{x}) := \mathrm{Aut}(\mathrm{Fib}_{\overline{x}})$.

Recall that an automorphism of a functor $F : \mathcal{C} \to \mathcal{C}'$ is a compatible collection of isomorphisms $\{\sigma_C : F(C) \to F(C),\ \sigma_C \text{ is an isomorphism in } \mathcal{C}' \mid \forall C \in \mathcal{C}\}$.

In our case, $\mathcal{C}'$ is the category of sets, so for every finite étale map $Y \to X$, the isomorphisms σ_Y are just permutations of $\mathrm{Fib}_{\overline{x}}(Y)$. Therefore our compatible collection forms a projective system (all the axioms are automatically fulfilled) of groups which are finite, since from proposition 4.6 it is not difficult to see that each set $\mathrm{Fib}_{\overline{x}}(Y)$ is finite, which implies that its group of permutations stays finite. The projective limit of this projective system is precisely $\pi_1^{ét}(X, \overline{x})$, so it is a profinite group.

Remark 4.8.

1. In [SGA1], Grothendieck assumes that X is noetherian, but it is not necessary: c.f. [Sza09, Def. 5.4.1] for example.

2. We don't need that X is connected, but we impose the condition because in this way, the isomorphism class of $\pi_1^{ét}(X, \overline{x})$ doesn't depend on the geometric point (c.f. [Sza09, Prop. 5.5.1 and Cor. 5.5.2]).

Example 4.9.

1. If $X := \mathrm{Spec}(\mathbb{Q})$, then a geometric point $\overline{x} : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(\mathbb{Q})$ corresponds with and embedding $\mathbb{Q} \hookrightarrow \Omega$, and the étale coverings correspond with the finite field extensions $K/\mathbb{Q}$ (because $\mathbb{Q}$ is perfect and therefore all its finite extensions are separable).

   If we consider $\mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt[3]{2})$, then $\mathrm{Fib}_{\overline{x}}(\mathrm{Spec}(\mathbb{Q}(\sqrt[3]{2}))) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}), \Omega)$ can be identified with the three roots of $T^3 - 2$ in Ω. Note that since $\mathbb{Q}(\sqrt[3]{2})$ is a 3-dimensional vector space over $\mathbb{Q}$, then $Y_{\overline{x}} = \mathrm{Spec}(\mathbb{Q}(\sqrt[3]{2})) \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\Omega) =$

$\bigsqcup_{i=1}^3 \mathrm{Spec}(\Omega)$. Hence, as sets, they are isomorphic (i.e. they have the same cardinality).

Finally, to give an automorphism of $\mathrm{Fib}_{\overline{x}}$ is the same as giving a compatible collection of isomorphisms of finite field extensions $K/\mathbb{Q}$ for all finite extensions K. Since they are compatible, they glue to an isomorphism of the algebraic closure $\overline{\mathbb{Q}}$ induced by $\mathbb{Q} \hookrightarrow \Omega$. In other words,

$$\pi_1^{ét}(\mathrm{Spec}(\mathbb{Q}), \overline{x}) = \mathrm{Aut}(\mathrm{Fib}_{\overline{x}}) \cong \varprojlim_{K/\mathbb{Q} \text{ finite}} \mathrm{Aut}_{\mathbb{Q}}(K) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

2. The same argument works in general: the fundamental group of any field k is isomorphic to its absolute Galois group, i.e. $\pi_1^{ét}(\mathrm{Spec}(k), \overline{x}) \cong \mathrm{Gal}(k^{sep}/k)$, where k^{sep} is the separable of k with respect to $\overline{x}$.

In the case of the fields, we know that any element in $\mathrm{Gal}(k^{sep}/k)$ can be determined by the Galois extensions of k contained in k^{sep}, i.e. we have

$$\pi_1^{ét}(\mathrm{Spec}(k), \overline{x}) \cong \mathrm{Gal}(k^{sep}/k) = \varprojlim_{\substack{k^{sep}/k'/k \\ \text{finite separable}}} \mathrm{Aut}_k(k') = \varprojlim_{\substack{k^{sep}/k'/k \\ \text{finite Galois}}} \mathrm{Aut}_k(k'),$$

so instead of looking at all the finite separable extensions of k, it is enough to study the finite Galois extensions. This makes the computations easier. In general, we can do the same, but first we need to introduce the notion of *Galois covering*.

Definition 4.10. Let $Y \to X$ be a finite étale covering. We say that this is a *Galois covering* if (i) Y is connected and (ii) $\mathrm{Aut}_X(Y)$ acts on $\mathrm{Fib}_{\overline{x}}(Y)$ transitively. If $Y \to X$ is Galois, then we call $\mathrm{Aut}_X(Y)$ the *Galois group of* $Y \to X$ and we denote it G_Y.

Remark 4.11. 1. Note that $\mathrm{Aut}_X(Y)$ acts transitively on $\mathrm{Fib}_{\overline{x}}(Y)$ if and only if $|\mathrm{Aut}_X(Y)| = \deg(Y/X)$.

2. If $Y \to X$ corresponds to a finite separable field extension L/K, then the covering is Galois if and only if L/K is Galois.

3. If $Y \to X$ is a Galois cover, with both Y, X irreducible varieties over a field k, then the finite field extension $K(Y)/K(X)$ is Galois. The converse, in general, is not true, because there can be some ramification in the extension $K(Y)/K(X)$ (and therefore the morphism $Y \to X$ would not be étale). However, we can fix this by imposing the ramification condition: $Y \to X$ is a Galois cover if and only if $K(Y)/K(X)$ is a finite Galois extension (this implies that $Y \to X$ is finite and flat) which is *unramified* for all the valuations in $\mathcal{O}_X(X)$.

In the following example, which continues the last part of example 4.3, we compute explicitly that the extension is Galois.

Example 4.12. Consider the finite surjective étale morphism

$$\begin{tikzcd} Y := \operatorname{Spec}(\mathbb{Q}(i)[s,s^{-1}]) \arrow[d, "f"] \\ X := \operatorname{Spec}(\mathbb{Q}(i)[t,t^{-1}]) \end{tikzcd}$$

corresponding to $t \mapsto s^4$. We know that Y is connected, so in order to show that this is a Galois covering we just need to show that given a geometric point $\overline{x}$, the group $\operatorname{Aut}_X(Y)$ acts on $\operatorname{Fib}_{\overline{x}}(Y)$ transitively. Consider the geometric point

$$\overline{x} : \operatorname{Spec}(\overline{\mathbb{Q}}) \to \operatorname{Spec}(\mathbb{Q}(i)[t,t^{-1}])$$

given by the evaluation $t \mapsto -1$. On the ring level, we have so far this setting:

$$\begin{tikzcd} & \mathbb{Q}(i)[s,s^{-1}] & & s^4 \\ \overline{\mathbb{Q}} & \mathbb{Q}(i)[t,t^{-1}] \arrow[l] \arrow[u] & -1 & t \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

and now we want to lift the geometric point $\overline{x}$, which means giving a diagonal arrow such that the diagram commutes. This means that $s^4 \mapsto -1$, so the image of s must be an 8th-primitive root of 1, i.e. $s \mapsto \{\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7\}$. This gives us $\operatorname{Fib}_{\overline{x}}(Y) = \operatorname{Hom}_X(\operatorname{Spec}(\overline{\mathbb{Q}}), Y)$, which consists on the geometric points $\overline{x}_{Y,j}$, for $j = 0, \ldots, 3$.

The group $G_Y := \operatorname{Aut}_X(Y)$ is given by the $\sigma : Y \to Y$ such that $f \circ \sigma = f$. On the ring level, this means that $\sigma^* : \mathbb{Q}(i)[s,s^{-1}] \to \mathbb{Q}(i)[s,s^{-1}]$ has to map $s^4 \mapsto s^4$, so the only possibilities are $s \mapsto \{s, -s, is, -is\}$. In particular, $G_Y \cong \mathbb{Z}/4\mathbb{Z}$, where the isomorphism is given by fixing a generator of G_Y, for example $\sigma_0^* : s \mapsto is$.

Finally, we see that the action of $\operatorname{Aut}_X(Y)$ on $\operatorname{Fib}_{\overline{x}}(Y)$ is transitive. This is easy, because $(\overline{x}_{Y,0}^* \circ (\sigma_0^*)^j)(s) = \zeta_8^{2j+1} = \overline{x}_{Y,j}^*(s)$ for $j = 0, \ldots, 3$, and similarly with the other geometric points.

Alternatively, we can use the remark to conclude directly that the covering is Galois, since $|\operatorname{Aut}_X(Y)| = 4 = \deg(Y/X)$.

In the case of fields, given a finite separable extension K'/K, we know that there exists a Galois closure $L/K'/K$. The following proposition is a generalization in our case of study:

Proposition 4.13. *Let $f : Y \to X$ be a connected étale cover. Then, there is a morphism $\pi : P \to Y$ such that $f \circ \pi : P \to X$ is a Galois cover. Moreover, if $P' \to X$ is another Galois cover factoring through $Y \to X$, then it factors through $P \to X$:*

$$\begin{tikzcd} P' \arrow[r, dotted, "\exists"] \arrow[dr] \arrow[ddr] & P \arrow[d, "\pi"] \\ & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

A proof can be found in [Sza09, Prop. 5.3.9].

In the definition of a Galois covering we impose the connectivity condition because it allows us to use proposition 4.6. The idea is the following: first we choose a geometric point $\overline{x} : \mathrm{Spec}(\Omega) \to X$, and then for any P_α Galois covering of X, we choose a lift of the geometric point, say $p_\alpha \in \mathrm{Hom}_X(\mathrm{Spec}(\Omega), P_\alpha)$. Then, the previous proposition tells us that this system is directed, and by proposition 4.6 there is *at most one* map between $\phi_{\beta,\alpha} : P_\beta \to P_\alpha$. Then, we have a projective system of Galois coverings $(P_\alpha, \phi_{\beta,\alpha})$. This projective system contains the information of the étale fundamental group:

Proposition 4.14. *With the notation above,*

$$\pi_1^{ét}(X, \overline{x}) \cong \varprojlim \mathrm{Aut}_X(P_\alpha)^{op}.$$

Check [Sza09, Cor. 5.4.8] for a proof.

Example 4.15. Let $Y \to X$ be a Galois covering of irreducible varieties, as in remark 4.11, part 3. Then, this proposition tells us that there is an isomorphism

$$\mathrm{Gal}(K(X)^{unr}/K(X)) \cong \pi_1^{ét}(X, \overline{\eta}),$$

where $K(X)^{unr}$ is the biggest extension of $K(X)$ in $K(X)^{sep}$ which is unramified for all the valuations in $\mathcal{O}_X(X)$.

The étale fundamental group $\pi_1^{ét}(X, \overline{x})$ encodes all the information of the finite étale covers of X, as we can see in the following deep theorem by Grothendieck:

Theorem 4.16. *Let X be a connected scheme, and let $\overline{x} : \mathrm{Spec}(\Omega) \to X$ be a geometric point. The functor $\mathrm{Fib}_{\overline{x}}$ induces an equivalence between the category of finite étale covers of X and the category of finite sets with a continuous left-action of $\pi_1^{ét}(X, \overline{x})$.*

Note that we don't assume that X is locally noetherian. A proof can be found in [Sza09, Thm. 5.4.2].

We conclude the section with a few more properties about the étale fundamental group. The proofs are in [KR15, Prop. 6.8], [Sza09, Prop. 5.6.1] and [SGA1, Exp. XII, Cor. 5.2]:

Proposition 4.17 (Functoriality)**.** *If $f : X' \to X$ is a morphism of noetherian connected schemes, and $\overline{x}' : \mathrm{Spec}(\Omega) \to X'$ a geometric point, then f induces a continuous homomorphism of groups*

$$\pi_1^{ét}(X', \overline{x}') \to \pi_1^{ét}(X, f\overline{x}').$$

Theorem 4.18 (Homotopy exact sequence)**.** *Let X be a quasi-compact and geometrically integral scheme over a field k. Let $\overline{x} : \mathrm{Spec}(\Omega) \to X$ be a geometric point, which induces the extensions $\overline{k}/k^{sep}/k$. Then the exact sequence*

$$1 \to \pi_1^{ét}(X_{k^{sep}}, \overline{x}) \to \pi_1^{ét}(X, \overline{x}) \to \mathrm{Gal}(k^{sep}/k) \to 1$$

induced by the maps $X_{k^{sep}} \to X$ and $X \to \mathrm{Spec}(k)$ is exact (note that here we abuse a little bit the notation, since we denote by $\overline{x}$ both the geometric point of X and the geometric point induced in $X_{k^{sep}}$).

