The theorem of the cube

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Abstract

In this talk of the seminar Abelian Varieties (FU Berlin, SS 2014) we present the theorem of the cube, a purely algebraic result that results very important in the study of Abelian Varieties. Following the notes *Abelian Varieties*, by Gerard van der Geer and Ben Moonen, we complete and correct some of the proofs and we try to prove the theorem. There is a missing step in the proof that I was not able to understand.

We offer a section of prerequisites to make the script as self-contained as possible, and we discuss some important corollaries of the theorem.

1 Prerequisites

Here we write some important definitions and facts (all of them with proof or an exact reference) that will be later used.

Definition 1. Let $f : X \to Y$ be a morphism of schemes.

1. f is **projective** if it factors into a closed immersion $i : X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$, where we define $\mathbb{P}^n_Y$ as $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$.

2. f is **locally of finite type** if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec}(B_i)$ such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec}(A_{ij})$, where each A_{ij} is a finitely generated B_i-algebra. f is of **finite type** if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the U_{ij}.

3. f is **separated** if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion. We also say that X is separated over Y.

4. f is **closed** if the image of any closed subset is closed. f is **universally closed** if it is closed, and for any morphism $Y' \to Y$, the extension $f' : X' \to Y'$ is also closed.

5. f is **proper** if it separated, of finite type, and universally closed.

Proposition 1. *We have that proper morphisms are stable under base extension (see for example Hartshorne, Cor. II.4.8 c)).*

Definition 2. Let $y \in Y$, and let $k(y)$ be the residue field of y, $\operatorname{Spec}(k(y)) \to Y$ the natural morphism. Then the **fibre** of the morphism f over the point y is the scheme $X \times_Y \operatorname{Spec}(k(y))$.

Remark 1. As topological spaces, $X_y \cong f^{-1}(y)$.

We can see the morphism $f : X \to Y$ as a family of schemes of the shape X_y parametrized by the points of Y.

Remark 2. Given X, Y S-schemes, then we can consider $X \times_S Y$. Given $y \in Y$, we can define the subscheme X_y as $X \times_S \operatorname{Spec}(k(y))$, where $k(y)$ is the residue field. Caution: Note that this *is not* the fibre of any morphism, since we don't have in general a morphism $X \to Y$. This has to be seen as the topological subset $X \times_S \{y\}$, since they are isomorphic as topological spaces. *But* (mixing a little bit the notations), we have that the fibre of the projection $p : X \times_S Y \to Y$ over the point y is just $(X \times_S Y) \times_Y \operatorname{Spec}(k(y)) \cong X \times_S \operatorname{Spec}(k(y))$. Therefore this abuse of notation should be read as $(X \times_S Y)_y$, and we will also call this the **fibre** of $X \times_S Y$ over the point $y \in Y$, and will denote it by X_y when there is no risk of confusion.

Note that here we have the natural morphism $id \times \iota : X_y \to X \times_S Y$ coming from $\iota : \operatorname{Spec}(k(y)) \to Y$. Then, if we have a line bundle L over $X \times_S Y$, we will denote L_y for the restriction $L_{|X \times \{y\}}$, i.e. for the pullback $(id \times \iota)^* L$.

We will be also interested in the fibre of line bundles, specially in the case of line bundles defined over a product variety $X \times Y$.

Definition 3. Let $X \times Y$ be a variety, and let $x \in X$. Let L be a line bundle over $X \times Y$. Then the **fibre** of L over the point x is the restriction $L_{|\{x\} \times Y}$, i.e. the pullback $(\iota \times id)^* L$, where $\iota \times id : \{x\} \times Y \to X \times Y$ is the obvious morphism. When there is no risk of confusion, we will denote $L_x = L_{|\{x\} \times Y}$, and $L_y = L_{|X \times \{y\}}$.

Definition 4. Let k be a field. A k-scheme is **geometrically integral** if for some algebraically closed field K containing k the scheme $X_K := X \times_k K$ is integral, i.e. irreducible and reduced. By EGA IV, if it holds for some alg. closed field K over k, then X_K is integral for every field K containing k.

