Ruled quartic surfaces (I)

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Abstract

This is a talk of the seminar on Algebraic Geometry at FU Berlin, organized by Joana Cirici, on the Winter Semester 2013-14. The aim of the talk is to partially describe the results of [1].

1 Some properties of the Grassmannian

In this section we will mention some properties of the Grassmannian $Gr(2,4)$ that will be useful in the following sections.

We begin with a vector space V of dimension 4 over an algebraically closed field K. As usual, the points of $Gr := Gr(2,4)$ will be the lines of $\mathbb{P}^3 := \mathbb{P}(V)$.

We fix a basis e_1, e_2, e_3, e_4 of V, and this allows us to identify K and the exterior product of V, so to say we have the isomorphism $\bigwedge^4 V \to K : \ e_1 \wedge e_2 \wedge e_3 \wedge e_4 \mapsto 1$.

Note that given a plane $W = \langle v_1, v_2 \rangle \subset V$ we can define a line in $\bigwedge^2 V$ by considering $\langle v_1 \wedge v_2 \rangle$. Recall that by definition, $V \wedge V = (V \otimes V)/(v \otimes v)$. Hence, different planes of V are mapped to different lines of $\bigwedge^2 V$. Note that $w \in \bigwedge^2 V$ different from zero can be identified with a plane if and only if[1] w is decomposable (ie $w = v_1 \wedge v_2$), so this identification can't be extended to the whole $\bigwedge^2 V$. However, we can write the following identifications, where we will understand that we are identifying just what we can identify:

$$\begin{array}{ccccc} \mathbb{P}(V) & \longleftrightarrow & V & \longleftrightarrow \\ \widetilde{w} \text{ line} & & W = \langle v_1, v_2 \rangle \text{ plane} & \end{array}$$

$$\begin{array}{ccccc} \longleftrightarrow & \bigwedge^2 V & \longleftrightarrow & \mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^5 \\ & \bigwedge^2 W = \langle v_1 \wedge v_2 \rangle = \langle w \rangle \text{ line} & & \overline{w} = Kw \text{ point} \end{array}$$

With these identifications, we have that Gr consists of all the points $\overline{w}$ coming from a line $\widetilde{w}$ of $\mathbb{P}(V)$, and by the previous paragraph, we have that a point $\overline{w} \in \mathbb{P}(\bigwedge^2 V)$ belongs to Gr if and only if w is decomposable. But w is decomposable if and only if $w \wedge w = 0$. Hence, we have a criterion to decide wether a point of $\mathbb{P}(\bigwedge^2 V)$ belongs to Gr or not. Let's see what happens if we introduce coordinates.

[1] One direction is clear. For the other, note that the identification is injective, so it will be bijective with its image.

Denote $e_{ij} := e_i \wedge e_j$ for $1 \leq i < j \leq 4$. They form a basis for $\bigwedge^2 V$ with the lexicographic ordering. The coordinates of $w \in \bigwedge^2 V$ with respect to this basis are the Plücker coordinates: $w = \sum_{i<j} p_{ij} e_{ij}$. What happens to the coordinates if we want $\overline{w}$ to be in Gr? The condition is equivalent to say $w \wedge w = 0$. We know that $e_{ij} \wedge e_{kl} \neq 0$ if and only if i, j, k, l are different, and if this is the case, the (wedge) product is equal to 1 or -1, depending of the order. Hence,

$$\begin{aligned} 0 &= w \wedge w \\ &= \textstyle\sum_{i<j} p_{ij} e_{ij} \wedge \sum_{i<j} p_{ij} e_{ij} \\ &= p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} + p_{23}p_{14} - p_{24}p_{13} + p_{34}p_{12} \\ &= 2p_{12}p_{34} - 2p_{13}p_{24} + 2p_{14}p_{23} \end{aligned}$$

So, as we already knew from Vincent's talk, $w = [p_{12} : \ldots : p_{34}] \in Gr$ if and only if w lies on Kleins quadric $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.

From now on, we will abuse the notation and we will use $\overline{w}$ with $w \wedge w = 0$ for the point of Gr and for the line $\widetilde{w}$ defined in $\mathbb{P}(V)$.

