Pyramid Mysteries

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In this article I describe the 3-dimensional version of the phenomenon explored in "Triangle Mysteries" ([1]). There we start with a row of n squares arbitrarily colored red, green, or blue. By a simple rule they generate a row of $n-1$ colored squares, and we continue until finally one square remains. The mystery: for certain n the color of the final square can easily be predicted.

In the present article we begin with an $n \times n$-grid of colored cubes. By a certain rule this gives rise to an $(n-1) \times (n-1)$-grid of colored cubes that form the second layer of a pyramid. One continues with an $(n-2) \times (n-2)$-grid, then an $(n-3) \times (n-3)$-grid, etc., until finally a pyramid is constructed. We will show that for suitable n we can predict the color of the top cube easily just by studying the first $n \times n$-grid.

We include the discussion of a variant where we start with a triangle formed by colored cubes that is considered to be the base of a pyramid. The various layers are again constructed by a simple rule. And also here we can predict in certain cases the color of the cube on the top.

The proofs are elementary, using only known properties of binomial coefficients $\binom{n}{k}$ when n is a power of a prime.

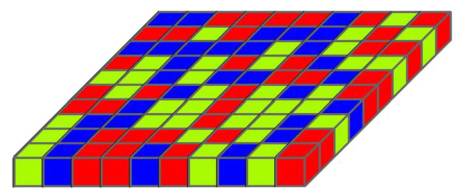
Our results cover the case of arbitrarily many dimensions, but we will mainly concentrate on the 3-dimensional case. For the formal approach we simply need a finite set Δ (the set of "colors") and a map that associates with 3 (resp. 4) colors another one, in other words, ϕ is a map from Δ^3 (resp. Δ^4) to Δ; it will be used to determine the color of the next cube that will be put on top of three resp. four colored cubes of the preceding layer of the pyramid.

After ϕ is given we can start to build pyramids.

Pyramids where the base is a square

Let Δ, a map $\phi : \Delta^4 \to \Delta$, and an $n \geq 2$ be given. We start with the base of our pyramid, the first layer. It consists of a square made of n^2 cubes. The cubes are colored with the elements of Δ. Formally such an arrangement is a square matrix $(x_{i,j})_{i,j=0,\ldots,n-1}$ with $x_{i,j} \in \Delta$.

Here is an example with $n = 10$ and $\Delta = \{r, g, b\}$ where these letters stand for the colors red, green, and blue. The first layer could look like this.



The first layer, the colors are chosen at random.

Next we are going to construct the second layer. We will use the following rule:

> Where four cubes with colors c_1, c_2, c_3, c_4 of the first layer meet, put on top of them in the middle another cube the color of which is $\phi(c_1, c_2, c_3, c_4)$. (It will be convenient here to consider only such ϕ where all permutations of c_1, c_2, c_3, c_4 are mapped to the same element of Δ. Thus it is not necessary to specify which of the four cubes is associated with c_1, etc.)

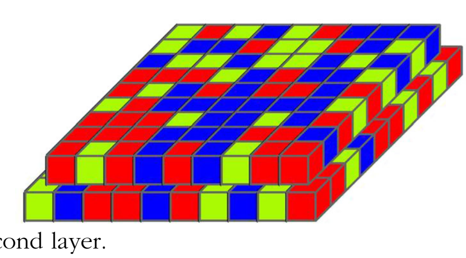
This means that we generate a new square of cubes with colors $(y_{i,j})_{i,j=0,\ldots,n-2}$ by the formula

$$y_{i,j} := \phi(x_{i,j}, x_{i+1,j}, x_{i,j+1}, x_{i+1,j+1}).$$

For our example (see the preceding picture) we consider the following map:

> The elements of Δ are identified with the numbers 0, 1, 2 of the group $\mathbb{Z}_3$ by $0 = r$, $1 = g$, $2 = b$, and $\phi(c_1, c_2, c_3, c_4) := c_1 + c_2 + c_3 + c_4 \bmod 3$. (Thus, e.g., $\phi(b, b, r, g) = b$.)