Theorem 4.19 (Comparison). *Let X be a connected scheme of finite type over $\mathbb{C}$. The functor $(Y \to X) \mapsto (Y^{an} \to X^{an})$ induces an equivalence of the category of finite étale covers of X with the category of finite topological covers of X^{an}. Consequently, for every $\mathbb{C}$-point $\overline{x} : \mathrm{Spec}(\mathbb{C}) \to X$, this functor induces an isomorphism*

$$\pi_1^{top}(X^{an}, \overline{x})^\wedge \cong \pi_1^{ét}(X, \overline{x}),$$

where the left-hand side is the profinite completion of the topological fundamental group of X^{an} with base point the image of $\overline{x}$.

4.2 Cohomology of ℓ-adic sheaves

The aim of this master thesis is to understand the Grothendieck-Ogg-Shafarevich formula, which measures the wild ramification of an ℓ-adic sheaf on a curve and puts it in term of the Euler characteristic of the curve and of the ℓ-adic sheaf. This sentence is still too vague because we haven't defined yet what an ℓ-adic sheaf, or its wild ramification or even its Euler characteristic is. Hence, we need to describe what are ℓ-adic sheaves. In this section we assume that our schemes are separated and noetherian.

4.2.1 Étale cohomology

Let $(ét/X)$ be the category of étale X-schemes with morphisms over X. Étale coverings define a Grothendieck topology on this category (c.f. the beautiful description in [Mum63, §1] or the more detailed approach in [Mil80, II. §1]). Recall that if A is a ring, an *étale presheaf* of A-modules on X is just a contravariant functor $\mathcal{F} : (ét/X) \to (A\text{-mod})$, and an *étale sheaf* is a presheaf that satisfies the sheaf axioms with respect to the mentioned Grothendieck topology. We denote the category of étale sheaves over X by $X_{ét}$.

If $\overline{x} : \mathrm{Spec}(\Omega) \to X$ is a geometric point, then the stalk of $\mathcal{F}$ at $\overline{x}$ is

$$\mathcal{F}_{\overline{x}} := \varinjlim_{\mathrm{Spec}(\Omega) \to U} \mathcal{F}(U),$$

where the limit is taken over the étale neighbourhoods of $\overline{x}$ (indeed we can choose a representative U for each isomorphism class to avoid set theoretical problems), i.e. all the lifts of $\overline{x}$ to a finite étale covering $U \to X$. If we assume that X is connected, then the map $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ defines an equivalence between the category of locally constant sheaves of sets (resp. abelian groups) with finite stalks on X and the category of finite $\pi_1^{ét}(X, \overline{x})$-sets (resp. $\pi_1^{ét}(X, \overline{x})$-modules). For a proof of this, see for example [Mil13, Prop. I.6.16].

Example 4.20.
1. If M is an A-module, the constant sheaf M_X is given by $U \mapsto M^{\pi_0(U)}$.
2. The sheaf $\mathbb{G}_{a,X}$ is given by $U \mapsto \Gamma(U, \mathbb{G}_a(U)) = \Gamma(U, \mathcal{O}_U)$. It is easy to see that it is indeed a sheaf, c.f. [KR15, Ex. 7.3, (b)].
3. The sheaf $\mathbb{G}_{m,X}$ is given by $U \mapsto \Gamma(U, \mathbb{G}_m(U)) = \Gamma(U, \mathcal{O}_U)^\times$.

4. The sheaf $\mu_{n,X}$ is given by the kernel of the multiplication by n on $\mathbb{G}_{m,X}$. If n is invertible in the ring R and the polynomial $t^n - 1$ decomposes in $R[t]$, then for any X an R-scheme we have a non-canonical isomorphism $\mu_{n,X} \cong (\mathbb{Z}/n\mathbb{Z})_X$ on $X_{ét}$ (c.f. [KR15, Ex. 7.3 (d)]).

Now we want to briefly introduce the notion of direct and inverse image in this setting. Let $\pi : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ a sheaf of A-modules on $X_{ét}$:

$$\begin{tikzcd} \mathcal{F} \arrow[d, squiggly] & \\ X \arrow[r] & Y. \end{tikzcd}$$

Then we obtain the *direct image of* $\mathcal{F}$ *under* π on $Y_{ét}$, denoted $\pi_*\mathcal{F}$, via $\pi_*\mathcal{F}(V) := \mathcal{F}(V \times_Y X)$ for any étale morphism $V \to Y$ in $(ét/Y)$. This definition of π_* defines a left exact functor from the category of A-modules on $X_{ét}$ to the category of A-modules on $Y_{ét}$.

Now, if we have a sheaf $\mathcal{G}$ in $Y_{ét}$,

$$\begin{tikzcd} & \mathcal{G} \arrow[d, squiggly] \\ X \arrow[r] & Y, \end{tikzcd}$$

we want to define an *inverse image of* $\mathcal{G}$ *under* π, $\pi^*\mathcal{G}$. This turns out to be the left adjoint[5] of π_*, i.e. $\mathrm{Hom}_{X_{ét}}(\pi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{Y_{ét}}(\mathcal{G}, \pi_*\mathcal{F})$. More concretely, if $\overline{x}$ is a geometric point of X, then we have that $(\pi^*\mathcal{G})_{\overline{x}} = \mathcal{G}_{(\pi\circ\overline{x})}$. In particular, π^* is an exact functor. Sometimes we use the restriction notation $\mathcal{G}|_X := \pi^*\mathcal{G}$, because if π itself is étale, then $\pi^*\mathcal{G}$ coincides with the restriction of $\mathcal{G}$ to the category $(ét/X)$. This allows us to define the notion of locally constant sheaf:

Definition 4.21. Let M be an A-module. An étale sheaf $\mathcal{F}$ of A-modules on X is said to be *locally constant with stalk* M if there exists a family $\{u_i : U_i \to X\}$ of étale morphisms with $\bigcup u_i(U_i) = X$ such that $\mathcal{F}|_{U_i} = M_{U_i}$ for all i.

We introduce one more functor, the *extension by zero*. Given $j : U \hookrightarrow X$ an open immersion and denoting $i : Z \hookrightarrow X$ the closed immersion of the complement, we define the extension by zero of a sheaf of A-modules $\mathcal{F}$ on U by

$$j_!\mathcal{F} := \ker(j_*\mathcal{F} \to i_*i^*j_*\mathcal{F}),$$

where the morphism comes from the right adjoint of the identity morphism

$$\mathrm{id} : i^*j_*\mathcal{F} \to i^*j_*\mathcal{F}$$

via the adjunction formula. Locally, if $\overline{x}$ is a geometric point of X, this is just

$$(j_!\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } \mathrm{im}(\overline{x}) \in U, \\ 0 & \text{if } \mathrm{im}(\overline{x}) \in Z. \end{cases}$$

[5]The existence of such a left adjoint is a standard result in category theory: one can check [Mil80, Prop. II.2.2], which is proven in [HS71, Thm IX.5.1].

In particular, $j_!$ is an exact functor from the category of A-modules on $U_{ét}$ to the category of A-modules on $X_{ét}$. We also have that it is left adjoint to j^*, i.e. that there exists a functorial isomorphism $\mathrm{Hom}_{X_{ét}}(j_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{U_{ét}}(\mathcal{F}, j^*\mathcal{G})$.

Remark 4.22. The category of A-modules on $X_{ét}$ is abelian. For this, check [Stacks, Tag 03D9], where the ringed site is given in our case by the site $X_{ét}$ and we take as the sheaf of rings the constant sheaf associated to A. Moreover, it has enough injectives: check [Stacks, Tag 01DU]. Hence, we are able to define higher direct images of of sheaves (and in particular, cohomology).

Definition 4.23. Let k be a perfect field, and $f : X \to Y$ a k-morphism between schemes of finite type over k. Let $\mathcal{F}$ be a sheaf of A-modules on $X_{ét}$, and let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then, the *i-th higher direct image of $\mathcal{F}$ under f* is the i-th right derived functor of f_*:

$$R^i f_* \mathcal{F} := H^i(f_* \mathcal{I}^\bullet).$$

Let's choose a compactification of f (using Nagata's theorem, c.f. [Lüt93, Thm. 3.2]), i.e. a proper morphism $\widetilde{f} : \widetilde{X} \to Y$ together with a dominant open immersion $j : X \hookrightarrow \widetilde{X}$ such that $f = \widetilde{f} \circ j$. Then, we define

$$R^i f_! \mathcal{F} := R^i \widetilde{f}_* j_! \mathcal{F}.$$

Remark 4.24. If k is an algebraically closed field and $\pi : X \to \mathrm{Spec}(k)$ is the structure morphism of X, then $R^i\pi_*\mathcal{F}$ coincides with the i-th right derived functor of the global section functor $\Gamma(X_{ét}, -)$, i.e. $R^i\pi_*\mathcal{F} = R^i(\Gamma(X_{ét}, -))\mathcal{F}$. This motivates the following definition:

Definition 4.25. Let k be an algebraically closed field, X a k-scheme and $\mathcal{F}$ a sheaf of A-modules on X. We define the *étale cohomology* as

$$H^i(X_{ét}, \mathcal{F}) := R^i\pi_*\mathcal{F},$$

and the *étale cohomology with compact support*, given by

$$H^i_c(X_{ét}, \mathcal{F}) := H^i(\widetilde{X}_{'et}, j_!\mathcal{F}) = R^i\pi_!\mathcal{F}.$$

Remark 4.26. Étale cohomology works nicely when A is assumed to be a torsion ring. For example, if X is a nonsingular variety over $\mathbb{C}$, we have the isomorphism

$$H^i(X_{ét}, (\mathbb{Z}/n\mathbb{Z})_X) \cong H^i_{sing}(X^{an}, \mathbb{Z}/n\mathbb{Z}),$$

where in the right hand side we consider the singular cohomology of X considered as a complex manifold with coefficients in $\mathbb{Z}/n\mathbb{Z}$ (c.f. [Mil13, Thm. 21.1]). But this is no longer true if we consider, for example, $A = \mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$ (c.f. [FK88, I. §12]. Nonetheless, we have the following isomorphism:

$$\varprojlim_n H^i(X_{ét}, (\mathbb{Z}/\ell^n\mathbb{Z})_X) \cong H^i_{sing}(X^{an}, \mathbb{Z}_\ell).$$

This motivates us to look not just to sheaves, but to projective systems of sheaves. We make this precise when we define the notion of constructible $\mathbb{Z}_\ell$-, $\mathbb{Q}_\ell$- and $\overline{\mathbb{Q}_\ell}$-sheaf, see definition 4.29 and the next ones below, and will lead us to the definition of ℓ-adic cohomology, which is a projective limit of étale cohomology. Recently, Bhatt and Scholze developed in [BS15] the notion of pro-étale topology for schemes, which is a new site that allows to define the pro-étale cohomology. With this cohomology, we have the isomorphism

$$H^i(X_{pro\acute{e}t}, (\overline{\mathbb{Q}_\ell})_X) \cong H^i_{sing}(X^{an}, \overline{\mathbb{Q}_\ell}),$$

but we will not follow this recent development and we will just stick to the classical setting.

4.2.2 Constructible sheaves and ℓ-adic cohomology

In this section we extend the notion of étale sheaf.

Definition 4.27. Let A be a noetherian ring which is torsion (i.e. $mA = 0$ for some $m > 0$) and $\mathcal{F}$ a sheaf of A-modules on $X_{\acute{e}t}$. Then we say that $\mathcal{F}$ is *constructible* if there exist finite type A-modules $M_1, \dots, M_n$ and locally closed subsets $X_1, \dots, X_n \subset X$ such that

- The scheme X is a disjoint union of the X_i, i.e. $X = \bigsqcup_i X_i$.
- The restrictions $\mathcal{F}|_{X_i}$ are locally constant with stalks M_i.