Now, we write some remarks that will be useful in the following sections. Some of them were mentioned in Gabriel's talk:

(i) Two lines $\overline{w_1}, \overline{w_2}$ of $\mathbb{P}(V)$ intersect if and only if $\dim(\langle v_{11}, v_{12}, v_{21}, v_{22} \rangle) = 3$, and this is equivalent to saying that $w_1 \wedge w_2 = v_{11} \wedge v_{12} \wedge v_{21} \wedge v_{22} = 0$.

(ii) Every hyperplane of $\mathbb{P}(\bigwedge^2 V)$ has the form $\{\overline{z} |\ w \wedge z = 0\}$, where $\overline{w}$ is a fixed point. It is clear that this is an hyperplane because this set is the projectivization of the kernel of the linear map $\gamma_w : \bigwedge^2 V \to K : z \mapsto w \wedge z$. We will denote this hyperplane as H_w.

(iii) If w is indecomposable, then the intersection of H_w and Gr is a non degenerate quadric, because Kleins quadric is non degenerate and $w \notin Gr$, so H_w is not a tangent plane[2]. We are intersecting Gr with an hyperplane in general position.

If w is decomposable, then H_w is the tangent space of Gr at the point w, $T_{Gr,\overline{w}}$, so the intersection will have a singularity. In particular, $T_{Gr,\overline{w}} \cap Gr$ can be identified[3] with the cone in $\mathbb{P}^4$ over a non singular quadric in $\mathbb{P}^3$ with vertex $\overline{w}$. Note that if K is algebraically closed, all the non singular quadrics are equivalent, so the cone is always the same. This intersection can be identified with $\sigma_1(\overline{w}) :=$ the collection of all lines $l \subset \mathbb{P}(V)$ with $l \cap \overline{w} \neq \emptyset$, because of (i).

(iv) Let p_0 be a point of $\mathbb{P}(V)$. Then, $\sigma_2(p_0) :=$ the collection of all lines through p_0. If we consider the lines as points, we obtain a $\mathbb{P}^2$, because we can consider a sphere with center p_0 and then the lines identify the antipodal points. When

[2] Recall that a non degenerate quadric give us a canonical way of going from a vector space to its dual space. In this particular case, the isomorphism is $w \mapsto H_w$, where Kleins quadric defines the wedge product. We are doing the inverse process: we start with a quadric and we obtain a bilinear form (uniquely determined up to scalar product). In this way, we can define the wedge product geometrically starting from Kleins quadric

[3] The reader can think in a lower dimensional example: if we start with an hyperboloid H, then $T_{H,p} \cap H$ is a cone in $\mathbb{P}^2$ over a non singular quadric in $\mathbb{P}$, ie, a pair of lines.

we go to the grassmannian, we obtain again a 2-plane. For example, if $p_0 = \overline{e_1}$, then $\sigma_2(\overline{e_1}) = \{\overline{\sum_{1<j\leq 4} p_{1j} e_1 \wedge e_j}\}$ is a 2-plane in Gr.

(v) Let h_0 be a plane of $\mathbb{P}(V)$. Then, $\sigma_{1,1}(h_0) :=$ the collection of all lines in the plane h_0. Again, it has the structure of a $\mathbb{P}^2$ (just consider, in h_0, the dual space. Then, you obtain the lines in h_0), and when you look at Gr, you have again a plane: indeed, if you choose a basis such that $h_0 = \overline{\langle e_1, e_2, e_3 \rangle}$, then $\sigma_{1,1}(h_0) = \{\overline{\sum_{1\leq i<j\leq 3} p_{ij} e_i \wedge e_j}\}$.

(vi) $\sigma_{2,1}(p_0, h_0) :=$ the collection of all lines in h_0 through p_0. Note that $\sigma_{2,1}(p_0, h_0) = \sigma_2(p_0) \cap \sigma_{1,1}(h_0)$, so this is a line on Gr.