For this particular ϕ the second layer of our pyramid looks like this:



The second layer.

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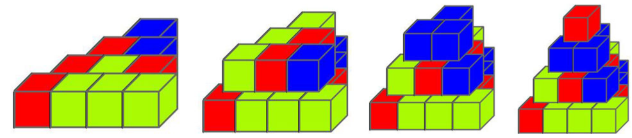
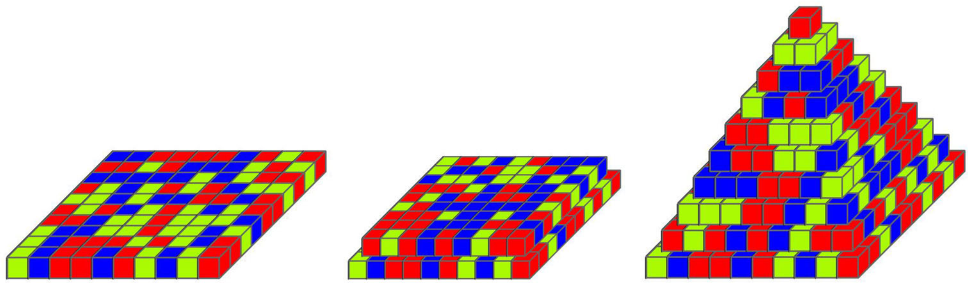
To continue, we repeat the process. We construct smaller and smaller square layers of cubes until finally we arrive at the single top cube.

Now the question becomes: Is it possible to predict in a simple way the color of this top cube just by checking the colors of the first layer?

In the picture below we see on the right the final pyramid. The cube on the last layer is red: is it possible to know this in advance?



The layers of the pyramid with triangular base in the case $n = 4$.



One, two, and ten layers (from left to right).

Motivated by our investigations in [1] we will call an integer n *ϕ-simple* if the top color c is always just the ϕ-value of the corner cubes of the ground layer, i.e., if $c = \phi(x_{0,0}, x_{n-1,0}, x_{0,n-1}, x_{n-1,n-1})$ holds. Thus, trivially, $n = 2$ is always ϕ-simple.

We will show below that $n = 10$ is ϕ-simple for the ϕ defined above so that – since $\phi(b, g, r, r) = r$ – it is no surprise that red is the top color.

Pyramids where the base is a triangle

This time we start with Δ, a $\phi : \Delta^3 \to \Delta$, and an $n \geq 2$. First we choose any triangular base of our pyramid, the first layer. It consists of a triangle made of cubes: 1 in the row 0, then 2 in row 1, etc., until finally there is a row of n cubes. The cubes are colored with the elements of Δ, i.e., we are given a triangular matrix $(x_{i,j})_{0 \leq j \leq i \leq n-1}$ with $x_{i,j} \in \Delta$. ($x_{i,j}$ is the color of cube j in the row i.)

As before, we work with $\Delta = \{r, g, b\}$, and in the example we are going to discuss we have $n = 4$. The first layer was generated with the help of a random generator; it can be seen on the left-hand side in the next picture. We continue in a similar way as in the case of pyramids with a square base:

Where three cubes with colors c_1, c_2, c_3 of the first layer meet put on top of them another cube the color of which is $\phi(c_1, c_2, c_3)$.

This means that we generate a new triangle of cubes with colors $(y_{i,j})_{0 \leq j \leq i \leq n-2}$ by the formula $y_{i,j} := \phi(x_{i,j}, x_{i+1,j}, x_{i+1,j+1})$.

In our example we identify Δ with $\mathbb{Z}_3$ and define ϕ by $\phi(c_1, c_2, c_3) := c_1 + c_2 + c_3 \mod 3$ (so that, e.g., $\phi(b, b, r) = g$.)