Remark 4.28. The notion of constructible sheaf generalizes the notion of locally constant sheaf. We do this generalization because in this way, the pullback of a constructible sheaf is again constructible (because this is still true for locally constant sheaves: if we consider the pullback of M_Y along the morphism $\pi : X \to Y$, we obtain $\pi^* M_Y = M_X$, c.f. [KR15, Ex. 7.4]), and more distinctively, the pushforward under a proper morphism of a constructible sheaf is again constructible (c.f. [SGA4, Exp. XIV, Thm. 1.1]), something which is not true for locally constant sheaves (c.f. [KR15, Ex. 7.4]).

Definition 4.29. Let X be a scheme, and R a complete local discrete valuation ring with maximal ideal $\mathfrak{m}$ with residue field of characteristic $\ell > 0$.

1. A *constructible R-sheaf* on X is a projective system of R-modules $\mathcal{F} = (\mathcal{F}_n)_{n\geq 1}$ on $X_{\acute{e}t}$ such that:
   - Each $\mathcal{F}_n$ is a constructible $R/\mathfrak{m}^n$-module on $X_{\acute{e}t}$ such that $\mathfrak{m}^n \cdot \mathcal{F}_n = 0$.
   - For all $n \geq 1$, $\mathcal{F}_n = \mathcal{F}_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n$.
2. A *lisse R-sheaf* on X is a constructible R-sheaf $\mathcal{F} = (\mathcal{F}_n)$ such that each $\mathcal{F}_n$ is a locally constant sheaf of $R/\mathfrak{m}^n$-modules.

Example 4.30. 1. Let $R = \mathbb{Z}_\ell$. The projective system given by $(\mathbb{Z}/\ell^n)_X$ for each n gives us a lisse $\mathbb{Z}_\ell$-sheaf. We write $\mathbb{Z}_{\ell,X} := ((\mathbb{Z}/\ell^n)_X)_n$, and we call it the constant lisse $\mathbb{Z}_\ell$-sheaf. This shouldn't be confused with the constant sheaf $(\mathbb{Z}_\ell)_X$. If X is a scheme over a ring containing all ℓ^n-roots of unity, and there are precisely ℓ^n of them, this sheaf coincides with $\mathbb{Z}_\ell(1) := (\mu_{\ell^n,X})_n$.

2. Let R be as in the definition, let n_0 be a natural number and $\mathcal{F}$ a locally constant sheaf of finitely generated $R/\mathfrak{m}^{n_0}$-modules on $X_{ét}$. Then we see it as a lisse R-sheaf by defining $\mathcal{F}_n := \mathcal{F}$ for $n \geq n_0$, and $\mathcal{F}_n := \mathcal{F} \otimes_{R/\mathfrak{m}^{n_0}} R/\mathfrak{m}^n$ for $n < n_0$.

Note that given two constructible R-sheaves $\mathcal{F} = (\mathcal{F}_n)$ and $\mathcal{G} = (\mathcal{G}_n)$, we have a morphism

$$\mathrm{Hom}(\mathcal{F}_{n+1}, \mathcal{G}_{n+1}) \to \mathrm{Hom}(\mathcal{F}_n, \mathcal{G}_n) : \ \varphi \mapsto \varphi \otimes R/\mathfrak{m}^n,$$

so we can take the projective limit in order to define morphisms between constructible R-sheaves:

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) := \lim_{\leftarrow} \mathrm{Hom}(\mathcal{F}_n, \mathcal{G}_n).$$

This makes the category of constructible sheaves abelian (c.f. [SGA5, V, Thm. 5.2.3]).

Remark 4.31. We have defined constructible R-sheaves for R a complete discrete valuation ring, but we will need different coefficient rings when we work with cohomology. Hence, we are going to define the notion of constructible E-sheaf, with $E/\mathbb{Q}_\ell$ a finite field extension, and constructible $\overline{\mathbb{Q}}_\ell$-sheaves in a similar way.

Definition 4.32. Let X be a scheme and ℓ an invertible prime on X, and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

- A *constructible E-sheaf*, where $E/\mathbb{Q}_\ell$ is a finite field extension, is a constructible $\mathcal{O}_E$-sheaf $\mathcal{F}$, but we denote it like $\mathcal{F} \otimes_{\mathcal{O}_E} E$. The reason for this is because we define the morphisms to be

  $$\mathrm{Hom}(\mathcal{F} \otimes_{\mathcal{O}_E} E, \mathcal{G} \otimes_{\mathcal{O}_E} E) := \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_E} E,$$

  which means that we identify to morphisms if both are equal after localizing with respect to $- \otimes_{\mathcal{O}_E} E$.

  We say that a constructible E-sheaf $\mathcal{F} \otimes E$ is *lisse* if there exists an étale covering $\{U_i \to X\}$ and lisse $\mathcal{O}_E$-sheaves $\mathcal{F}_i$ on U_i such that $\mathcal{F}|_{U_i} \otimes E \cong \mathcal{F}_i \otimes E$.

- A *constructible $\overline{\mathbb{Q}}_\ell$-sheaf* is an object of the category which consists of the colimit of the categories of constructible E-sheaves for all the finite extensions $E/\mathbb{Q}_\ell$. More precisely, if we have two finite field extensions $E'/E/\mathbb{Q}_\ell$, there is a natural functor from the category of constructible E-sheaves to the category of constructible E'-sheaves, given by

  $$\mathcal{F} \otimes_{\mathcal{O}_E} E \mapsto (\mathcal{F} \otimes_{\mathcal{O}_E} E) \otimes_E E' = \mathcal{F} \otimes_{\mathcal{O}_{E'}} E',$$

  and therefore we can take the inductive 2-colimit. This 2-colimit is the category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves.

  Finally, lisse $\overline{\mathbb{Q}}_\ell$-sheaves are those which étale locally are of the form $\mathcal{F} \otimes_E \overline{\mathbb{Q}}_\ell$, with $\mathcal{F}$ a lisse E-sheaf.

Convention 4.33. We will say that A is an *ℓ-adic coefficient ring* if it is either $\overline{\mathbb{Q}_\ell}$, a finite extension E of $\mathbb{Q}_\ell$, or equal to $\mathcal{O}_E$ or $\mathcal{O}_E/\mathfrak{m}^n$ for some $n \geq 1$. In this way, in the above setting we can talk about constructible A-sheaves. Note that we can write any constructible A-sheaf as $(\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$ for some finite field extension $E/\mathbb{Q}_\ell$ and $(\mathcal{F}_n)$ a constructible $\mathcal{O}_E$-sheaf.

Before defining the ℓ-adic cohomology, let's extend the definitions of the push-forward of a sheaf and of constant sheaf for an arbitrary ℓ-adic coefficient ring A:

Definition 4.34. Let $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$ be a constructible A-sheaf over X, and let $\pi : X \to Y$ be a proper morphism. Then $\pi_*\mathcal{F} := (\pi_*\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$.

Remark 4.35. The pushforward of a constructible sheaf under a proper morphism is again constructible.

Definition 4.36. Let A be an ℓ-adic coefficient ring, X a scheme as before such that ℓ is invertible on X. Let $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$ be a lisse A-sheaf, with $E/\mathbb{Q}_\ell$ finite.

1. If $A = \mathcal{O}_E$, then we say that $\mathcal{F}$ is *constant* if there exists a finitely generated $\mathcal{O}_E$-module $\mathcal{V}$ such that $(\mathcal{F}_n) \cong ((\mathcal{V} \otimes_{\mathcal{O}_E} \mathcal{O}_E/\mathfrak{m}^n)_X)$ as projective systems. In this case, we write $\mathcal{F} = \mathcal{V}_X$.

2. If $A = E$, we say that $\mathcal{F}$ is constant if there is a finite dimensional vector space V over E and an $\mathcal{O}_E$-lattice $\mathcal{V} \subset V$ such that $\mathcal{F} = \mathcal{V}_X \otimes_{\mathcal{O}_E} E$ as before. Note that as an E-sheaf, it only depends on V up to isomorphism, so we write $\mathcal{F} = V_X$.

3. If $A = \overline{\mathbb{Q}_\ell}$, we say that $\mathcal{F}$ is constant if we can write it as $\mathcal{F} = (V_E)_X \otimes_E \overline{\mathbb{Q}_\ell}$. It is again independent of all the choices up to isomorphism, so we write $\mathcal{F} = V_X$.

Definition 4.37. Let X be a separated k-scheme of finite type, where k is an algebraically closed field of characteristic $p \geq 0$, and let A be an ℓ-adic coefficient ring (with $\ell \neq p$). Let $\mathcal{F}$ be a constructible A-sheaf on $X_{ét}$, so that we can find a complete discrete valuation ring $\mathcal{O}_E$ such that A is an $\mathcal{O}_E$-algebra and $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$, with $(\mathcal{F}_n)$ a constructible $\mathcal{O}_E$-sheaf. Then, we define the *i-th étale cohomology group of* $\mathcal{F}$ as

$$H^i(X_{ét}, \mathcal{F}) := \left(\lim_{\leftarrow} H^i(X_{ét}, \mathcal{F}_n) \right) \otimes_{\mathcal{O}_E} A,$$

where each $H^i(X_{ét}, \mathcal{F}_n)$ is an $\mathcal{O}_E/\mathfrak{m}^n$-module, so the group $H^i(X_{ét}, \mathcal{F}_n)$ is the one from definition 4.25. Similarly, we define the cohomology groups with compact support of a constructible A-sheaf $\mathcal{F}$ as

$$H^i_c(X_{ét}, \mathcal{F}) := \left(\lim_{\leftarrow} H^i_c(X_{ét}, \mathcal{F}_n) \right) \otimes_{\mathcal{O}_E} A.$$

This definition is independent of the choice of the complete discrete valuation ring $\mathcal{O}_E$ and the constructible $\mathcal{O}_E$-sheaf $(\mathcal{F}_n)$ (c.f. [SGA5, pp. VI, 2.2]).

Remark 4.38. With this definition, in the situation of remark 4.26, we get the isomorphism

$$H^i(X_{ét}, \mathbb{Z}_{\ell,X}) \cong H^i_{sing}(X^{an}, \mathbb{Z}_\ell).$$

4.2.3 Some properties of ℓ-adic cohomology

Here we list some more definitions, properties and theorems of ℓ-adic cohomology that we will use later. As usual, schemes are still assumed to be separated and of finite type over a field k.

Remark 4.39. Let k be an algebraically closed field, $f : X \to Y$ a morphism of k-schemes, A an ℓ-adic coefficient ring and $\mathcal{F}$ a constructible A-sheaf on $X_{ét}$.

1. The A-modules $H^i(X_{ét}, \mathcal{F})$ and $H^i_c(X_{ét}, \mathcal{F})$ are finitely generated (c.f. [SGA5, Exposée VI, 2.2] or [Fu11, Thm. 9.5.2]).

2. A short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of constructible A-sheaves induces a long exact sequence of cohomology

   $$\cdots \to H^i(X_{ét}, \mathcal{F}) \to H^i(X_{ét}, \mathcal{F}'') \to H^{i+1}(X_{ét}, \mathcal{F}') \to \cdots ,$$

   and the same holds true if we replace H^i by H^i_c (c.f. [KR15, Par. 8.1.2]).

3. If $f : X \to Y$ is finite, then $H^i(X_{ét}, \mathcal{F}) = H^i(Y_{ét}, f_*\mathcal{F})$ (c.f. [op. cit.]).

4. Assume that X is proper, $j : U \hookrightarrow X$ an open embedding with complement $i : Z \hookrightarrow X$. Then there is a long exact sequence

   $$\cdots \to H^i_c(U_{ét}, j^*\mathcal{F}) \to H^i(X_{ét}, \mathcal{F}) \to H^i(Z_{ét}, i^*\mathcal{F}) \to H^{i+1}_c(U_{ét}, j^*\mathcal{F}) \to \cdots ,$$

   which is induced, if we write $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$, by the exact sequences $0 \to j_!j^*\mathcal{F}_n \to \mathcal{F}_n \to i_*i^*\mathcal{F}_n \to 0$ (c.f. [op. cit.]).