(vii) Every plane in Gr has the form $\sigma_2(p_0)$ or $\sigma_{1,1}(h_0)$, and every line in Gr has the form $\sigma_{2,1}(p_0, h_0)$.

(viii) We can classify three dimensional projective subspaces P of $\mathbb{P}(\wedge^2 V)$ with respect to their relation with Gr:

(a) $Gr \cap P$ is a non degenerate quadric surface. Then, the matrix associated to this quadric surface has rank 4, so we can choose the basis in such a way that the equations of P are $p_{12} = p_{34} = 0$. But the hyperplane $p_{12} = 0$ is precisely[4] $H_{e_{34}}$, and this is the tangent space $T_{Gr,\overline{e_{34}}}$. Analoguely, $p_{34} = 0$ defines the hyperplane $T_{Gr,\overline{e_{12}}}$. Hence, P is the intersection of two tangent spaces.

(b) $Gr \cap P$ is an irreducible degenerate quadric surface. In this case, the equations of P are $p_{34} = 0$ and $p_{13} + p_{24} = 0$ for a suitable basis of V. In this case, P lies on only one tangent space, $T_{Gr,\overline{e_{12}}}$ and $Gr \cap P$ is the cone $p_{13}^2 + p_{14}p_{23} = 0$, $p_{34} = 0$ and $p_{13} + p_{24} = 0$. The cone is over the quadric curve $p_{13}^2 + p_{14}p_{23} = 0$.

(c) $Gr \cap P$ is reducible. The equations for P in this case are $p_{12} = p_{13} = 0$ for a basis of V, and $Gr \cap P$ is the union of the planes $p_{14} = 0$ and $p_{23} = 0$.

2 Ruled surfaces and curves on Gr

Given a ruled surface S in $\mathbb{P}(V)$, we can consider the set of lines of the surface. This set, in general, can be viewed as family of lines, and therefore as family of points $\widetilde{C}$ in Gr. In this section we will study the relation between ruled surfaces and curves on Gr, and will answer the following question: given the ruled surface S, what do we obtain in Gr?

We start with a naive example that we have already mentioned. If we consider a plane $h_0 \subset \mathbb{P}(V)$, then we have a lot of lines. Indeed, as seen in (v), this set is precisely $\sigma_{1,1}(h_0)$, and this is a 2-plane on Gr. Note that in general, one starts with a ruled surface that consists on all the lines passing through a curve, but in this configuration there are a lot of lines that appear in the surface, and therefore we obtain a two-parameter family. We expect that this is not the general case.

[4] If $\overline{z} = [p_{12} : \ldots : p_{34}]$, then $\overline{z} \in H_{e_{34}}$ iff $0 = z \wedge e_{34} = p_{12}$.

We are also interested in the inverse problem, so to say: given a curve $C \subset Gr$, do we obtain a ruled surface? We will start with this second question, and afterwards we will study the first one.

2.1 From a curve in Gr to a ruled surface in $\mathbb{P}(V)$

Again, we can also look at an easy example. If we start with a line C on Gr, we know from (vi) that the only lines on Gr have the form $\sigma_{2,1}(p_0, h_0)$, and we obtain a plane in $\mathbb{P}(V)$ (all the lines on h_0 that pass through p_0 define the plane h_0).

Now, let's assume that the degree of C is $d \geq 2$. How can we define a set S on $\mathbb{P}(V)$? We do the following. Consider $\widetilde{S} := \{(\overline{w}, \overline{v}) \in C \times \mathbb{P}(V) |\, w \wedge v = 0\}$. The condition $w \wedge v = 0$ is equivalent as saying that given a point $w = w_1 \wedge w_2 \in C \subset Gr$, the point $v \in \mathbb{P}(V)$ must not be linearly independent with respect to w_1 and w_2, ie, v must lie on the line $\langle w_1, w_2 \rangle$. This is equivalent as saying that the fibres of the projection $pr_1 : \widetilde{S} \to C$ are lines in $\mathbb{P}(V)$.

If we consider the projection $pr_2 : \widetilde{S} \to \mathbb{P}(V)$, its image, S, will be the set of points lying on the lines that form C.