A k**-variety** X is a k-scheme of finite type and geometrically integral. If it is proper over k, we say that X is **complete**.

Remark 3. Let X and Y be k-varieties, with X complete. Then the fibre X_y (coming from $X \times_k Y$) is also complete (over $k(y)$). Indeed, here $X_y = X \times_k \operatorname{Spec}(k(y))$, and we have that X is of finite type over k, so it will also be of finite type over $k(y)$ (note that this is just a finite extension of k, and if A_{ij} is a finitely generated k-algebra, then of course $A_{ij} \otimes_k k(y)$ is a finitely generated $k(y)$-algebra), and it is also geometrically closed, because if K is an alg. closed overfield of $k(y)$, then it is also an overfield of k. (c.f. Qing Liu, Ex. III.2.14) Finally, since properness is stable under base change (c.f. Prop. 1), then X_y is proper over $k(y)$.

Proposition 2. *We have that if X is proper over a ring A, then the ring of global sections $\mathcal{O}(X)$ is integral over A (c.f. Qing Liu, Prop. 3.18 of chapter 3). As a corollary we obtain that if X is complete over a (not necessarily alg. closed) field k, then $\mathcal{O}(X) = k$ (Qing Liu, Cor. 3.21).*

Proposition 3. *Let X be a complete variety. An invertible sheaf L over X is trivial if and only if both it and its dual have non trivial global sections, i.e. $H^0(M) \neq 0 \neq H^0(M^{-1})$. (Milne, AG, Prop. 13.3)*

Definition 5. Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is **flat** over Y at a point $x \in X$, if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. We say that $\mathcal{F}$ is flat over Y if it is flat on every point of X, and we say that X is flat over Y if $\mathcal{O}_X$ is.

Remark 4. Note that with this definition, it is clear that locally free sheaves are flat.

Proposition 4. *Flatness is stable under base extension. (c.f. Hartshorne, III.9.2)*

Proposition 5. *Let $f : X \to Y$ be a separated morphism of finite type of noetherian schemes, and let $\mathcal{F}$ be a quasicoherent sheaf on X. Let $u : Y' \to Y$ be a flat morphism of noetherian schemes.*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

Then (c.f. Hartshorne, III.9.3) for all $i \geq 0$ there are natural isomorphisms

$$u^* R^i f_*(\mathcal{F}) \cong R^i g_*(v^* \mathcal{F}).$$

Proposition 6 (Künneth formula)**.** *If X, Y are locally noetherian schemes of finite type over k, then*

$$H^n(X \times Y, \mathcal{O}_{X \times Y}) \cong \bigoplus_{i+j=n} H^i(X, \mathcal{O}_X) \otimes H^j(Y, \mathcal{O}_Y)$$

(c.f. EGA III_2, Theorem 6.7.8)

Definition 6. Let Y be a top. space. A function $\phi : Y \to \mathbb{Z}$ is **upper semicontinuous** if for each $y \in Y$, there is an open neighbourhood U of y st for all $y' \in U$, $\phi(y') \leq \phi(y)$. Intuitively: this means that ϕ gets bigger in special points.

For example, given a curve X, define $\phi(x) = 0$ if x is regular or generic, and $\phi(x) = 1$ if it is singular. Then ϕ is upper semicontinuous, because we know that the set of singular points is closed.

Theorem 1 (Semicontinuity)**.** *Let $f : X \to Y$ be a projective morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Then for each $i \geq 0$, the function*

$$h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is an upper semicontinuous functions on Y. In particular, the set $\{y \in Y | h^i(y, \mathcal{F}) \geq n\}$ is closed for each n (see Hartshorne, III.12.8).

Corollary 1 (Grauert)**.** *With the same hypotheses as the theorem, suppose furthermore that Y is integral, and that for some i, the function $h^i(y, \mathcal{F})$ is constant on Y. Then $R^i f_*(\mathcal{F})$ is locally free on Y, and for every y the natural map*

$$R^i f_*(\mathcal{F}) \otimes k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism, where $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}_{|f^{-1}(V)})$.

2 The theorem of the cube

The aim of this section is to clarify the contents of the notes as much as possible.