5. If X is affine, then $H^i(X_{ét}, \mathcal{F}) = 0$ for all $i > \dim X$, see [SGA4, XIV, Cor. 3.2].

6. In general, if X has pure dimension d, $H^i(X_{ét}, \mathcal{F}) = 0$ for $i > 2 \dim X$ (see [SGA4, X, Cor. 4.3]).

Definition 4.40. Let X be a proper separated scheme over an algebraically closed field k, and let $j : U \hookrightarrow X$ be an open subset of X with complement $i : Z \hookrightarrow X$. Let $\mathcal{F}$ be a constructible A-sheaf, where A is an ℓ-adic coefficient ring.

- If A is a finite ℓ-adic coefficient ring (think of $\mathcal{O}_E/\mathfrak{m}^n$, with $n \geq 1$), we define the *cohomology with support on Z* as follows: let $j^*\mathcal{F} \to j^*\mathcal{F}$ be the identity morphism, which corresponds via the adjunction formula to $\mathcal{F} \to j_*j^*\mathcal{F}$, and now we pullback this morphism to Z so we obtain $i^*\mathcal{F} \to i^*j_*j^*\mathcal{F}$. Define

  $$i^!\mathcal{F} := \ker(i^*\mathcal{F} \to i^*j_*j^*\mathcal{F}),$$

  which defines a left exact functor $\Gamma_Z(X_{ét}, \mathcal{F}) := \Gamma(Z_{ét}, i^!\mathcal{F})$. Then, the cohomology with support in Z is defined as

  $$H^i_Z(X_{ét}, \mathcal{F}) := R^i\Gamma_Z(X_{ét}, \mathcal{F}).$$

- If A is a general ℓ-adic coefficient ring, the we write $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$ and set

$$H^i_Z(X_{\acute{e}t}, \mathcal{F}) := \left(\lim_{\leftarrow} H^i_Z(X_{\acute{e}t}, \mathcal{F}_n)\right) \otimes_{\mathcal{O}_E} A.$$

Remark 4.41. In the same setting as in the definition, we have an exact sequence

$$\cdots \to H^i(X_{\acute{e}t}, \mathcal{F}) \to H^i(U_{\acute{e}t}, \mathcal{F}|_U) \to H^{i+1}_Z(X_{\acute{e}t}, \mathcal{F}) \to \cdots,$$

see [KR15, Par. 8.1.2].

Recall that we defined in example 4.20 the sheaf $\mu_{\ell^n,X}$ on $X_{\acute{e}t}$, and now we want to use it in order to define the Tate twist of a constructible sheaf.

Definition 4.42. Let X be a scheme, and define, for $i \geq 0$, the sheaf $\mathbb{Z}/\ell^n(i)$ on $X_{\acute{e}t}$ given by

$$U \mapsto \mu_{\ell^n,X}(U) \otimes_{\mathbb{Z}/\ell^n} \cdots \otimes_{\mathbb{Z}/\ell^n} \mu_{\ell^n}(U),$$

and for $i < 0$, we have that the sheaf $\mathbb{Z}/\ell^n(i)$ is given by

$$U \mapsto \mathrm{Hom}_U(\mathbb{Z}/\ell^n(-i)|_U, \mathbb{Z}/\ell^n|_U).$$

Now, if A is an ℓ-adic coefficient ring and $\mathcal{F}$ a constructible sheaf, then he *i-th Tate twist of $\mathcal{F}$, $\mathcal{F}(i)$*, is defined as follows: we write $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$, where $\mathcal{O}_E$ is a complete discrete valuation ring finite over $\mathbb{Z}_\ell$ and $(\mathcal{F}_n)$ is a constructible $\mathcal{O}_E$-sheaf. The $\mathcal{F}_n$ are not just $\mathcal{O}_E/\mathfrak{m}^n$-modules, but also $\mathbb{Z}/\ell^n$-modules on $X_{\acute{e}t}$. Hence, we define $\mathcal{F}_n(i); U \mapsto \mathcal{F}_n(U) \otimes_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^n(i)(U)$, and

$$\mathcal{F}(i) := (\mathcal{F}_n(i)) \otimes_{\mathcal{O}_E} A.$$

This definition doesn't depend on the choices.

Remark 4.43. 1. If $\mathcal{F}$ is lisse, so is $\mathcal{F}(i)$.

2. We have $\mathcal{F}(i)(j) = \mathcal{F}(i+j)$.

Definition 4.44. If $\mathcal{F}$ is a constructible A-sheaf for some ℓ-adic coefficient ring A, then we define the *dual of* $\mathcal{F}$, denoted $\mathcal{F}^\vee$, as follows: first we write $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$ as usual, and then $\mathcal{F}^\vee := (\mathcal{H}om(\mathcal{F}_n, \mathcal{O}_E/\mathfrak{m}^n)) \otimes_{\mathcal{O}_E} A$. Here $\mathcal{H}om(\mathcal{F}_n, \mathcal{O}_E/\mathfrak{m}^n)$ plays the role of $\mathcal{F}_n^\vee$, and as usual $(\mathcal{F}_n^\vee)$ denotes the limit.

Theorem 4.45 (Poincaré duality)**.** *Let X be a smooth k-scheme of pure dimension d, k a perfect field of characteristic $p \geq 0$ and $\overline{k}$ an algebraic closure of k. Let $\ell \neq p$, A an ℓ-adic coefficient ring and $\mathcal{F}$ a lisse A-sheaf. Then for all $i \in \mathbb{Z}$, there is a natural (i.e. functorial in $\mathcal{F}$) and* $\mathrm{Gal}(\overline{k}/k)$*-equivariant isomorphism*

$$H^{2d-i}((X \otimes_k \overline{k})_{\acute{e}t}, \mathcal{F}^\vee(d)) \to H^i_c((X \otimes_k \overline{k})_{\acute{e}t}, \mathcal{F})^\vee,$$

where $H^i_c((X \otimes_k \overline{k})_{\acute{e}t}, \mathcal{F})^\vee := \mathrm{Hom}_A(H^i_c((X \otimes_k \overline{k})_{\acute{e}t}, \mathcal{F}), A)$.

A proof can be found on [Fu11, Cor. 8.5.3], [KR15, Thm 8.4] or [Mil13, Thm. 24.1].

Definition 4.46. Let X be an (as usual, separated of finite type) smooth k-scheme, and let $f : X \to X$ be a k-morphism. Let Γ_f be the graph of f and Δ_X the diagonal. Then, the degree of the intersection product $\Gamma_f \cdot \Delta_X$ is

$$(\Gamma_f \cdot \Delta_X) := \sum_{x \in \Gamma_f \cap \Delta_X} \mathrm{length}(\mathcal{O}_{X,x}/I_f),$$

where $I_f \subset \mathcal{O}_{X,x}$ is the ideal generated by the elements $a - f^*(a)$ in $\mathcal{O}_{X,x}$ (note that if $x \in \Gamma_f \cap \Delta_X$, f induces an endomorphism of $\mathcal{O}_{X,x}$).

Theorem 4.47 (Lefschetz trace formula)**.** *Let X be a smooth projective scheme over an algebraically closed field k of characteristic $p > 0$, and let $\ell \neq p$ be a prime number. Let $f : X \to X$ be a k-morphism, and assume that the intersection scheme $\Gamma_f \cap \Delta_X$ is either empty or zero dimensional (i.e. Γ_f and Δ_X intersect properly). Then,*

$$(\Gamma_f \cdot \Delta_X) = \sum_i (-1)^i \mathrm{Tr}(f^* | H^i(X_{\acute{e}t}, \mathbb{Q}_\ell)).$$

A proof can be found in [SGA4.5, Cycle, Cor. 3.7].

Definition 4.48. We say that a group G *acts on a scheme* X if there exists a group homomorphism $G \to \mathrm{Aut}(X)$. We say that G acts *admissibly* on X if X is a union of open affines $U = \mathrm{Spec}(A)$ such that the action of G restricts to an action on U.

Remark 4.49. If G is finite and X is a quasi-projective scheme over an affine scheme, then G acts admissibly (c.f. [SGA1, V, Prop. 3.1]).

If G acts admissibly on X, then we can form the quotient $\pi : X \to X/G$ by glueing the schemes $U/G := \mathrm{Spec}(A^G)$. Then, $\mathrm{Hom}(X,Y)^G = \mathrm{Hom}(X/G, Y)$ for all schemes Y.

Definition 4.50. Let G be a finite group acting admissibly on X, and let $\mathcal{F}$ be a sheaf of A-modules in $X_{\acute{e}t}$ together with morphisms

$$\mathcal{F}(\sigma) : \mathcal{F} \to \sigma^* \mathcal{F}, \quad \sigma \in G$$

such that $\mathcal{F}(1_G) = \mathrm{id}_{\mathcal{F}}$ and $\mathcal{F}(\sigma\tau) = \tau^*(\mathcal{F}(\sigma)) \circ \mathcal{F}(\tau)$. Then we say that $\mathcal{F}$ is a *sheaf with G-action on X*.

Remark 4.51. If $\pi : X \to X/G$ as above, then by the adjunction formula we have morphisms $\sigma_* \mathcal{F} \to \mathcal{F}$, and we can push forward in order to obtain $\pi_* \sigma_* \mathcal{F} \to \pi_* \mathcal{F}$. But note that $\pi \circ \sigma = \pi$, so if we equip X/G with the trivial action of G, we get that G acts on $\pi_* \mathcal{F}$. This allows us to define the *sheaf of G-invariant elements* $(\pi_* \mathcal{F})^G$ as follows:

Definition 4.52. In the above situation, we define the sheaf $(\pi_* \mathcal{F})^G$ on $(X/G)_{\acute{e}t}$, whose sections for $U \to X/G$ étale are given by

$$(\pi_* \mathcal{F})^G(U) = \{a \in \mathcal{F}(U \times_{X/G} X) |\, \mathcal{F}(\sigma)(a) = a \text{ for all } \sigma \in G\}.$$

We now extend the definition to constructible A-sheaves, where A is an ℓ-adic coefficient ring.

Definition 4.53. Let X be a scheme and ℓ an invertible prime on X, and let $\mathcal{F}$ be a constructible A-sheaf on X, so we can write $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$ for some constructible $\mathcal{O}_E$-sheaf $(\mathcal{F}_n)$. Then, we say that a finite group G (with the trivial action on X) *acts on* $\mathcal{F}$ if we can choose $(\mathcal{F}_n)$ such that the $\mathcal{F}_n$ form a projective system of $\mathcal{O}_E[G]$-modules. Then, we define the *sheaf of G-invariants* $\mathcal{F}^G$ by

$$\mathcal{F}^G := (\mathcal{F}_n^G) \otimes_{\mathcal{O}_E} A.$$

Proposition 4.54. *Let k be an algebraically closed field, X an integral k-scheme, $E/\mathbb{Q}_\ell$ a finite extension and $\mathcal{F}$ a constructible E-sheaf on X. If there is a finite group G acting on $\mathcal{F}$, then*

$$H^i(X_{\acute{e}t}, \mathcal{F}^G) = H^i(X_{\acute{e}t}, \mathcal{F})^G.$$

This is proved in [KR15, Lem. 8.8], for example.

4.3 Wild ramification of an ℓ-adic sheaf

In this section, k will still be a perfect field of characteristic p, $\ell \neq p$ a different prime and A an ℓ-adic coefficient ring. In this section we want to define the wild ramification on a point $x \in X$ of an ℓ-adic sheaf $\mathcal{F}$, which is measured by a number $\mathrm{Swan}_x(\mathcal{F})$ that will appear in the Grothendieck-Ogg-Shafarevich formula. The idea to define this number is to relate lisse A-sheaves with representations of $\pi_1^{\acute{e}t}(X, \overline{\eta})$ and then use the machinery already developed to talk about wild ramification.

Given a connected scheme X with a geometric point $\overline{x}$, an ℓ-adic coefficient ring A and a constructible A-sheaf $\mathcal{F}$ on X, we can find a complete discrete valuation ring $\mathcal{O}_E$ such that $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$. If $\overline{x} : \mathrm{Spec}(\Omega) \to X$ is a geometric point, then we define the stalk of $\mathcal{F}$ at $\overline{x}$ as $\mathcal{F}_{\overline{x}} := \left(\lim\limits_{\leftarrow} \mathcal{F}_{n,\overline{x}}\right) \otimes_{\mathcal{O}_E} A$.