What happens if C lies on a 2-plane $\sigma_2(p_0)$? Here, all the lines of S will pass through p_0, so we will have a cone. Indeed, if you consider a plane $h \subset \mathbb{P}(V)$ with equation $\sum \lambda_i x_i = 0$ that doesn't contain p_0 and you take $S \cap h$, you obtain a curve with equations $w \wedge v = 0$ and $\sum \lambda_i v_i = 0$. Then, S will be exactly the set of all the lines passing through p_0 and the points of this curve.

In general, we have the following

Lemma 1. *Let $C \subset Gr$ be an irreducible curve of degree $d \geq 2$ not lying in some 2-plane $\sigma_2(p_0)$. Then, $\widetilde{S}$ is an irreducible variety of dimension 2, and S is an irreducible surface.*

Proof. □

Lemma 2. *With the above assumptions, the surface S has some degree $e \leq d$. Suppose that through a general point of S there are f lines $\overline{w} \in C$. Then, $d = e \cdot f$.*

Before we proof the lemma, recall from Christ's talk (the first one) that the degree of an hypersurface coincides with the intersection number of a line with it.

Proof. A general line $\overline{w}_0$ in $\mathbb{P}(V)$ intersects S in e points. Through each of these e points there are f lines $\overline{w} \in C$. Thus the intersection of C with the general hyperplane $\{\overline{w} \in \mathbb{P}(V) |\, w \wedge w_0 = 0\}$ consists of $e \cdot f$ points and therefore $d = e \cdot f$. □

Given a curve $C \subset Gr$, define $P(C)$ as the smallest projective subspace of $\mathbb{P}(\bigwedge^2 v)$ containing C. We have the following

Lemma 3. *If $d \geq 3$ and S is not a cone, a plane or a quadric, then* $\dim P(C) \geq 3$.

Proof. Since $d > 1$, $\dim P(C) > 1$. Now, assume that $\dim P(C) = 2$. If $P(C) \subset Gr$, then, by (vii), either $P(C)$ is a $\sigma_2(p_0)$ and S is a cone, or $P(C)$ is a $\sigma_{1,1}(h_0)$ and S is the plane h_0. Now, if $P(C) \not\subseteq Gr$, then $C \subset P(C) \cap Gr$, but the degree of the curve $P(C) \cap Gr$ is at most 2, and then S consists of a plane or a quadric. Therefore $\dim P(C) \geq 3$.

□

Now we move on to the first question.

2.2 From a ruled surface in $\mathbb{P}(V)$ to a curve in Gr

Consider a ruled (irreducible and reduced) surface $S \subset \mathbb{P}(V)$ of some degree $d \geq 3$ (note that we already know how to classify surfaces of degree 2, and we also know which ones are ruled). Assume that S is not a cone. We define $\widetilde{C}$ to be the subset of Gr corresponding to the lines on S. Then,

Lemma 4. *$\widetilde{C}$ is the union of an irreducible curve C (not lying in some 2-plane $\sigma_2(p_0)$) of degree d and a finite, possibly empty, set.*

Proof. Consider the affine open part A_{12} of Gr given by $p_{12} \neq 0$. The points of this affine part, actually $\cong \mathbb{A}^4$, are precisely the planes in V given by the generators $\langle e_1 + ae_3 + be_4, e_2 + ce_3 + de_4 \rangle$, which correspond to the points $\overline{w} \in A_{12}$ with coordinates $[1 : c : d : -a : -b : (ad - bc)]$.

Let $F(t_1, \dots, t_4) = 0$ be the homogeneous equation of S. The intersection of $\widetilde{C}$ with this affine part consists of the tuples (a, b, c, d) such that $F(s, t, as+ct, bs+dt) = 0$ for all $(s, t) \neq (0, 0)$ (see example below). Then, the ideal generated by this polynomials in a, b, c, d defines the intersection of $\widetilde{C}$ with A_{12}. Thus $\widetilde{C}$ is Zarisky closed.