Theorem 2. *Let X and Y be varieties. Suppose X is complete. Let L and M be two line bundles on $X \times Y$. If for all closed points $y \in Y$ we have $L_y \cong M_y$, there exists a line bundle N on Y such that $L \cong M \otimes p^*N$, where $p = pr_Y : X \times Y \to Y$ is the projection.*

Proof. We have that $L_y \otimes M_y^{-1}$ is the trivial bundle of X_y. By remark 3, X_y is complete, so we can apply prop. 2 to conclude that $H^0(X_y, L_y \otimes M_y^{-1}) \cong k(y)$.

By Grauert's corollary, this implies that $p_*(L \times M^{-1}) \otimes k(y)$ is isomorphic to $k(y)$, and therefore $p_*(L \otimes M^{-1})$ is a line bundle over Y which we denote N. Note that we just now that it has rank 1, so it may not be trivial. Indeed, in general, it will not be trivial.

So if we prove that $L \otimes M^{-1} \cong p^*N$, we will be done. For this, we will show that the natural map

$$\alpha : p^*p_*(L \otimes M^{-1}) \to L \otimes M^{-1}$$

is an isomorphism.

For this we proceed in two steps: first we restrict α to the fibres X_y (where $y \in Y$ is a closed point) and see that there is trivial, and secondly we see that this implies that $L \otimes M^{-1} \cong p^*(p_*(L \otimes M^{-1}))$. Caution: note that this is different from saying that if two line bundles are isomorphic in the fibres, then they are isomorphic: first we need to fix a morphism, and afterwards we have to check that this morphism is an isomorphism in the fibres (as we will do now).

So consider the commutative diagram:

$$\begin{array}{ccc} X_y & \xrightarrow{\phi} & X \times Y \\ \downarrow{\scriptstyle \pi_y} & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}(k(y)) & \xhookrightarrow{\iota} & Y \end{array}$$

where $\phi = id_{X \times Y} \times \iota$. How does α looks like in the fibre? Note that since the diagram commutes, $\phi^*p^*p_*(L \otimes M^{-1}) = \pi_y^*\iota^*p_*(L \otimes M^{-1})$, and by Prop. **??**, $\iota^*p_*(L \otimes M^{-1}) \cong \pi_{y*}\phi^*(L \otimes M^{-1})$. Hence the restriction is $\phi^*(\alpha) : \pi_y^*\pi_{y*}(L \otimes M^{-1})_{|y} \to (L \otimes M^{-1})_{|y}$.

Note that the pushforward is by definition $\pi_{y*}(L \otimes M^{-1})(U) = H^0(X_y, L_y \otimes M_y^{-1})$ for every non empty U, since we are going to a single point. But we have already seen that this is isomorphic to $k(y)$, so $\pi_y^*\pi_{y*}(L \otimes M^{-1}) \cong \pi_y^*\mathcal{O}_{k(y)} \cong \mathcal{O}_{X_y} \otimes_{\mathcal{O}_{X_y}} \mathcal{O}_{X_y} \cong \mathcal{O}_{X_y}$.

By hypothesis, we have that $(L \otimes M^{-1})_{|y} \cong \mathcal{O}_{X_y}$, so writing everything together we have that $\phi^*(\alpha)$ is an isomorphism.

We now go for the second step. It is enough to show that given a morphism $f : \mathcal{E} \to \mathcal{O}_{X \times Y}$ s.t. f_y is an isomorphism for every closed point y, then $\mathcal{E}$ is trivial.

Let $\widetilde{x} \in X \times Y$ be a closed point. Since we can see $(X \times Y)$ as an algebraic family of the fibres $X_y = (X \times Y)_y$ parametrized by Y, there is a closed point $y \in Y$ s.t. x "lies" in X_y, so to say, that there exists a point that maps to $\widetilde{x}$ via $\phi : X_y \to X \times Y$. We call this point x.