Remark 4.55. If $\mathcal{F}$ is a lisse sheaf, then $\mathcal{F}_{\overline{x}}$ is a finite type A-module. This comes from the fact that $\mathcal{F}_{1,\overline{x}}$ is finitely generated, and after lifting the system of generators and tensoring with A, we are done, c.f. [KR15, p. 7.1.10].

We say the a lisse A-sheaf $\mathcal{F}$ is *free* if its stalks are free A-modules.

Definition 4.56. Let M be a finitely generated A-module (it will play the role of $\mathcal{F}_{\overline{x}}$), which has the induced ℓ-adic topology from A, and let X be a (separated, noetherian) connected scheme with a geometric point $\overline{x} : \mathrm{Spec}(\Omega) \to X$. If $A \neq \overline{\mathbb{Q}_\ell}$, an *$A$-representation of* $\pi_1^{\acute{e}t}(X, \overline{x})$ is a continuous group homomorphism

$$\pi_1^{\acute{e}t}(X, \overline{x}) \to \mathrm{Aut}_A(M).$$

If $A = \overline{\mathbb{Q}_\ell}$, then a $\overline{\mathbb{Q}_\ell}$-representation of $\pi_1^{\acute{e}t}(X, \overline{x})$ is a continuous group homomorphism $\rho_V : \pi_1^{\acute{e}t}(X, \overline{x}) \to \mathrm{Aut}_{\overline{\mathbb{Q}_\ell}}(V)$ coming (via base change with $- \otimes \overline{\mathbb{Q}_\ell}$) from an E-representation of $\pi_1^{\acute{e}t}(X, \overline{x})$, where $E/\mathbb{Q}_\ell$ is a finite extension.

Theorem 4.57. *Let X be a connected scheme and $\overline{x}$ a geometric point. Let A be an ℓ-adic coefficient ring. Then, the functor $\mathcal{F} \rightsquigarrow \mathcal{F}_{\overline{x}}$ induces a natural equivalence of categories between*

$$(\text{lisse } A\text{-sheaves on } X) \to (A\text{-representations of } \pi_1^{\acute{e}t}(X, \overline{x})).$$

Furthermore, let $\pi : X' \to X$ be a finite Galois cover with Galois group G. Then, the functor induces another equivalence between the subcategories

$$\text{(lisse } A\text{-sheaves on } X, \text{ constant on } X') \to \text{(finitely generated } A[G]\text{-modules)}.$$

In this case, the inverse is given by

$$(\pi_* M_{X'})^G \leftarrow\!\!\!\sim\!\!\sim\!\!\sim M.$$

The action of $\pi_1^{ét}(X, \overline{x})$ on $\mathcal{F}_{\overline{x}}$ is as follows: once we fix an inverse system of finite Galois coverings of X over $\overline{x}$, denoted (P_α, p_α), then any element $\sigma \in \pi_1^{ét}(X, \overline{x}) \cong \varprojlim \mathrm{Aut}(P_\alpha)^{op}$ is the same as a compatible system of automorphisms of (P_α, p_α) fixing $\overline{x}$. Let U be an étale neighbourhood of $\overline{x}$. Then, for every α, we denote $\sigma_\alpha^* : \mathcal{F}(P_\alpha) \to \mathcal{F}(P_\alpha)$ the automorphism of the compatible system, and $U_\alpha := U \times_X P_\alpha$ the fiber product. Then we have

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(U_\alpha) \xrightarrow{\sigma_\alpha^*|_{U_\alpha}} \mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}_{\overline{x}},$$

which forms a compatible system, and taking the limit over all the étale neighbourhoods U of $\overline{x}$ gives us the action $\sigma^* : \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$. A proof of this theorem can be found in [KR15, Thm. 7.13].

Remark 4.58. In the situation of the second part of the theorem, recall that if A is infinite and M a finitely generated $A[G]$-module, then there exists a complete discrete valuation ring $\mathcal{O}_E$ finite over $\mathbb{Z}_\ell$ and a finitely generated $\mathcal{O}_E$-module N such that $M = N \otimes_{\mathcal{O}_E} A$. Then, M corresponds to

$$\big((\pi_* N_{X'})^G \otimes_{\mathcal{O}_E} \mathcal{O}_E/\mathfrak{m}^n\big) \otimes_{\mathcal{O}_E} A \leftarrow\!\!\!\sim\!\!\sim\!\!\sim M.$$

Now, we have all the tools to define the wild ramification. In order to make things easier, we are going to restrict now to the situation of the Grothendieck-Ogg-Shafarevich formula and we will fix some notation:

- We will work with C a smooth proper and geometrically connected curve over a perfect field k of characteristic $p > 0$, and U will be an affine open subset of C (any non-trivial open subset will work).
- We fix $\ell \neq p$ a different prime number.
- We denote $K := k(C)$ the function field of C, and we fix an algebraic closure $\overline{K}$ of K. Let $\eta : \mathrm{Spec}(K) \to C$ be the generic point of C and $\overline{\eta} : \mathrm{Spec}(\overline{K}) \to C$ the geometric point corresponding to the chosen algebraic closure. We denote by K^{sep} the separable closure of K in $\overline{K}$.
- Given a closed point $x \in C$, K_x will denote the completion of K with respect to the valuation corresponding to x, and for every x we choose an embedding $\iota_x : K^{sep} \hookrightarrow K_x^{sep}$ over K. Here K_x^{sep} denotes a separable closure of K_x.

- The absolute Galois group of K will be denoted by $G := \mathrm{Gal}(K^{sep}/K)$, and for a closed point $x \in C$ we write D_x for the decomposition subgroup of G with respect to ι_x, i.e. the image of the inclusion $\mathrm{Gal}(K_x^{sep}/K_x) \hookrightarrow \mathrm{Gal}(K^{sep}/K)$. We denote by $P_x \subset I_x$ the wild inertia and inertia subgroup respectively.

Definition 4.59. Let A be an ℓ-adic coefficient ring and $\mathcal{F}$ a free lisse A-module on U. By theorem 4.57, $\mathcal{F}$ corresponds to a representation of $\pi_1^{\acute{e}t}(U, \overline{\eta})$, namely $\mathcal{F}_{\overline{\eta}}$. Since $\pi_1^{\acute{e}t}(U, \overline{\eta}) \cong \mathrm{Gal}(K^{unr}(U)/K(U))$ (see example 4.15), there is a natural surjection $G \to \pi_1^{\acute{e}t}(U, \overline{\eta})$, so we obtain a G-action on $\mathcal{F}_{\overline{\eta}}$

$$G \to \pi_1^{\acute{e}t}(U, \overline{\eta}) \to \mathrm{Aut}(\mathcal{F}_{\overline{\eta}}).$$

Since P_x is a pro-p-group, lemma 3.18 tells us that the restriction of this action factors over a finite quotient of P_x. Therefore we have a break decomposition of the P_x-representation $\mathcal{F}_{\overline{\eta}}$, and we can define its Swan conductor, denoted $\mathrm{Swan}_x(\mathcal{F})$: this is the *wild ramification of $\mathcal{F}$ at the point x*.

Remark 4.60. Here we state and recall some facts, c.f. section 3.2.2:

1. Although P_x depends on the chosen embedding $\iota_x : K^{sep} \hookrightarrow K_x^{sep}$, $\mathrm{Swan}_x(\mathcal{F})$ doesn't, because a different embedding leads to a conjugate in G, and then we get isomorphic break decompositions.

2. If $A \to A'$ is a homomorphism between ℓ-adic coefficient rings, then

$$\mathrm{Swan}_x(\mathcal{F}) = \mathrm{Swan}_x(\mathcal{F} \otimes_A A').$$

3. The wild ramification of $\mathcal{F}$ at a point x is a non-negative integer, i.e.

$$\mathrm{Swan}_x(\mathcal{F}) \in \mathbb{Z}_{\geq 0}.$$

4. We can see $\pi_1^{\acute{e}t}(U, \overline{\eta})$ as the quotient of $\mathrm{Gal}(K^{sep}/K)$ by the smallest closed normal subgroup containing I_x for all $x \in U$, because we don't admit ramification on U. Hence, I_x (and in particular P_x) acts trivially on $\mathcal{F}_{\overline{\eta}}$ for all $x \in U$. In particular, $\mathrm{Swan}_x(\mathcal{F}) = 0$ for all $x \in U$.

5. For any exact sequence of free lisse sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have $\mathrm{Swan}_x(\mathcal{F}) = \mathrm{Swan}_x(\mathcal{F}') + \mathrm{Swan}_x(\mathcal{F}'')$.

6. If $\overline{x} \in C \otimes_k \overline{k}$ is a point above $x \in C$, then $\mathrm{Swan}_{\overline{x}}\left(\mathcal{F}|_{U \otimes_k \overline{k}}\right) = \mathrm{Swan}_x(\mathcal{F})$ (c.f. [KR15, Lem. 9.2]).

We are now in conditions to state the Grothendieck-Ogg-Shafarevich formula and prove it.

4.4 Grothendieck-Ogg-Shafarevich formula

In this section we also fix the situation and the notation of the previous section. The Grothendieck-Ogg-Shafarevich formula measures the Euler characteristic of a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on a curve U via its rank and its wild ramification. First, recall the definition of the compactly supported Euler characteristic:

Definition 4.61. Let $\mathcal{F}$ be a $\overline{\mathbb{Q}_\ell}$-sheaf on U, and let $\overline{U} := U \otimes_k \overline{k}$. The *compactly supported Euler characteristic* of $\mathcal{F}$ is

$$\chi_c(\overline{U}, \mathcal{F}) := \sum_{i=0}^{2} (-1)^i \dim H^i_c(\overline{U}_{\acute{e}t}, \mathcal{F}),$$

where we abuse notation and write $\mathcal{F}$ instead of $\mathcal{F}|_{\overline{U}}$.

Remark 4.62. Note that $H^0_c(\overline{U}_{\acute{e}t}, \mathcal{F}) = 0$, since the extension by zero of $\mathcal{F}$ to the compactification C doesn't have any non-zero global section. The alternate sum goes until 2 because of the vanishing theorems of étale cohomology. Hence,

$$\chi_c(\overline{U}, \mathcal{F}) = -\dim_{\overline{\mathbb{Q}_\ell}} H^1_c(\overline{U}_{\acute{e}t}, \mathcal{F}) + \dim_{\overline{\mathbb{Q}_\ell}} H^2_c(\overline{U}_{\acute{e}t}, \mathcal{F}).$$

Theorem 4.63 (Grothendieck-Ogg-Shafarevich formula)**.** *Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on U. Then,*

$$\chi_c(\overline{U}, \mathcal{F}) = \mathrm{rk}(\mathcal{F}) \cdot \chi_c(\overline{U}, \overline{\mathbb{Q}_\ell}) - \sum_{x \in C \setminus U} [k(x) : k] \mathrm{Swan}_x(\mathcal{F}).$$

We prove the theorem in several steps following the ideas of [KR15].

Step 1. It is enough to consider the case $k = \overline{k}$. Indeed, given a closed point $x \in C$, the number of points $\overline{x} \in C \otimes_k \overline{k}$ above x is precisely $[k(x) : k]$. Hence, since $\mathrm{Swan}_{\overline{x}}(\mathcal{F}) = \mathrm{Swan}_x(\mathcal{F})$,

$$\chi_c(\overline{U}, \mathcal{F}) = \mathrm{rk}(\mathcal{F}) \cdot \chi_c(\overline{U}, \overline{\mathbb{Q}_\ell}) - \sum_{x \in C \setminus U} [k(x) : k] \mathrm{Swan}_x(\mathcal{F})$$

$$\Updownarrow$$

$$\chi_c(\overline{U}, \mathcal{F}) = \mathrm{rk}(\mathcal{F}) \cdot \chi_c(\overline{U}, \overline{\mathbb{Q}_\ell}) - \sum_{\overline{x} \in \overline{C} \setminus \overline{U}} \mathrm{Swan}_x(\mathcal{F}),$$

so from now on we assume that $k = \overline{k}$ and we get rid off the bars, i.e. we want to prove

$$\chi_c(U, \mathcal{F}) = \mathrm{rk}(\mathcal{F}) \cdot \chi_c(U) - \sum_{x \in C \setminus U} \mathrm{Swan}_x(\mathcal{F}),$$

where $\chi_c(U) := \chi_c(U, \overline{\mathbb{Q}_\ell})$.