Now, we have that $\widetilde{C}$ has dimension 1, and by the above discussion, it can be written as a finite union of irreducible curves C_i, $i = 1, \dots, r$ and a finite set of points. It is only left to prove that $r = 1$. Note that the image of the projection $\{(\overline{w}, \overline{v}) \in C_1 \times \mathbb{P}(V) |\ w \wedge v = 0\} \to \mathbb{P}(V)$ is a ruled surface contained in S. But S is irreducible, so this image is precisely S. Now, assume that $r \geq 2$. Then through every point of a line $\overline{w}_2 \in C_2 \setminus C_1$ passes a line $\overline{w}_1 \in C_1$, since the above projection gives S. Hence $w_1 \wedge w_2 = 0$ for all $\overline{w}_1 \in C_1$, and therefore $w \wedge w_2 = 0$ for all $\overline{w} \in P(C_1)$ (you can take as generators of $P(C_1)$ all the points of C_1). By symmetry, $w_1 \wedge w_2 = 0$ for all $\overline{w}_i \in P(C_i)$. Since the symmetric bilinear form $(w_1, w_2) \mapsto w_1 \wedge w_2$ is not degenerate, we have that $5 = \dim(\mathbb{P}(\bigwedge^2 V)) \geq \dim(P(C_1)) + \dim(P(C_2))$, and from lemma 3 we have that $\dim(P(C_i)) \geq 3$, and this makes a contradiction. Hence, $r = 1$ and the factor f of lemma 2 is 1, so the degree of C is d.

□

As a concrete example of the proof, let's do it with the cubic ruled surface S defined by $F(t_1, \dots, t_4) = t_3 t_1^2 + t_4 t_2^2$. For convenience, we will choose the affine open part A_{14} of Gr given by $p_{14} \neq 0$. We have that $\overline{w} \in A_{14}$ iff the associated plane $W \subset V$ is generated by $\langle e_1 + ae_2 + be_3, ce_2 + de_3 + e_4 \rangle = \langle v_1, v_2 \rangle$. Hence,

$\overline{w} = \overline{v_1 \wedge v_2} = [c : d : 1 : (ad - bc) : a : b]$ are the coordinates of a point at our affine open part A_{14}.

Now, we want to look at $\widetilde{C} \cap A_{14}$. The line $\overline{w}$ lies on S iff $F(s, as+ct, bs+dt, t) = bs^3 + (d + a^2)ts^2 + 2acst^2 + c^2t^3 = 0$ for all $(s, t) \neq (0, 0)$. Then, (a, b, c, d) must satisfy $b = 0, c = 0$ and $d + a^2 = 0$. These polinomials define the intersection of $\widetilde{C} \cap A_{14}$, so to say, $\overline{w} \in \widetilde{C} \cap A_{14}$ iff $\overline{w} = [0, -a^2 : 1 : -a^3 : a : 0]$ for some $a \in K$.

In the following lemma we look how do the points in $\widetilde{C} \setminus C$ look like in our initial surface S. The answer will be that the lines $\overline{w}_0 \in \widetilde{C} \setminus C$ intersect all the lines of C.

Lemma 5. *Let $\overline{w}_0 \in \widetilde{C} \setminus C$. Then C lies in the tangent space of Gr at $\overline{w}_0$.*

Before proving the lemma, note that this is the same as saying that $\overline{w}_0$ intersect all the lines of C, because we identified $T_{Gr,\overline{w}_0} \cap Gr$ with $\sigma_1(\overline{w}_0)$, the collection of all lines $\overline{w}$ with $\overline{w}_0 \cap \overline{w} \neq \emptyset$.

Proof. Assume the opposite: if the tangent space at $\overline{w}_0$ does not contain C, then the intersection $C \cap T_{Gr,\overline{w}_0}$ consists of d points (counting multiplicity). Thus the line $\overline{w}_0$ on S intersects d lines of S, corresponding to points of C. Let $H \subset \mathbb{P}(V)$ be a plane through $\overline{w}_0$. The intersection $H \cap S$ consists of $\overline{w}_0$ and a curve Γ of degree $d-1$ (note that the union of both curves is a reducible curve of degree d). If we now look what happens in $H \subset \mathbb{P}(V)$, we have that $\Gamma \cap \overline{w}_0$ consists of $d-1$ points (counted with multiplicity), instead of the d points that we expect from above. Therefore we have a contradiction.