We continue to construct smaller and smaller layers until finally there is only one cube on the top. The four layers of the pyramid of our example can be seen here:

As for pyramids with a square base, we are interested in situations where the top color can directly be determined from the corner colors of the base. More specifically we will call the number n *ϕ-simple* if for arbitrary choices of $(x_{i,j})_{0 \leq j \leq i \leq n-1}$ the top color c is given by the formula $c = \phi(x_{0,0}, x_{n-1,0}, x_{n-1,n-1})$. We will prove that 4 is ϕ-simple for our ϕ, so that we can predict immediately in our example that the top color must be red.

Our main results concerning ϕ-simple n can be found in the next two sections. In the last section we discuss some generalizations.

Pyramids Where the Base Is a Square

Let us now set the stage for our theoretical development. From now on we will assume that

- Δ is a nontrivial finite abelian group with respect to the operation "+".
- The mapping $\phi : \Delta^4 \to \Delta$ is defined by $(c_1, c_2, c_3, c_4) \mapsto c_1 + c_2 + c_3 + c_4$.

In the above example we worked with $(\mathbb{Z}_3, +)$ as the group Δ.

To prepare what follows we introduce some notation. Here $m,n \geq 2$ denote arbitrary integers.

The mappings Φ_m

$\Phi_m : \Delta^{m^2} \to \Delta^{(m-1)^2}$ is the map that defines the next layer: we have $\Phi_m\left((x_{i,j})_{i,j=0,\ldots,m-1}\right) := (y_{i,j})_{i,j=0,\ldots,m-2}$, where

$$y_{i,j} := \phi(x_{i,j}, x_{i+1,j}, x_{i,j+1}, x_{i+1,j+1}).$$

With this definition the top color c when starting with $(x_{i,j})_{i,j=0,\ldots,n-1}$ is

$$c = \Psi_n\Big((x_{i,j})_{i,j=0,\ldots,n-1}\Big) := \Phi_2 \circ \cdots \circ \Phi_{n-1} \circ \Phi_n\Big((x_{i,j})_{i,j=0,\ldots,n-1}\Big).$$

(In other words, Ψ_n maps the pattern in the first layer to the top color.)

The $\sigma^m_{k,l;x}$

For $x \in \Delta$ and $k, l = 0, \ldots, m-1$, we denote by $S^m_{k,l;x}$ that element $(x_{i,j}) \in \Delta^{m^2}$ where $x_{k,l} = x$ and the other $x_{i,j}$ are zero. $\sigma^m_{k,l;x} \in \Delta$ is defined to be the top color when we work with ϕ, i.e., $\sigma^m_{k,l;x} = \Psi_m(S^m_{k,l;x})$.

It will be clear soon why it is convenient to extend this definition: we put $\sigma^m_{k,l;x} := 0$ for $(k,l) \in \{-1, m\} \times \{-1, \ldots, m\}$ and $(k,l) \in \{-1, \ldots, m\} \times \{-1, m\}$. (These indices extend the $\{0, \ldots, m-1\}^2$-pattern to a $\{-1, \ldots, m\}^2$-pattern, and the σ for the new indices are zero.)

The $\sigma^n_{k,l;x}$ can be explicitly determined:

Lemma 1 For $k, l = 0, \ldots, n-1$ we have $\sigma^n_{k,l;x} = \binom{n-1}{k} \cdot \binom{n-1}{l} x$.

Proof The proof is by induction on n. In the case $n = 2$ we have $\sigma^2_{k,l;x} = \phi(x, 0, 0, 0) = x = \binom{1}{k} \cdot \binom{1}{l} x$ for $k, l \in \{0, 1\}$. Now suppose that the lemma is proved for some number $n-1$ with $n \geq 3$. Let us analyze the first step in the calculation of $\sigma^n_{k,l;x}$: we pass from the first layer $S^n_{k,l;x}$ to $\Phi_n(S^n_{k,l;x})$. This is an $(n-1) \times (n-1)$-matrix with the entry x at one, two, or four positions (the other entries are zero). For example, if $0 < k,\ l < n-1$, then the x are at the four positions (k, l), $(k-1, l-1)$, $(k-1, l)$, and $(k, l-1)$. Because Φ_{n-1} is a homomorphism (from $\Delta^{(n-1)^2}$ to Δ), we arrive at the equation

$$\sigma^n_{k,l;x} = \sigma^{n-1}_{k,l;x} + \sigma^{n-1}_{k-1,l;x} + \sigma^{n-1}_{k,l-1;x} + \sigma^{n-1}_{k-1,l-1;x}.$$