Step 2. We want to change the ℓ-adic coefficient ring (until now, we are working with a lisse $\overline{\mathbb{Q}_\ell}$-sheaf) in order to use more tools, and for this we need to extend the notion of compactly supported Euler characteristic for an arbitrary ℓ-adic coefficient ring A. Given $\mathcal{F}$, first we see that we can build in a functorial way a two term complex of A-modules that will encode the cohomology of $\mathcal{F}$, and we will use this complex to define the Euler characteristic.

Lemma 4.64. *Recall that $k = \overline{k}$. Let A be an ℓ-adic coefficient ring and $\mathcal{F}$ a lisse A-sheaf on U. Then there is a functor from the category of lisse A-sheaves to the category of complexes of A-modules*

$$C(-) : \mathcal{F} \rightsquigarrow C(\mathcal{F}) := \cdots \to 0 \to C^1(\mathcal{F}) \to C^2(\mathcal{F}) \to 0 \to \cdots$$

with the following properties:

1. *$H^i_c(U, \mathcal{F}) = H^i(C(\mathcal{F}))$ for all i.*
2. *If $\mathcal{F}$ is free, then $C^i(\mathcal{F})$ is a free A-module of finite rank.*
3. *The functor $C(-)$ is exact.*
4. *If $\mathcal{F}$ is free and $A \to A'$ is a morphism of ℓ-adic coefficient rings, then we have an isomorphism*

$$C(\mathcal{F}) \otimes_A A' \cong C(\mathcal{F} \otimes_A A').$$

Proof. One defines the A-modules of the complex by choosing a closed point $P \in U$ and setting $C^1(\mathcal{F}) := H^1((C \setminus P)_{ét}, (j_!\mathcal{F})|_{C \setminus P})$ and $C^2(\mathcal{F}) := H^0(P_{ét}, (j_!\mathcal{F})(-1)|_P)$, where $j : U \hookrightarrow C$ is the inclusion. One uses tools from étale cohomology in order to prove the theorem, check [KR15, Lem. 9.3] for the details. □

Hence, we can extend the definition of the compactly supported Euler characteristic of free lisse sheaves $\mathcal{F}$ to arbitrary ℓ-adic coefficient rings via

$$\chi_c(U, \mathcal{F}) := -\mathrm{rk}_A(C^1(\mathcal{F})) + \mathrm{rk}_A(C^2(\mathcal{F})),$$

and we have the following corollary:

Corollary 4.65. *In the situation of the lemma, if we assume that $\mathcal{F}$ is a free lisse A-sheaf, we have:*

1. *If $H^i_c(U_{ét}, \mathcal{F})$ is a free A-module for $i = 1, 2$, then*

$$\chi_c(U, \mathcal{F}) = \sum_i (-1)^i \mathrm{rk}_A H^i_c(U_{ét}, \mathcal{F}).$$

2. *If $A \to A'$ is a morphism of ℓ-adic coefficient rings, then*

$$\chi_c(U, \mathcal{F}) = \chi_c(U, \mathcal{F} \otimes_A A').$$

3. *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence, then*

$$\chi_c(U, \mathcal{F}) = \chi_c(U, \mathcal{F}') + \chi_c(U, \mathcal{F}'').$$

Hence, going back to the formula that we want to prove, note that the lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ comes from a free $\mathcal{O}_E$-sheaf $\mathcal{F}'$, where $\mathcal{O}_E$ is as usual the ring of integers of a finite field extension $E/\mathbb{Q}_\ell$, via $\mathcal{F} = (\mathcal{F}' \otimes_{\mathcal{O}_E} E) \otimes_E \overline{\mathbb{Q}_\ell}$. Hence, from remark 4.60, it is enough to prove the formula for $A = \mathbb{F}_\lambda$ the residue field of E and $\mathcal{F}$ a lisse $\mathbb{F}_\lambda$-sheaf. This is a reduction because now $\mathcal{F}$ corresponds to a continuous homomorphism $\pi_1^{ét}(U, \overline{\eta}) \to \mathrm{GL}_{\mathrm{rk}(\mathcal{F})}(\mathbb{F}_\lambda)$, and the target is a finite group. Let's see how can we obtain benefit from this in the next step:

Step 3. Now we describe $\mathrm{Swan}_x(\mathcal{F})$ more explicitly. The idea is to consider a (finite) Galois cover of U that trivializes $\mathcal{F}$, and then we put it in terms of the Swan representation (c.f. 3.1) of the Galois group of the cover.

First, note that if $\mathcal{F}$ is a lisse $\mathbb{F}_\lambda$-sheaf, then it corresponds to a continuous homomorphism $\pi_1^{ét}(U, \overline{\eta}) \to \mathrm{Aut}(\mathcal{F}_{\overline{\eta}}) \cong \mathrm{GL}_{\mathrm{rk}(\mathcal{F})}(\mathbb{F}_\lambda)$. The target is a finite group, so we have a finite group with a $\pi_1^{ét}(U, \overline{\eta})$ action. This corresponds, via theorem 4.16, to a finite Galois covering (it is not just an étale covering because the finite set is itself a group, and hence the subgroup corresponding to the cover is the kernel of a group homomorphism: hence, it is a normal subgroup and the covering is Galois) trivializing $\mathcal{F}$: indeed, $\pi_1^{ét}(U', \overline{\eta}')$ is precisely the kernel of $\pi_1^{ét}(U, \overline{\eta}) \to \mathrm{Aut}(\mathcal{F}_{\overline{\eta}})$, so the restriction to U' $\mathcal{F}|_{U'}$ is constant.

Let $G_{U'}$ denote the Galois group of this cover. We can extend the morphism $U' \to U$ to $C' \to C$, but this one will not be in general a Galois cover since there might be some ramification at $C' \setminus U'$. In order to understand this ramification, we can use the Swan conductor to measure it: given $x' \in C' \setminus U'$ above $x \in C \setminus U$, consider the decomposition group $G_{U',x'}$ of $G_{U'}$ at x', which corresponds to the Galois extension $k(C')_{x'}/k(C)_x$ (that is, the completions of the function fields). By theorem 3.13, there exists a finitely generated and projective $\mathbb{Z}_\ell[G_{U',x'}]$-module $\mathrm{Sw}_{G_{U',x'}}$ underlying the Swan representation of $G_{U',x'}$. We consider the induced representation with respect to $\iota_{x'} : G_{U',x'} \hookrightarrow G_{U'}$ and denote it $\mathrm{Sw}_{G_{U',x}} := \mathrm{Ind}_{\iota_{x'}}(\mathrm{Sw}_{G_{U',x'}}) = \mathrm{Sw}_{G_{U',x'}} \otimes_{\mathbb{Z}_\ell[G_{U',x'}]} \mathbb{Z}_\ell[G_{U'}]$. We denote its character by $\mathrm{sw}_{G_{U',x}}$. This measures the wild ramification of $\mathcal{F}$ at x:

Proposition 4.66. *Let* $\mathrm{sw}_{G_{U',x}}$ *denote the character of the above representation.*

1. *If* $U' \to U$ *is a Galois cover trivializing* $\mathcal{F}$*, then, for all closed points* $x \in C$*,*

$$\mathrm{Swan}_x(\mathcal{F}) = \frac{1}{|G_{U'}|} \sum_{\sigma \in G_{U'}} \mathrm{sw}_{G_{U',x}}(\sigma) \cdot \mathrm{Tr}(\sigma^* | \mathcal{F}_{\overline{\eta}}).$$

2. *With the same notation,*

$$\mathrm{Swan}_x(\mathcal{F}) = \dim_{\mathbb{F}_\lambda} \mathrm{Hom}_{\mathbb{F}_\lambda[G_{U'}]}(\mathrm{Sw}_{G_{U',x}} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \mathcal{F}_{\overline{\eta}}).$$

3. The character $\mathrm{sw}_{G_{U',x}}$ *is given by*

$$\mathrm{sw}_{G_{U',x}}(\sigma) = \begin{cases} \sum\limits_{\substack{y\mapsto x\\ \sigma(y)=y}} (1 - i_{G_{U',y}}(\sigma)), & \text{if } \sigma \neq 1, \\ \left(\sum\limits_{y\mapsto x} (1 + v_y(\mathfrak{D}_{C'/C}))\right) - |G_{U'}|, & \text{if } \sigma = 1. \end{cases}$$

Recall that $i_{G_{U',y}}(\sigma) = v_y(\sigma(\alpha) - \alpha)$, *where* α *is a local parameter of* $k(C')_y$. *The different* $\mathfrak{D}_{C'/C}$ *is just the different of the ring of integers of the extension* $k(C')_y/k(C)_x$, *c.f.* [Ser79].

In particular, the definition of $\mathrm{Sw}_{G_{U',x}}$ *doesn't depend on the choice of* x'.

Proof. Along the proof, we denote $G := G_{U'}$ to make the notation easier. We start by proving the third part. Since $\iota_{x'} : G_{x'} \hookrightarrow G$ is injective, the character of the induced representation is given, according to definition 3.6, by

$$\mathrm{sw}_{G_x}(\sigma) = \frac{1}{|G_{x'}|} \sum_{\substack{\tau\in G\\ \tau\sigma\tau^{-1}\in G_{x'}}} \mathrm{sw}_{G_{x'}}(\tau\sigma\tau^{-1}),$$

where $\mathrm{sw}_{G_{x'}}$ is the Swan character of $G_{x'}$ given by

$$\mathrm{sw}_{G_{x'}} = a_{G_{x'}} - u_{G_{x'}},$$

c.f. definition 3.11 (note that the extension is totally ramified since the base field is after step 1 assumed to be algebraically closed).

- Case $\sigma \neq 1$: we have that

  $$\tau\sigma\tau^{-1} \in G_{x'} \Leftrightarrow \tau\sigma\tau^{-1}(x') = x' \Leftrightarrow \sigma(\tau^{-1}(x')) = \tau^{-1}(x') \Leftrightarrow \sigma \in G_{\tau^{-1}(x')},$$

  and also that

  $$\mathrm{sw}_{G_{x'}}(\tau\sigma\tau^{-1}) = \mathrm{sw}_{G_{\tau^{-1}(x')}}(\sigma),$$

  because we can see $\mathrm{Sw}_{G_{x'}}$ and $\mathrm{Sw}_{G_{\tau^{-1}(x')}}$ as terms of the direct sum $\mathrm{Sw}_G := \bigoplus_{\tau\in G/G_{x'}} \mathrm{Sw}_{G_{\tau(x')}}$, and here we have that

  $$\mathrm{sw}_{G_{x'}}(\tau\sigma\tau^{-1}) = \mathrm{sw}_G(\tau\sigma\tau^{-1}) = \mathrm{sw}_G(\tau^{-1}\tau\sigma) = \mathrm{sw}_{G_{\tau^{-1}(x')}}(\sigma),$$

  where we use the fact from linear algebra that the trace function satisfies $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. With this, we have that

  $$\mathrm{sw}_{G_x}(\sigma) = \frac{1}{|G_{x'}|} \sum_{\substack{\tau\in G\\ G_{\tau^{-1}(x')}\ni\sigma}} \mathrm{sw}_{G_{\tau^{-1}(x')}}(\sigma).$$

  Since $G = \mathrm{Aut}_U(U')$, G acts transitively on the points over x. Hence we have a bijection $G/G_{x'} \longleftrightarrow \{y \in C' \mid y \mapsto x\}$, so we get

  $$\mathrm{sw}_{G_x}(\sigma) = \sum_{\substack{y\mapsto x\\ \sigma(y)=y}} \mathrm{sw}_{G_y}(\sigma).$$

  Finally, since for $\sigma \neq 1$ we have that $a_{G_y}(\sigma) = -i_{G_y}(\sigma)$ and $u_{G_y}(\sigma) = -1$, we obtain the formula.