□

In the following, we will call the lines on S corresponding to the points of $\widetilde{C} \setminus C$ *isolated lines*. A line $\overline{w}_1$ on S is, classically, called a *directrix* if $\overline{w}_1$ meets every line $\overline{w} \in C$. Hence, an isolated line is a directrix. The opposite is, in general, false: we can think on a cone, where all the lines intersect and therefore they are all directrices.

The (Zariski closure of the) set of points of S lying on at least two non isolated lines of S will be called "*double curve*". Not that it can be something different from a curve: in an hyperboloid, is the whole surface. In a cone, is just the vertex. We will not use this concept in this talk, but it can be found in [2], p. 8.

As a remark, note that if we consider only the curve C and we consider $\widetilde{S} = \{(\overline{w}, \overline{v}) \in C \times \mathbb{P}(V) |\ w \wedge v = 0\}$, we recover the whole surface by projecting on the second component, so to say, we can recover the isolated lines from the curve C.

Corollary 1. *A ruled surface of degree $d \geq 3$ different from a cone, a plane or a quadric can have at most two isolated lines. If S has two isolated lines $\overline{w}_1, \overline{w}_2$, then $\overline{w}_1 \cap \overline{w}_2 = \emptyset$.*

Proof. From lemma 3, we have that $\dim P(C) \geq 3$. Since C lies on $\bigcap_{\overline{w} \in \widetilde{C} \setminus C} T_{Gr,\overline{w}}$ by previous lemma, we have that there can be at most two lines in $\widetilde{C} \setminus C$. This actually gives us much more information, but we will not use it.

For the second part, assume that $\overline{w}_1 \cap \overline{w}_2 \neq \emptyset$. The, C lies on $Gr \cap T_{Gr,\overline{w}_1} \cap T_{Gr,\overline{w}_2}$. According to the list of properties of Gr, (viii) part (c) tells us that $Gr \cap T_{Gr,\overline{w}_1} \cap T_{Gr,\overline{w}_2}$ is the union of two planes. One of them contains C because it is irreducible, and then we have a contradiction with $\dim P(C) \geq 3$.

□

We have developed until now some theory regarding the ruled surfaces, so we may ask ourselves what does a modern algebraic geometer understands when you tell him/her that you have a ruled surface. As everything, the answer is in Hartshorne's book, and is the first definition of Ch. V.2: a ruled surface is a surface X, together with a surjective morphism $\pi : X \to C$ to a (non singular) curve C, such that the fibre X_y is isomorphic to $\mathbb{P}^1$ for every $y \in C$. In our setting, we have a surface S

that give us a curve $C \subset Gr$ and (possibly) some isolated lines. Then we define $\widetilde{S} = \{(\overline{w}, \overline{v}) \in C \times \mathbb{P}(V) |\ w \wedge v = 0\}$, that gives us a morphism onto C. But C may have singularities, so in order to get a ruled surface (in the modern sense) we need to change the basis. The natural candidate for the change of basis is C^{norm}. Indeed, it can be proven that $\widetilde{\widetilde{S}} = \widetilde{S} \times_C C^{norm}$ together with $\widetilde{\widetilde{S}} \twoheadrightarrow C^{norm}$ is a ruled surface in the modern sense. You can find the proof (not so difficult) in [1]. Moreover, in [1] we have the following proposition, that gives us much more information:

Proposition 1. *1. $pr_2 : \widetilde{S} \to S$ is a birational morphism. Let $C^{norm} \to C$ be the normalization of C and let $\widetilde{\widetilde{S}} = \widetilde{S} \times_C C^{norm}$ be the pullback of $\widetilde{S} \to C$. Then $\widetilde{\widetilde{S}} \to C^{norm}$ is a ruled surface (in the modern sense) and $\widetilde{\widetilde{S}} \to S$ is the normalization of S.*