We note that we can use the same formula if we adopt the extended definition of the $\sigma^n_{k,l;x}$: e.g., $\sigma^n_{0,0;x} = \sigma^{n-1}_{0,0;x}$, and this is covered by the formula since $\sigma^n_{-1,0;x} = \sigma^n_{0,-1;x} = \sigma^n_{-1,-1;x} = 0$. By the induction hypothesis we know that the σ on the right-hand side can be expressed by binomial coefficients so that

$$\sigma^n_{k,l;x} = \left(\sum_{k'=k-1,k;\ l'=l-1,l} \binom{n-2}{k'} \binom{n-2}{l'} \right) x.$$

But this sum coincides with $\binom{n-1}{k}\binom{n-1}{l} x$, as can easily be deduced from the identity $\binom{n-2}{m-1} + \binom{n-2}{m} = \binom{n-1}{m}$.

We now are going to show that the ϕ-simple integers can be characterized:

Proposition 2 Suppose that $(\Delta, +)$ is isomorphic to $(\mathbb{Z}_p, +)^d$ for a prime p and an integer d. Then a number n is ϕ-simple iff there is an $s \in \mathbb{N}$ such that $n = p^s + 1$.

Proof The key tool will be – as in [1] – Balak Ram's result [4] on binomial coefficients:

- Let p be a prime and m an integer. Then all $\binom{m}{l}$ for $l = 1, \ldots, m-1$ are divisible by p iff there is an s such $m = p^s$.
- Let m, r be integers such that $m > r > 1$. If r divides all $\binom{m}{l}$ for $l = 1, \ldots, m-1$ then r is a prime and – by the first part – m is of the form r^s.

A proof can be found in [1] and [4], and for a far-reaching generalization we refer the reader to [3].

Now let n be given. We observe that $\Psi_n : \Delta^{n^2} \to \Delta$ is a group homomorphism when we consider Δ^{n^2} as a product group. This has the following consequence:

$$\begin{aligned}\Psi_n\Big((x_{i,j})_{i,j}\Big) &= \Psi_n\left(\sum_{i,j} S^n_{i,j;x_{i,j}}\right) \\ &= \sum_{i,j} \Psi_n\Big(S^n_{i,j;x_{i,j}}\Big) \\ &= \sum_{i,j} \sigma^n_{i,j,x_{i,j}} \\ &= \sum_{i,j} \binom{n-1}{i}\binom{n-1}{j} x_{i,j}.\end{aligned}$$

By definition, an n is ϕ-simple iff the preceding sum always coincides with $\phi(x_{0,0}, x_{n-1,0}, x_{0,n-1}, x_{n-1,n-1}) = x_{0,0} + x_{n-1,0} + x_{0,n-1} + x_{n-1,n-1}$, i.e., if $\sum_{(i,j)\notin A} \binom{n-1}{i} \cdot \binom{n-1}{j} x_{i,j} = 0$, where A denotes the set consisting of the four elements $(0, 0)$, $(n-1, 0)$, $(0, n-1)$, $(n-1, n-1)$. And this is obviously true iff $\binom{n-1}{i}\binom{n-1}{j} x = 0$ for all x and for all i, j with $(i, j) \notin A$.

Suppose that n is of the form $p^s + 1$. Then, by the first part of Ram's result, p divides all $\binom{n-1}{l}$ for $l = 1, \ldots, n-2$, and consequently all $\binom{n-1}{i}\binom{n-1}{j}$ with $(i, j) \notin A$ are divisible by p. But $px = 0$ for all x since $(\Delta, +)$ is isomorphic with $(\mathbb{Z}^p, +)^d$. This shows that numbers of the form $p^s + 1$ are ϕ-simple.