We want to show that $\mathcal{E}_{\widetilde{x}} = 0$. We have that

$$0 = (\mathcal{E}_y)_x = (\phi^*\mathcal{E})_x = (\phi^{-1}\mathcal{E} \otimes_{\phi^{-1}\mathcal{O}_{X\times Y}} \mathcal{O}_{X_y})_x = \mathcal{E}_{\widetilde{x}} \otimes_{\mathcal{O}_{X\times Y,\widetilde{x}}} \mathcal{O}_{X_y,x}$$

and if I is the kernel of $\mathcal{O}_{X\times Y,\widetilde{x}} \to (\phi_*\mathcal{O}_{X_y})_{\widetilde{x}} \cong \mathcal{O}_{X_y,x}$, then we have

$$0 = \mathcal{E}_{\widetilde{x}} \otimes_{\mathcal{O}_{X\times Y,\widetilde{x}}} \mathcal{O}_{X_y,x} \cong \mathcal{E}_{\widetilde{x}} \otimes_{\mathcal{O}_{X\times Y,\widetilde{x}}} \mathcal{O}_{X\times Y,\widetilde{x}}/I \cong \mathcal{E}_{\widetilde{x}}/I\mathcal{E}_{\widetilde{x}}$$

and Nakayama's lemma implies that $\mathcal{E}_{\widetilde{x}} = 0$.

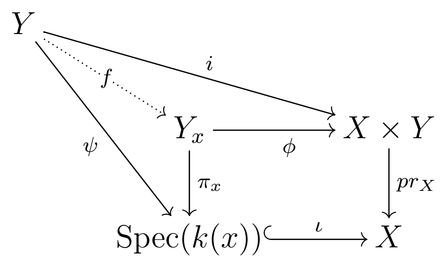
□

This theorem induces an equivalence relation in the set $\mathrm{Pic}(X\times Y)$, and therefore induces the following

Definition 7. We say that a line bundle L over $X \times Y$ is **trivial (in the fibres) over** Y if it is the pullback of a line bundle on Y. This is equivalent to being trivial (in the usual sense) over Y, i.e. that $L_y \cong \mathcal{O}_{X_y}$. We say that two line bundles are **isomorphic (in the fibres) over** Y if $L \otimes M^{-1}$ is trivial in the fibres.

Theorem 3 (See-saw Principle)**.** *Let X and Y be as above, and L, M two line bundles on $X \times Y$ isomorphic over Y. If in addition we have $L_x = M_x$ for some $x \in X(k)$, then $L \cong M$, i.e. L and M are isomorphic in the usual sense.*

Proof. We have that $L \cong M \otimes pr_Y^* N$ for some line bundle N over Y. If we restrict to $\{x\} \times Y$, we have that $M_x = L_x \cong M_x \otimes (pr_Y^* N)_x$, so $(pr_Y^* N)_x$ is trivial. Hence, N is trivial, because we have the following commutative diagram:



where ψ is just the structure morphism of Y (note that x is k-rational!), and i arises from the fibre product of $\iota \circ \psi$ and the identity on Y. Since the diagram is commutative, we have that $id_Y = pr_Y \circ \phi \circ f$, and $f^*\mathcal{O}_{Y_x} = \mathcal{O}_Y$, so

$$\mathcal{O}_Y = f^*\mathcal{O}_{Y_x} \cong f^*(\phi^* pr_Y^* N) = id_Y^* N = N$$

and therefore N is trivial. And we are done, because $pr_Y^*\mathcal{O}_Y \cong \mathcal{O}_{X\times Y} \Rightarrow L \cong M$.

□

Now we want to prove the main theorem of the talk, the theorem of the cube. It says (under certain conditions) that given a line bundle L over $X \times Y \times Z$, if it is trivial over $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$ and $X \times Y \times \{z\}$ for some k-rational points x, y and z, then the line bundle L is trivial.

Before we prove (and state correctly) this, we need a lemma.