2. *The singular locus of S is purely 1-dimensional or empty.*

3. *Suppose that the line $\overline{w}$ belongs to the singular locus of S and does not correspond to a singular point of C. Then C lies in the tangent space of Gr at the point $\overline{w}$.*

Lemma 6. *Suppose that $\dim P(C) = 3$ and that $P(C)$ is the intersection of two tangent spaces of Gr at points $\overline{w}_1 \neq \overline{w}_2$. Then the lines $\overline{w}_1$ and $\overline{w}_2$ do not intersect. For a suitable choice of coordinates (t_1, t_2, t_3, t_4) of $\mathbb{P}(V)$, the equation F of S is bi-homogeneous of degree (a_1, a_2), with $a_1 + a_2 = d$, in the pairs t_1, t_2 and t_3, t_4. Further, $\widetilde{C} \setminus C = \{\overline{w}_1, \overline{w}_2\}$.*

The lines $\overline{w}_1$, $\overline{w}_2$ are "directrices". The singular locus of S consists of the lines $\overline{w}_i$ with $a_i > 1$ and for each singular point $\overline{w} \in C$, the line $\overline{w} \subset S$.

We only prove part of the lemma.

Proof. As in Corollary 1, the assumption that the lines $\overline{w}_1, \overline{w}_2$ intersect leads to a contradiction. Now, take $w_1 = e_{12}$ and $w_2 = e_{34}$, then the elements of $P(C) = T_{Gr,\overline{e_{12}}} \cap T_{Gr,\overline{e_{34}}}$ have coordinates $[0 : p_{13} : p_{14} : p_{23} : p_{24} : 0]$ (cf property (iii) of the list of properties), so C lies on the quadric surface $Gr \cap P(C)$, given by $-p_{13}p_{24} + p_{14}p_{23} = 0$. Note that this is the equation of the Segre embedding, so we can identify this quadric with $\mathbb{P}^1 \times \mathbb{P}^1$, and then we have that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree (a_1, a_2), with $a_1 + a_2 = d$.

Now, consider the rational map $f : \mathbb{P}(V) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by $(t_1, t_2, t_3, t_4) \mapsto ((t_1, t_2), (t_3, t_4))$. It is defined outside the lines $\overline{w}_1, \overline{w}_2$. The surface S is the Zarisli closure of $f^{-1}(C)$ and so the equation F of S is bi-homogeneous and coincides with the equation for $C \subset \mathbb{P}^1 \times \mathbb{P}^1$. The other statments are not proved.

□

As a remark, assume now that $\dim P(C) = 3$ but $P(C)$ is only in a single tangent space of Gr. In this case, we can also choose coordinates in order to get that $P(C) \subset T_{Gr,\overline{e_{12}}}$ and $P(C)$ is given by the equations $p_{34} = 0, p_{13} + p_{24} = 0$, and here we have that a point of Gr lies on $P(C)$ if it has the form $[p_{12} : p_{13} : p_{14} : p_{23} : -p_{13} : 0]$, so we can take $p_{12}, p_{13}, p_{14}, p_{23}$ as coordinates of $P(C)$. Then, $Gr \cap P(C)$ is the cone with equation $p_{13}^2 + p_{14}p_{23} = 0$ with vertex $\overline{e_{12}}$. Since C lies on this cone, we have the rational map $f : C \dashrightarrow E := \{p_{13}^2 + p_{14}p_{23} = 0\}$. This map can be identified with the rational map $C \dashrightarrow \overline{e_{12}}$ given by $\overline{w} \mapsto \overline{w} \cap \overline{e_{12}}$.

The rational map f is a morphis if $\overline{w_{12}} \notin C$ or if $\overline{e_{12}} \in C$ and this is a regular point of C. Let e be the degree of the morphism f.

In case $\overline{e_{12}} \notin C$, take two unramified points $e_1, e_2 \in E$ (ie points such that their preimage via f gives e different points) and the plane through the corresponding two lines. This plane meets C in $2e$ points. Hence, $d = 2e$. In case $\overline{e_{12}} \in C$ and is not a singular points, we have that $d - 1 = 2e$.