- Case $\sigma = 1$. In this case,

$$\mathrm{sw}_{G_x}(1) = \frac{1}{|G_{x'}|} \sum_{\substack{\tau \in G \\ \tau 1 \tau^{-1} \in G_{x'}}} \mathrm{sw}_{G_{x'}}(\tau 1 \tau^{-1}) = \frac{|G|}{|G_{x'}|} \mathrm{sw}_{G_{x'}}(1) = \sum_{y \mapsto x} \mathrm{sw}_{G_y}(1)$$
$$= \sum_{y \mapsto x} \left(\left(\sum_{\sigma \neq 1} i_{G_y}(\sigma) \right) - (|G_y| - 1) \right).$$

By [Ser79, IV, Prop. 4], $\sum_{\sigma \neq 1} i_{G_y}(\sigma) = v_y(\mathfrak{D}_{C'/C})$, and by the above bijection $\sum_{y \mapsto x} |G_y| = |G|$, so we obtain the formula.

For the second part, note that $\mathrm{Swan}_x(\mathcal{F}) = \mathrm{Swan}_x(\mathcal{F} \otimes_{\mathcal{O}_E} \mathbb{F}_\lambda)$, and the Swan conductor coincides with the $b(-)$ invariant, so we get that

$$\begin{aligned}\mathrm{Swan}_x(\mathcal{F} \otimes_{\mathcal{O}_E} \mathbb{F}_\lambda) &= b(\mathrm{Res}_{\iota_{x'}}(\mathcal{F}_{\overline{\eta}} \otimes_{\mathcal{O}_E} \mathbb{F}_\lambda)) \\ &= \dim_{\mathbb{F}_\lambda} \mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\mathrm{Sw}_{G_{x'}} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \mathrm{Res}_{\iota_{x'}}(\mathcal{F}_{\overline{\eta}} \otimes_{\mathcal{O}_E} \mathbb{F}_\lambda)),\end{aligned}$$

where $\mathrm{Res}_{\iota_{x'}}$ is the restriction with respect to the inclusion $\iota_{x'}$. Now using that $\mathrm{Ind}_{\iota_{x'}}$ is a left adjoint to $\mathrm{Res}_{\iota_{x'}}$, we get the formula.

Finally, for the first part, recall from remark 3.21 that

$$\dim_{\overline{\mathbb{Q}_\ell}} \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[G]}(\mathrm{Sw}_{G_x} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}, \mathcal{F}_{\overline{\eta}}) = \dim_{\mathbb{F}_\lambda} \mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\mathrm{Sw}_{G_x} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \mathcal{F}_{\overline{\eta}}),$$

so we can stay in the zero-characteristic case. Here, for two ℓ-adic Galois representations V_1, V_2 with characters χ_1, χ_2, we know that

$$\dim_{\overline{\mathbb{Q}_\ell}} \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[G]}(V_1, V_2) = \langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_1(\sigma^{-1}) \chi_2(\sigma),$$

so applying this to $V_1 = \mathrm{Sw}_x$ and $V_2 = \mathcal{F}_{\overline{\eta}}$ we get the equality, since $\mathrm{sw}_{G_x}(\sigma^{-1}) = \mathrm{sw}_{G_x}(\sigma)$ by the above computation.

□

Hence, having a Galois cover trivializing $\mathcal{F}$ allows us to describe explicitly $\mathrm{Swan}_x(\mathcal{F})$. In the next step, we are going to see the representation over a finite extension $E/\mathbb{Q}_\ell$, something that will allow us to use some facts from representation theory.

Step 4. The Grothendieck-Ogg-Shafarevich formula, using the functor $C(-)$ from lemma 4.64, looks at this point like this:

$$\mathrm{rk}_{\mathbb{F}_\lambda}(C^2(\mathcal{F})) - \mathrm{rk}_{\mathbb{F}_\lambda}(C^1(\mathcal{F})) = \mathrm{rk}_{\mathbb{F}_\lambda}(\mathcal{F}) \cdot \chi_c(U, \overline{\mathbb{Q}_\ell}) - \sum_{x \in C \setminus U} \mathrm{Swan}_x(\mathcal{F}).$$

Since $\mathcal{F}$ is trivialized by U', theorem 4.57 tells us that $\mathcal{F}$ corresponds to a finitely generated $\mathbb{F}_\lambda[G_{U'}]$-module. Now let $R_{\mathbb{F}_\lambda}(G_{U'})$ denote the Grothendieck group of the category of finitely generated $\mathbb{F}_\lambda[G_{U'}]$-modules. Let $E/\mathbb{Q}_\ell$ be a finite extension with ring of integers $\mathcal{O}_E$ and residue field $\mathbb{F}_\lambda$.

Lemma 4.67. *There is a group homomorphism $d : R_E(G_{U'}) \to R_{\mathbb{F}_\lambda}(G_{U'})$ which is a surjection.*

Proof. The homomorphism is constructed as follows: to give an $E[G_{U'}]$-module V is the same as giving an E-representation of $G_{U'}$, i.e. a group homomorphism $G_{U'} \to \mathrm{GL}(V)$. But by lemma 3.16, this map factors through a representation $\mathcal{V}$ over $\mathcal{O}_E$ such that $\mathcal{V} \otimes_{\mathcal{O}_E} E = V$:

$$G_{U'} \to \mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(V),$$

and tensoring with $\mathbb{F}_\lambda$ we obtain the $\mathbb{F}_\lambda$-module. One checks that this doesn't depend of the choice of $\mathcal{V}$. The surjectivity is not so easy, but in [Ser77, Ch. 17] there is a proof. Here we omit it.

□

Hence, if we prove for a lisse sheaf of E-modules $\mathcal{F}$ that

$$\mathrm{rk}_E(C^2(\mathcal{F})) - \mathrm{rk}_E(C^1(\mathcal{F})) = \mathrm{rk}_E(\mathcal{F}) \cdot \chi_c(U, \overline{\mathbb{Q}_\ell}) - \sum_{x \in C \setminus U} \mathrm{Swan}_x(\mathcal{F}),$$

we will be done, because the map d is a surjective group homomorphism. Hence, now our situation is the following: we have a lisse E-sheaf of rank r trivialized by the Galois covering $\pi : U' \to U$, so if we denote $V := \mathcal{F}_{\overline{\eta}}$, we have by theorem 4.57 that $\mathcal{F} \cong (\pi_* V_{U'})^{G_{U'}}$ and $\pi^* \mathcal{F} \cong V_{U'}$, where $V_{U'}$ denotes the constant lisse E-sheaf on U'.

In the next step, we will see how can we benefit from this.

Step 5. Now we want to write the cohomology groups $H^i_c(U_{\acute{e}t}, \mathcal{F})$ in terms of the cohomology of the constant sheaf $E_{U'}$ in $U'_{\acute{e}t}$. For this, we extend the morphism $\pi : U' \to U$ to a morphism of smooh poper curves:

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "\pi"] & C' \arrow[d, "\overline{\pi}"] \\ U \arrow[r, hook, "j"] & C \end{tikzcd}$$

and we have that

$$j_! \mathcal{F} \cong j_! (\pi_* V_{U'})^{G_{U'}} \cong (j_! \pi_* V_{U'})^{G_{U'}} \cong (\overline{\pi}_* j'_! V_{U'})^{G_{U'}}.$$

The second isomorphism comes from comparing them as presheaves from the definition of extension by zero and $G_{U'}$-invariants, and then sheafifying. The third isomorphism requires a little bit more of work: first, we have that the morphism $j'^* j'_! V_{U'} \to V_{U'}$ is the identity. Hence, we pull it back and obtain an isomorphism $\pi_* j'^* j'_! V_{U'} \cong \pi_* V_{U'}$. By base change, we have that $\pi_* j'^* j'_! V_{U'} \cong j^* \overline{\pi}_* j'_! V_{U'}$, so we have an isomorphism $\pi_* V_{U'} \to j^* \overline{\pi}_* j'_! V_{U'}$. Now extend by zero this isomorphism to get $j_! \pi_* V_{U'} \cong j_! j^* \overline{\pi}_* j'_! V_{U'}$, so we only need to check that the right hand side is isomorphic to $\overline{\pi}_* j'_! V_{U'}$. For this, if we regard $j_! j^* \overline{\pi}_* j'_! V_{U'}$ as a presheaf, then the morphism $j_!j^*\overline{\pi}_*j_!'V_{U'} \to \overline{\pi}_*j'!V_{U'}$ is an isomorphism on stalks, so the sheafification gives a morphism which is an isomorphism on stalks.

Hence when we compute the cohomology with support we get, by definition and using proposition 4.54,

$$H_c^i(U_{\acute{e}t}, \mathcal{F}) = H^i(C_{\acute{e}t}, j_!\mathcal{F}) \cong H^i\left(C_{\acute{e}t}, (\overline{\pi}_*j_!'V_{U'})^{G_{U'}}\right) \cong H^i(C_{\acute{e}t}, \overline{\pi}_*j_!'V_{U'})^{G_{U'}},$$

and since $\overline{\pi}$ is finite, this is equal to

$$H^i(C_{\acute{e}t}, \overline{\pi}_*j_!'V_{U'})^{G_{U'}} \cong H^i(C'_{\acute{e}t}, j_!'V_{U'})^{G_{U'}} \cong H_c^i(U'_{\acute{e}t}, V_{U'})^{G_{U'}},$$

and we conclude by noticing that there is a $G_{U'}$-invariant isomorphism $H_c^i(U'_{\acute{e}t}, V_{U'}) \cong H_c^i(U'_{\acute{e}t}, E_{U'}) \otimes_E V$: indeed, if we fix an $\mathcal{O}_E$-lattice $\mathcal{V} \subset V$ which is $G_{U'}$-invariant, then for any injective resolution $(\mathcal{O}_E/\mathfrak{m}^n)_{U'} \to \mathcal{I}^\bullet$, we have that $\mathcal{I}^\bullet \otimes_{\mathcal{O}_E} (\mathcal{V} \otimes_{\mathcal{O}_E} \mathcal{O}_E/\mathfrak{m}^n)_{U'}$ is an injective resolution of $(\mathcal{V} \otimes_{\mathcal{O}_E} \mathcal{O}_E/\mathfrak{m}^n)_{U'}$ compatible with the $G_{U'}$-action. Hence, we have

$$H_c^i(U_{\acute{e}t}, \mathcal{F}) \cong (H_c^i(U'_{\acute{e}t}, E_{U'}) \otimes_E V)^{G_{U'}}.$$

In the next step, we will use representation theory in order to compute the dimensions.

Step 6. Recall that given a representation V of a finite group G, the element $\frac{1}{|G|}\sum_{g \in G} g$ induces a projection $V \to V^G$, so the trace of this map gives us the dimension of the image, i.e.

$$\mathrm{Tr}\left(\frac{1}{|G|}\sum_{g \in G} g\right) = \dim_E V^G.$$

Applying this to our situation, and having in mind that the character of a tensor product is the product of the characters (c.f. remark 3.2), we have that

$$\begin{aligned}\chi_c(U, \mathcal{F}) &= \dim_E(H_c^2(U'_{\acute{e}t}, E) \otimes_E V)^{G_{U'}} - \dim_E(H_c^i(U'_{\acute{e}t}, E) \otimes_E V)^{G_{U'}} \\ &= \frac{1}{|G_{U'}|}\sum_{\sigma \in G_{U'}} (\mathrm{Tr}(\sigma^*|H_c^2(U'_{\acute{e}t}, E)) - \mathrm{Tr}(\sigma^*|H_c^1(U'_{\acute{e}t}, E)))\mathrm{Tr}(\sigma^*|_V).\end{aligned}$$

We can do more: setting $Z' := C' \setminus U'$, we can look at the long exact sequence on cohomology

$$\cdots \to H^i(C'_{\acute{e}t}, E) \to H^i(Z'_{\acute{e}t}, E) \to H_c^{i+1}(U'_{\acute{e}t}, E) \to \cdots,$$

and since $j_!'E$ has no global sections on C' and Z' is zero-dimensional we get an exact sequence

$$0 \to H^0(C'_{\acute{e}t}, E) \to H^0(Z'_{\acute{e}t}, E) \to H_c^1(U'_{\acute{e}t}, E) \to H^1(C'_{\acute{e}t}, E) \to 0$$

and an isomorphism

$$H_c^2(U'_{\acute{e}t}, E) \cong H^2(C'_{\acute{e}t}, E).$$

Therefore we obtain

$$
\begin{aligned}
&\mathrm{Tr}(\sigma^*|H^2_c(U'_{\acute{e}t},E)) - \mathrm{Tr}(\sigma^*|H^1_c(U'_{\acute{e}t},E)) = \\
&= \mathrm{Tr}(\sigma^*|H^2(C'_{\acute{e}t},E)) + \mathrm{Tr}(\sigma^*|H^0(C'_{\acute{e}t},E)) - \mathrm{Tr}(\sigma^*|H^0(Z'_{\acute{e}t},E)) - \mathrm{Tr}(\sigma^*|H^1(C'_{\acute{e}t},E)) \\
&= -\mathrm{Tr}(\sigma^*|H^0(Z'_{\acute{e}t},E)) + \sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\sigma^*|H^i(C'_{\acute{e}t},E))
\end{aligned}
$$

for every $\sigma \in G_{U'}$. The zero-dimensional part is easy to compute, since $H^0(Z'_{\acute{e}t},E) = \bigoplus_{x'\in Z'} E_{x'}$ and σ acts via $\sigma^* : E_{x'} \to E_{\sigma(x')}$ as the identity (note that $E_{x'} = E$), since $G_{U'}$ is just permuting the points on Z'. Hence we have, for any $\sigma \in G_{U'}$,

$$\mathrm{Tr}(\sigma^*|H^0(Z'_{\acute{e}t},E)) = \sum_{\substack{x'\in Z' \\ \sigma(x')=x'}} 1.$$

In the next step we see what happens with the one-dimensional part.