Lemma 1. *Let X and Y be varieties, with X complete. For a line bundle L on $X \times Y$, the set $\{y \in Y | L_y$ is trivial over $Y\}$ is closed in Y.*

Proof. By Prop. **??**, we have that

$$\{y \in Y | L_y \text{ is trivial}\} = \{y \in Y | h^0(L_y) > 0\} \cap \{y \in Y | h^0(L_y^{-1}) > 0\}.$$

Since the functions $y \mapsto h^0(L_y)$ and $y \mapsto h^0(L^{-1})$ are upper semi-continuous (c.f. Semicontinuity theorem), the sets on the right hand side are closed and we are done. □

There is a refinement of this lemma:

Lemma 2. *Let X be a complete variety over k, Y a k-scheme, and let L be a line bundle on $X \times Y$. Then there exists a closed subscheme $Y_0 \hookrightarrow Y$ which is the maximal subscheme of Y over which L is trivial. More concretely:*

1. *Triviality: the restriction of L to $X \times Y_0$ is the pullback of a line bundle on Y_0.*
2. *Maximality: for every morphism $\phi : Z \to Y$ such that $(id_X \times \phi)^*(L)$ is trivial over Z, then ϕ factors through Y_0. Note than in particular, if we take Z as closed subschemes of Y, we are asking for L to be trivial over this closed subscheme, so this Y_0 is the maximal one.*

Now we go for the big theorem:

Theorem 4. *Let X and Y be complete varieties and let Z be a connected, locally noetherian scheme. Let $x \in X(k)$, $y \in Y(k)$ and let z be a point of Z. If L is a line bundle on $X \times Y \times Z$ whose restriction to $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$ and to $X \times Y \times \{z\}$ is trivial, then L is trivial.*

Proof. Since the projection $X \times Y \times Z \to Z$ is flat, we can see L as a family of line bundles parametrized by Z. Now let Z' be the maximal closed subscheme of Z over which L is trivial, as in the lemma. By the maximality condition, taking $z \hookrightarrow Z$, then this factorizes through $j : Z' \hookrightarrow Z$: in other words, $z \in Z'$, so Z' is non empty. We want to show that $Z' = Z$, and for this it will be enough to show that it is open (because Z is connected and Z' is closed and non-empty).

Let ζ be a point of Z'. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{Z,\zeta}$ and $I \subset \mathcal{O}_{Z,\zeta}$ be the ideal defining the germ of Z', i.e. the kernel of $\mathcal{O}_{Z,\zeta} \to (j_*\mathcal{O}_{Z'})_\zeta$. Note that $I = (0)$ if and only if there is an open subset $\zeta \in V \subset Z$ s.t. $V \subset Z'$, because of the exact sequence $0 \to i_!(\mathcal{O}_U) \to \mathcal{O}_Z \to j_*\mathcal{O}_{Z'} \to 0$, where $i : U \subset Z$ is the complement of Z'. (c.f. Hartshorne, Ex. II.1.19) Therefore if we prove that (I) is zero, then $\mathcal{O}_Z \cong j_*\mathcal{O}_{Z'}$, and since Z' is a closed subscheme of Z, they must be equal and we will be done.

So assume that $I \neq (0)$. $\mathcal{O}_{Z,\zeta}$ is noetherian because Z is locally noetherian, so $\cap_n \mathfrak{m}^n = (0)$. Let n be the natural number s.t. $I \subset \mathfrak{m}^n$ and $I \not\subseteq \mathfrak{m}^{n+1}$. Write $a_1 = (I, \mathfrak{m}^{n+1})$. Claim: there exists an ideal a_2 s.t. $\mathfrak{m}^{n+1} \subset a_2 \subset a_1 \subset \mathfrak{m}^n$ and $\dim_{k(\zeta)}(a_1/a_2) = 1$. Indeed, if we quotient with $\mathfrak{m}^{n+1}$ we have the chain of $k(\zeta)$-vector spaces $0 \subset a_1 \subset \mathfrak{m}^n/\mathfrak{m}^{n+1}$, and since $I \not\subseteq \mathfrak{m}^{n+1}$, the dimension of $a_1/\mathfrak{m}^{n+1}$ is greater than zero. Hence, taking a subspace $\overline{a_2} \subset a_1/\mathfrak{m}^{n+1}$ of codimension one and going back to the chain of $\mathcal{O}_{Z,\zeta}$-modules, we obtain such an a_2. By construction, $I \not\subseteq a_2$.