Suppose now that n is *not* of the form $p^s + 1$. By Ram's result we find a $k \in \{1, \ldots, n-2\}$ such that $\binom{n-1}{k}$ is not divisible by p so that $\alpha := \binom{n-1}{k} \neq 0$ in $\mathbb{Z}_p$. Let x be any nonzero element in $(\mathbb{Z}_p)^d$. Then

$$\begin{aligned}\phi(0,0,0,0) &= 0 \\ &\neq \alpha^2 x \\ &= \Psi_n(S^n_{k,k;x}).\end{aligned}$$

Hence n is not ϕ-simple.

Now we understand why our examples in the first section worked with $n = 4$ and $n = 10$: the "good" n here are the integers of the form $3^s + 1$.

For another example, we work with $\Delta = (\mathbb{Z}_2, +)$ (0 is "red" and 1 is "green"). Note that in this case the ϕ-simple n are the integers of the form $2^s + 1$, and therefore it is no surprise that the top color is red after we have seen the base:

A pyramid where $n = 3$ with $\Delta = \mathbb{Z}_3$ and ϕ.

In this case it is simple to translate the definition of ϕ without using the arithmetic in $\mathbb{Z}_2$: find $c = \phi(c_1, c_2, c_3, c_4)$ such that the total number of green balls among c, c_1, c_2, c_3, c_4 is even.

Suppose that n is not ϕ-simple in the preceding case. Then it might happen that $\Psi_n\big((x_{i,j})\big) \neq \phi(x_{0,0}, x_{n-1,0}, x_{0,n-1}, x_{n-1,n-1})$. How often will this be the case? The answer can be found in

Proposition 3 Let $(\Delta, +)$ be again isomorphic to $(\mathbb{Z}_p, +)^d$ for a prime p. Suppose that n is not ϕ-simple. If $N = p^d$ denotes the cardinality of Δ, then the following is true: the cardinality of the $(x_{i,j}) \in \Delta^{n^2}$ where $\Psi_n\big((x_{i,j})\big) = \phi(x_{0,0}, x_{n-1,0}, x_{0,n-1}, x_{n-1,n-1})$ holds divided by the cardinality of Δ^{n^2} is precisely $1/N$. To state it otherwise: if we want to "predict" the top color by giving the guess $\phi(x_{0,0}, x_{n-1,0}, x_{0,n-1}, x_{n-1,n-1})$ we have a chance of $1/N$ of being correct.

Proof Let n be an integer and $(x_{i,j})_{i,j} \in \Delta^{n^2}$ be arbitrary. The prediction that the top color is $\phi(x_{0,0}, x_{n-1,0}, x_{0,n-1}, x_{n-1,n-1})$ will be correct iff

$$\sum_{i,j,(i,j)\notin A} \binom{n-1}{i}\binom{n-1}{j} x_{i,j} = 0.$$

(The set A of indices is as in the proof of proposition 2.)

We observe that the map $\Psi : (x_{i,j}) \mapsto \sum_{i,j,(i,j)\notin A} \binom{n-1}{i}\binom{n-1}{j} x_{i,j}$ is a group homomorphism (from Δ^{n^2} to Δ), and since n is not ϕ-simple it is not the trivial homomorphism. It follows that there must be a pair $(k, l) \notin A$ and an $x \in \Delta$ such that $\binom{n-1}{i}\binom{n-1}{j} x \neq 0$. This implies that $r := \binom{n-1}{i} \cdot \binom{n-1}{j} \bmod p \neq 0$, and since $\mathbb{Z}_p$ is a field, we may select $r' \in \mathbb{Z}_p$ with $rr' = 1p$. It is now easy to show that $(x_{i,j}) \mapsto \sum_{i,j,(i,j)\notin A} \binom{n-1}{i}\binom{n-1}{j} x_{i,j}$ from Δ^{n^2} to Δ is onto: a given $y \in \Delta$ has $S^n_{k,l;r'y}$ as a preimage. From this we may conclude that the number of elements in the kernel of Ψ is N^{n^2-4}/N, and because there are N^4 possible choices for the $x_{i,j}$ with $(i,j) \in A$, the proposition is proved.