Step 7. If $\sigma \neq 1$, we have that Γ_σ and $\Delta_{C'}$ intersect properly in $C' \times C'$, because $\sigma(x') = x'$ implies that $x' \in Z' = C' \setminus U'$: indeed, the Galois group $G_{U'}$ acts transitively and freely on the fibers over U, so there we will not find any fixed point. Hence we may apply the Lefschetz trace formula of theorem 4.47 to obtain

$$\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\sigma^*|H^i(C'_{\acute{e}t},E)) = (\Gamma_\sigma \cdot \Delta_{C'}).$$

Moreover, we can describe this product explicitly:

$$(\Gamma_\sigma \cdot \Delta_{C'}) = \sum_{\sigma(x')=x'} i_{G_{U',x'}}(\sigma),$$

where $i_{G_{U',x'}}$ is as in step 3. For this, recall that the definition of the intersection product $(\Gamma_\sigma \cdot \Delta'_C)$ is

$$(\Gamma_\sigma \cdot \Delta_{C'}) := \sum_{\sigma(x')=x'} \mathrm{length}(\mathcal{O}_{C',x'}/I_\sigma),$$

where the ideal I_σ is generated by the elements $a - \sigma^*(a), a \in \mathcal{O}_{C',x'}$. Hence, it is enough to show that

$$\mathrm{length}(\mathcal{O}_{C',x'}/I_\sigma) = i_{G_{U',x'}}(\sigma) = v_{x'}(\alpha - \sigma^*(\alpha)),$$

where α is a local parameter of A', the $\mathfrak{m}_{x'}$-adic completion of $\mathcal{O}_{C',x'}$. Since $\mathcal{O}_{C',x'}$ is a discrete valuation ring, I_σ is of the shape $\mathfrak{m}^l_{x'}$, where l is precisely the left hand side of the last equation (the length doesn't change when we take $\mathfrak{m}_{x'}$-adic completion). But this is equal to $\mathrm{length}(A'/I_\sigma A')$, and since I_σ is the ideal generated by $\alpha - \sigma^*(\alpha)$, we are done[6].

[6] To see this, note that $v_{x'}(\alpha - \sigma^*(\alpha))$ is the index of the smallest ramification subgroup containing σ, and by remark 2.2 it follows the equality of the ideals.

If $\sigma = 1$, then $\mathrm{Tr}(1^*|H^i(C'_{ét}, E)) = \dim_E H^i(C'_{ét}, E)$, so the alternating sum is just the Euler characteristic of C':

$$\sum_{i=0}^{2}(-1)^i \mathrm{Tr}(1^*|H^i(C'_{ét}, E)) = \chi(C').$$

We can write $\chi(C')$ in terms of $\chi(C)$ as follows:

Lemma 4.68. *In our situation, we have*

$$\chi(C') = 2 - 2g(C') = |G_{U'}| \cdot \chi(C) - \sum_{x' \in Z'} v_{x'}(\mathfrak{D}_{C'/C}),$$

with $v_{x'}(\mathfrak{D}_{C'/C})$ as in proposition 4.66. Here $\chi(C) = \chi_c(U) + \mathrm{card}(Z)$.

Proof. The second equality is just the Hurwitz formula, c.f. [Liu02, Thm 7.4.16], so we only need to prove the first one. The idea is to consider a model $\mathcal{C}'$ of C' over the ring of Witt vectors $W(k)$, and then change base to the complex numbers. By the comparison theorem, we can compare the cohomology of $\mathcal{C}'_{\mathbb{C}}$ with the cohomology of C'. In $\mathcal{C}'_{\mathbb{C}}$ we have the first equality, and this implies the equality in our case because the genus stays constant on the fibers of flat families. This is explained very well in [KR15, Lem. 9.8], and we refer the reader there for the details. □

In the next step, we put everything together in order to obtain the formula.

Step 8. We have, from step 6, that

$$\begin{aligned}\chi_c(U,\mathcal{F}) &= \frac{1}{|G_{U'}|}\sum_{\sigma \in G_{U'}} \big(\mathrm{Tr}(\sigma^*|H^2_c(U'_{ét}, E)) - \mathrm{Tr}(\sigma^*|H^1_c(U'_{ét}, E))\big)\,\mathrm{Tr}(\sigma^*|_V)\\ &= \frac{1}{|G_{U'}|}\sum_{\sigma \in G_{U'}} \left(-\mathrm{Tr}(\sigma^*|H^0(Z'_{ét}, E)) + \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\sigma^*|H^i(C'_{ét}, E))\right)\mathrm{Tr}(\sigma^*|_V).\end{aligned}$$

We know that

$$\mathrm{Tr}(\sigma^*|H^0(Z'_{ét}, E)) = \sum_{\substack{x' \in Z' \\ \sigma(x') = x'}} 1$$

for all $\sigma \in G_{U'}$, and in step 7 we saw that the behaviour of $\mathrm{Tr}(\sigma^*|H^i(C'_{ét}, E))$ is different for $\sigma = 1$ and $\sigma \neq 1$, so we study each of the terms separately:

- For $\sigma = 1$,

$$
\begin{aligned}
-\mathrm{Tr}(1^*|H^0(Z'_{\acute{e}t}, E)) + \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(1^*|H^i(C'_{\acute{e}t}, E))& \\
&= -\mathrm{card}(Z') + \chi(C') \\
&= -\mathrm{card}(Z') + |G_{U'}| \cdot \chi(C) - \sum_{x' \in Z'} v_{x'}(\mathfrak{D}_{C'/C}) \\
&= -\left(\sum_{x' \in Z'} 1\right) + |G_{U'}| \cdot (\chi_c(U) + \mathrm{card}(Z)) - \sum_{x' \in Z'} v_{x'}(\mathfrak{D}_{C'/C}) \\
&= |G_{U'}|\chi_c(U) - \sum_{x \in Z}\left(\left(\sum_{x' \mapsto x} 1 + v_{x'}(\mathfrak{D}_{C'/C})\right) - |G_{U'}|\right) \\
&= |G_{U'}|\chi_c(U) - \sum_{x \in Z} \mathrm{sw}_{G_{U',x}}(1).
\end{aligned}
$$

- For $\sigma \neq 1$,

$$
\begin{aligned}
-\mathrm{Tr}(\sigma^*|H^0(Z'_{\acute{e}t}, E)) + \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\sigma^*|H^i(C'_{\acute{e}t}, E))& \\
&= -\left(\sum_{\substack{x' \in Z' \\ \sigma(x')=x'}} 1\right) + (\Gamma_\sigma \cdot \Delta_{C'}) \\
&= -\left(\sum_{x \in Z} \sum_{\substack{x' \mapsto x \\ \sigma(x')=x'}} 1\right) + \sum_{x \in Z} \sum_{\substack{x' \mapsto x \\ \sigma(x')=x'}} i_{G_{U',x'}}(\sigma) \\
&= \sum_{x \in Z} \sum_{\substack{x' \mapsto x \\ \sigma(x')=x'}} \big(i_{G_{U',x'}}(\sigma) - 1\big) \\
&= -\sum_{x \in Z} \mathrm{sw}_{G_{U',x}}(\sigma).
\end{aligned}
$$

Since $\mathrm{Tr}(1^*|_V) = \mathrm{rk}(\mathcal{F})$ and

$$
\mathrm{Swan}_x(\mathcal{F}) = \frac{1}{|G_{U'}|} \sum_{\sigma \in G_{U'}} \mathrm{sw}_{G_{U',x}}(\sigma) \cdot \mathrm{Tr}(\sigma^*|_V),
$$

we get that

$$\begin{aligned}
&\frac{1}{|G_{U'}|}\sum_{\sigma\in G_{U'}}\left(-\mathrm{Tr}(\sigma^*|H^0(Z'_{\text{ét}},E))+\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\sigma^*|H^i(C'_{\text{ét}},E))\right)\mathrm{Tr}(\sigma^*|_V)=\\
&\quad=\frac{1}{|G_{U'}|}\sum_{\substack{\sigma\in G_{U'}\\ \sigma\neq 1}}\left(-\mathrm{Tr}(\sigma^*|H^0(Z'_{\text{ét}},E))+\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\sigma^*|H^i(C'_{\text{ét}},E))\right)\mathrm{Tr}(\sigma^*|_V)\\
&\qquad+\frac{1}{|G_{U'}|}\left(-\mathrm{Tr}(1^*|H^0(Z'_{\text{ét}},E))+\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(1^*|H^i(C'_{\text{ét}},E))\right)\mathrm{Tr}(1^*|_V)\\
&\quad=\frac{1}{|G_{U'}|}\sum_{\substack{\sigma\in G_{U'}\\ \sigma\neq 1}}\left(-\sum_{x\in Z}\mathrm{sw}_{G_{U',x}}(\sigma)\right)\mathrm{Tr}(\sigma^*|_V)\\
&\qquad+\frac{1}{|G_{U'}|}\left(|G_{U'}|\chi_c(U)-\sum_{x\in Z}\mathrm{sw}_{G_{U',x}}(1)\right)\mathrm{Tr}(1^*|_V)\\
&\quad=\mathrm{Tr}(1^*|_V)\chi_c(U)-\sum_{x\in Z}\frac{1}{|G_{U'}|}\sum_{\sigma\in G_{U'}}\mathrm{sw}_{G_{U',x}}(\sigma)\cdot\mathrm{Tr}(\sigma^*|_V)\\
&\quad=\mathrm{rk}(\mathcal{F})\chi_c(U)-\sum_{x\in Z}\mathrm{Swan}_x(\mathcal{F}).
\end{aligned}$$

This finishes the proof.

4.4.1 Recapitulation of the proof

Since the proof is a little bit long, let's recapitulate the main points and ideas here. We start with a lisse $\overline{\mathbb{Q}_\ell}$-sheaf and the first thing that we do is to reduce to the case where we have a lisse $\mathbb{F}_\lambda$-sheaf $\mathcal{F}$. The advantage of this reduction is that now we can find a Galois cover $U' \to U$ trivializing $\mathcal{F}$. This cover allows us to describe $\mathrm{Swan}_x(\mathcal{F})$ explicitly. In order to compute the Euler characteristic, we use the surjection of the Grothendieck groups $R_E(G_{U'}) \to R_{\mathbb{F}_\lambda}(G_{U'})$ coming from representation theory to be in the situation of a lisse E-sheaf that is trivialized by a Galois cover $U' \to U$, and here we are able to use tools coming from ℓ-adic cohomology and from representation theory in order to describe precisely the Euler characteristic of the sheaf.

Putting everything together yields the formula.

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