Let $Z_i \subset \operatorname{Spec}(\mathcal{O}_{Z,\zeta})$ be the closed subschemes defined by a_i, which topologically they are both just $\mathfrak{m}$, since $\overline{\mathfrak{p}} \in \operatorname{Spec}(\mathcal{O}_{Z,\zeta}/a_i)$ iff $\mathfrak{p} \supset a_i \supset \mathfrak{m}^{n+1}$ iff $\mathfrak{p} = \mathfrak{m}$. Let $\operatorname{Spec}(A) \subset Z$ be an open subset of Z containing ζ. Note that $A_\zeta = \mathcal{O}_{Z,\zeta}$, so the localization $A \to A_\zeta$ induces a morphism $Z_i \subset \operatorname{Spec}(\mathcal{O}_{Z,\zeta}) \to \operatorname{Spec}(A) \hookrightarrow Z$ for each i. Let ϕ_i denote the composition $\phi_i : Z_i \hookrightarrow Z$. Since $I \subset a_1$, we have that ϕ_1 factors through Z', so the restriction of L to $X \times Y \times Z_1$ is trivial. We will show now that the restriction of L to $X \times Y \times Z_2$ is trivial, which implies that ϕ_2 factors also through Z', but this implies that $I \subset a_2$, which is a contradiction, and therefore we will be done.

First we fix the notation. Let L_i be the restriction $j_i^*(L) := (id_X \times id_Y \times \phi_i)^* L$. Take a trivializing global section $s \in \Gamma(L_1)$, i.e. a global section s s.t. the morphism $\mathcal{O}_{X\times Y\times Z_1} \to L_1 : 1 \mapsto s$ is actually an isomorphism (this can be done because L_1 is trivial!). The inclusion $Z_1 \subset Z_2$ induces $i_{12} : X \times Y \times Z_1 \to X \times Y \times Z_2$. We also have the commutative diagram

$$\begin{array}{ccc} X\times Y\times Z_1 & & \\ \downarrow{\scriptstyle i_{12}} & \searrow^{j_1} & \\ X\times Y\times Z_2 & \xrightarrow[j_2]{} & X\times Y\times Z \end{array}$$

so we have the natural map $L_2 \to i_{12*}(i_{12}^* L_2) = i_{12*} L_1$ coming from the identity $i_{12}^* L_2 \to i_{12}^* L_2$ via the adjunction formula. (c.f. Hartshorne, p. 110) Now, taking global sections, we have a restriction map $\psi : \Gamma(L_2) \to \Gamma(L_1)$. Claim: L_2 is trivial if and only if s can be lifted to a global section s' of L_2.

Indeed, suppose first that we have a lift s' of the trivializing global section $s \in \Gamma(L_1)$. Since the underlying point set of both Z_1 and Z_2 is the same, $X \times Y \times Z_i$ are homeomorphic. We know that if the image $s'(P)$ of s' via $\mathcal{O}_{X\times Y\times Z_2} \to L_2 \to (L_2)_P \to (L_2)_P/\mathfrak{m}_P$, where $\mathfrak{m}_P$ is the maximal ideal of $(\mathcal{O}_{X\times Y\times Z_2})_P$, is zero, then also $s(P)$ is zero. Indeed, just look at the commutative diagram

$$\begin{array}{ccc} L_2 & \longrightarrow & i_{12*}L_1 \\ \downarrow & & \downarrow \\ (L_2)_P & \longrightarrow & (i_{12*}L_1)_P \\ \downarrow & & \downarrow \\ L_{2,P} \otimes_{\mathcal{O}_{X\times Y\times Z_2}} k(P) & \longrightarrow & (i_{12*}L_1)_P \otimes_{\mathcal{O}_{X\times Y\times Z_2}} k(P) \end{array}$$

and note that in the last row, the tensor product with $k(P)$ is isomorphic to the quotient $L_{2,P}/\mathfrak{m}_P L_{2,P}$, and something similar in the right hand side.