Only the groups considered in the preceding propositions admit ϕ-simple integers:

Proposition 4 Suppose that $(\Delta, +)$ is not isomorphic to any of the groups of the preceding proposition. Then there are no ϕ-simple $n > 2$.

Proof $(\Delta, +)$ is a finite commutative group so that it is a product of cyclic groups $\mathbb{Z}_{r_i}$ where the r_i are prime powers.

Case 1: There is a factor $\mathbb{Z}_{p^s}$ with $s > 1$. A ϕ-simple n w.r.t. Δ would also be ϕ-simple w.r.t. the subgroup $\mathbb{Z}_{p^s}$, i.e., $\binom{n-1}{i}$ would be divisible by p^s for $i = 1, \ldots, n-2$. But there are no such $n > 2$.

Case 2: $(\Delta, +)$ *is a product of at least two subgroups of the form* $(\mathbb{Z}_{p_\alpha})^{r_\alpha}$ with different primes p_α. Suppose that a ϕ-simple integer $n > 2$ exists. We conclude from the second part of Ram's result that n can be written as $p_\alpha^{s_\alpha} + 1$ for every α and suitable s_α. But this is surely not possible for more than one p_α.

Pyramids Where the Base Is a Triangle

All the results of the preceding paragraph have an analogue. The modifications are the following:

- For the finite abelian group $(\Delta, +)$ with at least two elements, we consider the mapping $\phi : \Delta^3 \to \Delta$ defined by $(c_1, c_2, c_3) \mapsto c_1 + c_2 + c_3$.
- A triangular pattern (the layers of the pyramid) of elements of Δ needs $m + (m-1) + \cdots + 1 = m(m+1)/2 =: \delta(m)$ entries. Thus a typical pattern is given by $(x_{i,j})_{0 \le j \le i \le m-1}$, this describes a triangle where the last row has m elements. Consequently the passage to the next layer is described by a map $\Phi_m : \Delta^{\delta(m)} \to \Delta^{\delta(m-1)}$; it is defined by $\Phi_m\Big((x_{i,j})_{0\le j\le i\le m-1}\Big) = (y_{i,j})_{0\le j\le i\le m-2}$, where $y_{i,j} = \phi(x_{i,j}, x_{i+1,j}, x_{i+1,j+1})$.
- Ψ_n, the map that assigns the top color to the pattern of the ground layer, is again defined as
$$\Phi_2 \circ \cdots \circ \Phi_n : \Delta^{\delta(n)} \to \Delta.$$
The number n will be called *ϕ-simple* if always $\Psi_n\Big((x_{i,j})_{0\le j\le i\le n-1}\Big) = \phi(x_{0,0}, x_{n-1,0}, x_{n-1,n-1})$ holds.
- For $0 \le l \le k \le m-1$, the pattern $\tilde{S}^m_{k,l;x} \in \Delta^{\delta(m)}$ is defined to have x at the position (k, l) and all other entries to be zero. $\tilde{\sigma}^m_{k,l;x}$ denotes the element $\Psi_m(\tilde{S}^m_{k,l;x}) \in \Delta$.

- It will then be crucial that $\tilde{\sigma}^m_{k,l;x} = \binom{m-1}{k}\binom{k}{l}x$ for all k, l. This is again proved by induction, using simple properties of binomial coefficients.

With these preparations at hand we can prove the following results for pyramids with a triangular base similarly as in the previous section:

Proposition 5 Suppose that $(\Delta, +)$ is isomorphic to $(\mathbb{Z}_p, +)^d$ for a prime p and an integer d. Then a number n is ϕ-simple iff there is an $s \in \mathbb{N}$ such that $n = p^s + 1$.

In the next picture we illustrate this proposition by a pyramid built with balls: note that 4 is ϕ-simple in this example. The colors r, g, b correspond to 0, 1, 2 as above.

A pyramid: triangular base, $n = 4, \Delta = \mathbb{Z}_3$.