2.3 The possibilities for the singular locus

Now, we want to study the singular locus of the curve $C \subset Gr$. In general, we do the following: if S is a ruled surface, we consider $Q := S \cap H$ with $H \subset \mathbb{P}(V)$ a general plane. By Bertini's theorem, Q is an irreducible reduced curve of degree d. The morphism $C \to Q$, given by $\overline{w} \mapsto \overline{w} \cap H \in Q$, is birational. Thus C^{norm} is also the normalization of Q. We can write the singular locus of S as a union of its irreducible components C_i, with degree d_i and multiplicity m_i. Note that the curve Q meets every C_i in d_i points with multiplicity m_i (if you prefer, you can think on the plane H intersecting the curve C_i, but since $C_i \subset S$, $H \cap C_i = Q \cap C_i$). Hence, with the Plücker formulae it is possible to write the genus of C^{norm} in terms of the degree of Q and the multiplicity of its singular points.

But for $d = 3$, the work is easy, since the only singularities of a cubic curve are a node or a cusp. Therefore the singular locus consists on one curve (C has at most one singularity) of degree 1 and of multiplicity 2 (the singularity is only a node or a cusp).

For $d = 4$ there are more possibilities.

2.4 Ruled surfaces of degree 3

Now we can classify the ruled surfaces of degree 3. For this, we will assume that the characteristic of the base field is different from 2. Now we know that the singular locus of S is a line, and from the Plücker formulae that we have ommited, it can be shown that C^{norm} has genus 0. From lemma 3, we know also that $P(C) = 3$. This implies that C is the twisted cubic in $P(C)$, and hence $C = C^{norm}$. We have two possibilities according to $P(C)$. In the first case, $P(C)$ is the intersection of two tangent spaces, and in the second one $P(C)$ lies just in one tangent space.

In the first case, where $P(C)$ lies on $T_{Gr,\overline{w}_1}$ and $T_{Gr,\overline{w}_2}$. We know from lemma 6 that S is given by a bi-homogeneous equation F in the pairs of variables t_1, t_2 and t_3, t_4 of bi-degree $(2, 1)$, corresponding to a morphism $f : \overline{w}_1 \to \overline{w}_2$ of degree 2. The line $\overline{w}_1$ is non singular and a "directrix". The line $\overline{w}_2$ is the singular locus. Further, $\widetilde{C} \setminus C = \{\overline{w}_1, \overline{w}_2\}$. If we assume that the field is algebraically closed, then we can find a basis such that f has the form $(t_1, t_2, 0, 0) \mapsto (0, 0, t_2^2, -t_1^2)$.

In the second case $P(C)$ lies in only one tangent space, namely at the point $\overline{w}_0$, which is the singular line of S. Then C lies on the quadratic cone in $P(C)$ and

$\overline{w}_0 \in C$. In this case, $\widetilde{C} = C$. Now C and S can be put in a standard form, and we arrive to the following result:

Proposition 2. *The standard equations for ruled cubic surfaces S over an algebraically closed fields which are not cones are the following:*

- $t_3t_1^2 + t_4t_2^2 = 0$.
- $t_3t_1t_2 + t_4t_1^2 + t_2^3 = 0$.

We already found a parametrization of the C associated to the first surfac. Let's write both parametrizations.

The first one is $[0 : -t^2 : 1 : -t^3 : t : 0]$, and the second one is $[0 : t^3 : t^2 : -t^2 : -t : -1]$.

If the field is $\mathbb{R}$, we obtain in the first case another equation, so to say $t_3(t_1t_2) + t_4(-t_1^2 + t_2^2)$.

References

[1] I. Polo-Blanco, M. van der Put and J. Top. *Ruled quartic surfaces, models and classification.* Geom Dedicata, 151-180. 2011.

[2] W. L. Edge. *The theory of ruled surfaces.* Cambridge University Press, Cambridge. 1931.

[3] H. T. Muhly and O. Zarisky. *The resolution of singularities of an algebraic curve.* Amer. J. Math., 61 (1): 107-114. 1939.

[4] Hartshorne. *Algebraic Geometry.* Springer. 1977.