Now assume that L_2 is trivial. Then we have that the sequence with L_2 is just

$$\mathcal{O}_{X\times Y\times Z_2} \to i_{12*}\mathcal{O}_{X\times Y\times Z_1} \to 0$$

and we can take global sections and use Künneth formula, i.e. $H^0(X \times Y \times Z_i, \mathcal{O}_{X\times Y\times Z_i}) \cong H^0(X \times Y, \mathcal{O}_{X\times Y}) \otimes H^0(Z_i, \mathcal{O}_{Z_i})$ and we have that the induced morphism is just

$$id_{X\times Y} \otimes \pi : H^0(X\times Y, \mathcal{O}_{X\times Y}) \otimes_k H^0(Z_2, \mathcal{O}_{Z_2}) \to H^0(X\times Y, \mathcal{O}_{X\times Y}) \otimes_k H^0(Z_1, \mathcal{O}_{Z_1})$$

where π is just the projection given by $\mathcal{O}_{Z,\zeta}/a_1 \to \mathcal{O}_{Z,\zeta}/a_2$. Hence this is surjective, so we will be able to lift s.

So we reduced the problem to finding a lift s' of s. For this, it is enough to show that $\Gamma(L_2) \to \Gamma(L_1)$ is surjective. From the exact sequence

$$0 \to \ker(\psi) \to L_2 \to i_{12*}L_1 \to 0$$

it is enough to show that $H^1(X \times Y \times Z_2, \ker(\psi)) = 0$.

The missing step: For this, it is enough to show that $H^1(X \times Y, \mathcal{O}_{X\times Y}) = 0$.

Assumed that this is enough, let's prove that it is indeed equal to zero. Let $\xi \in H^1(X \times Y, \mathcal{O}_{X\times Y})$. By hypothesis, the restrictions of L_2 to $\{x\} \times Y \times Z_2$ and $X \times \{y\} \times Z_2$ are trivial. As in the proof of the See-saw principle, since the points $x, y \in X(k)$, we find the maps $i_1 = (id_X, y) : X \hookrightarrow X \times Y$ and $i_2 = (x, id_Y) : Y \hookrightarrow X \times Y$. But being trivial implies that $i_1^*(\xi) = 0 = i_2^*(\xi)$, because of the direction of the claim that we haven't proved and the missing step. (Sorry for that) Finally, since X and Y are complete, we have that $H^1(X, \mathcal{O}_X) \otimes H^0(X, \mathcal{O}_X) \cong H^1(X, \otimes\mathcal{O}_X)$, and analog with $\mathcal{O}_Y$. Hence the Künneth formula gives us the following isomorphism

$$H^1(X \times Y, \mathcal{O}_{X\times Y}) \cong H^1(X, \mathcal{O}_X) \oplus H^1(Y, \mathcal{O}_Y)$$

so $\xi = 0$ and we are done (modulo the missing steps :P).

□

Remark 5. In the theorem as stated we require x and y to be k-rational points of X and Y, but we can generalize this. We used this fact in the definition of i_1 and i_2, but the theorem still holds without these assumptions. The point is that if $k \subset K$ is a field extension then a line bundle M on a k-variety V is trivial if and only if the line bundle M_K on V_K is trivial. (Exercise (2.1) of van der Geer and Moonen notes of AV)

Remark 6. The analogous statement for line bundles on a product of two complete varieties is false in general, so to say, if L is a line bundle on $X \times Y$, then we may have $L_x \cong \mathcal{O}_Y$ and $L_y \cong \mathcal{O}_X$, and still we will not have $L \cong \mathcal{O}_{X\times Y}$. For example, consider $X = Y$ an elliptic curve, and take the divisor

$$D = \Delta_X - (\{0\} \times X) - (X \times \{0\})$$

where $\Delta_X \subset X \times X$ is the diagonal. Then $L = \mathcal{O}_{X\times X}(D)$ restricts to the trivial bundle on $\{0\} \times X$ and $X \times \{0\}$, because the divisor $1 \cdot e_X$ is linearly equivalent (from the group law) to a divisor whose support doesn't contain e_X. But L is not the trivial bundle: if it were, $L_{|\{P\}\times X} = \mathcal{O}_X(P - e_x) \cong \mathcal{O}_X$, and then we would have that X is a rational curve, and we get a contradiction. (c.f. Hartshorne, Example II.6.10.1)

3 Consequences

This theorem has a lot of consequences which will give us information about line bundles. First, recall that if $f(x) = ax^2 + bx + c \in \mathbb{R}[x]$ is a quadratic form, then the polynomial

$$f(x+y+z) - f(x+y) - f(x+z) - f(y+z) + f(x) + f(y) + f(z)$$

is constant. We have an analogue of this for line bundles on abelian varieties. Before we state it, let's prove the following corollary which will be helpful.

Corollary 2. *Let X be an abelian variety, and let $I = \{i_1, \ldots, i_r\} \subset \{1, 2, 3\}$. We denote $p_I : X^3 \to X$ for the morphism sending (x_1, x_2, x_3) to $x_{i_1} + \ldots + x_{i_r}$, i.e. $p_{12} + p_1 + p_2$, and so on. Let L be a line bundle on X. Then the line bundle*

$$\begin{aligned} \Theta(L) &:= \bigotimes_{I \subset \{1,2,3\}} p_I^* L^{\otimes(-1)^{1+\#I}} \\ &= p_{123}^* L \otimes p_{12}^* L^{-1} \otimes p_{13}^* L^{-1} \otimes p_{23}^* L^{-1} \otimes p_1^* L \otimes p_2^* L \otimes p_3^* L \end{aligned}$$

on X^3 is trivial.

Proof. From the big theorem, it is enough to show that the restrictions to $\{0\} \times X \times X$, $X \times \{0\} \times X$ and $X \times X \times \{0\}$ are trivial. We do it for $\{0\} \times X \times X$. Let $j : \{0\} \times X \times X \hookrightarrow X^3$ be the obvious map. Then $j^* p_{123}^* L \cong p_{23}^* L$, and $j^* p_1^* L \cong \mathcal{O}_{\{0\} \times X \times X}$, and similarly $j_{12}^* L \cong p_2^* L$. Then, when we substitute, everything cancels and we get that the line bundle is trivial.

□

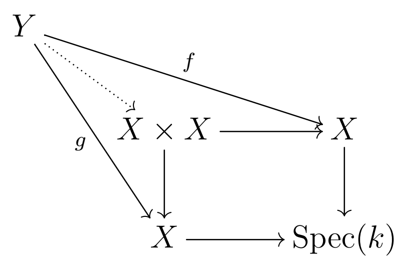
And now we get the analogue of our quadratic form:

Corollary 3. *Let Y be a scheme and let X be an abelian variety. For every triple f, g, h of morphisms $Y \to X$ and for every line bundle L on X, we have that the bundle*

$$(f+g+h)^* L \otimes (f+g)^* L^{-1} \otimes (f+h)^* L^{-1} \otimes (g+h)^* L^{-1} \otimes f^* L \otimes g^* L \otimes h^* L$$

on Y is trivial.

Proof. Consider $(f, g, h) : Y \to X \times X \times X$ and apply the previous corollary. Since from the definition of the fibre product we have that the diagrams



commute, so the above line bundle is the pullback of the constant line bundle $\Theta(L)$, and hence it is again constant.

□

And we finish with an important corollary of the theorem of the cube, the so called theorem of the square:

Corollary 4 (Theorem of the Square)**.** *Let X be an abelian variety and let L be a line bundle over it. Then, for all $x, y \in X(k)$,*

$$t^*_{x+y}L \otimes L \cong t^*_x L \otimes t^*_y L.$$

Proof. The statement follows from the previous corollary by taking $f = id_X$, and g and h the constant maps to x and y respectively.

□

Remark 7.
1. The previous corollary holds more generally: let T be a k-scheme and let L_T be the pullback of L to X_T. Then

$$t^*_{x+y}L_T \otimes L_T \cong t^*_x L_T \otimes t^*_y L_T \otimes pr^*_T((x+y)^*L \otimes x^*L^{-1} \otimes y^*L^{-1}).$$

2. If we tensor the isomorphism in the corollary with L^{-2} and look at the points x and e_X, we obtain the following important fact:

Corollary 5. *Given a line bundle L over an abelian variety X, then the map*

$$\varphi_L : X(k) \to Pic(X) : x \mapsto [t^*_x L \otimes L^{-1}]$$

is a group homomorphism.

This generalizes the well known fact for elliptic curves. (c.f. Hartshorne, IV.1.3.7)

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