Proposition 6 Let $(\Delta, +)$ be again isomorphic to $(\mathbb{Z},+)^d$ for a prime p. Suppose that n is not ϕ-simple. If $N = p^d$ denotes the cardinality of Δ, then the following is true: the cardinality of the $(x_{i,j}) \in \Delta^{\delta(n)}$ where $\Psi_n\big((x_{i,j})\big) = \phi(x_{0,0}, x_{n-1,0}, x_{n-1,n-1})$ holds divided by the cardinality of $\Delta^{\delta(n)}$ is precisely $1/N$.

Proposition 7 Suppose that $(\Delta, +)$ is not isomorphic to any of the groups of the preceding proposition. Then there are no ϕ-simple $n > 2$.

Magic in Hyperspace and More General ϕ

In the first part of this section we note that we can naturally generalize our theory to arbitrarily many dimensions D: magicians in hyperspace can present the same tricks! The preceding results of the present article and the results of [1] correspond to $D = 3$ and $D = 2$, respectively.

Hyperpyramids where the base is a hypersquare

Let us fix an integer D, an $n \in \mathbb{N}$, and a nontrivial finite abelian group $(\Delta, +)$. We define, with $D' := 2^{D-1}$, a mapping $\phi_{D'} : \Delta^{D'} \to \Delta$ by $(c_i)_{i \in D'} \mapsto \sum_i c_i$.

We are going to construct a hyperpyramid consisting of colored hypercubes. The "ground layer" is made from $n^{D'}$ hypercubes, their colors are given by $(x_{i_1,i_2,\dots,i_{D-1}}) \in \Delta^{n^{D-1}}$, $i_1, \dots, i_{D-1} = 0, \dots, n-1$. Wherever D' hypercubes meet we put "on top of them" another hypercube with color determined by $\phi_{D'}$. More formally: we map the pattern $(x_{i_1,i_2,\dots,i_{D-1}})_{i_1,\dots,i_{D-1}=0,\dots,n-1}$ to $(y_{i_1,i_2,\dots,i_{D-1}})_{i_1,\dots,i_{D-1}=0,\dots,n-2}$, where

$$y_{i_1,i_2,\dots,i_{D-1}} := \phi_{D'}\Big(\big(x_{i_1+j_1,i_2+j_2,\dots,i_{D-1}+j_{D-1}}\big)_{j_1,\dots,j_{D-1}\in\{0,1\}}\Big).$$

In this way we continue, after n steps we arrive at the top layer that consists of only one hypercube. If its color is always $\phi_{D'}\Big(\big(x_{i_1,i_2,\dots,x_{D-1}}\big)_{i_1,\dots,i_{D-1}\in\{0,n-1\}}\Big)$ then we will call n a $\phi_{D'}$-*simple integer*: the "top" color can be predicted easily from the corner colors of the first layer.

It can be shown with similar techniques as in the preceding sections that the $\phi_{D'}$-simple n are precisely the $p^s + 1$ if $(\Delta, +)$ is a power of $\mathbb{Z}_p$ and that there are no such $n > 2$ for other Δ.

The key result here is the fact that the final color is

$$\binom{n-1}{i_1}\binom{n-1}{i_2}\cdots\binom{n-1}{i_{D-1}}x$$

if the ground layer has color x at the position $(i_1, \dots, i_{D-1})$ and color zero at the other places.

Hyperpyramids where the base is hypertriangular

A *hypertriangular array of size n* is a family

$$(x_{i_1,\dots,i_{D-1}})_{0 \le i_{D-1} \le i_{D-2} \le \dots \le i_2 \le i_1 \le n-1}$$

where the x are in Δ. We map such an array to one of size $n-1$ by using a rule that generalizes the rule for 3-dimensional pyramids with a triangular base, and after $n-1$ steps we arrive at the "top" of a D-dimensional hyperpyramid with base colors $(x_{i_1,\dots,i_{D-1}})$. Again we are able to identify those n where we can predict the top color in a simple way: as before, precisely the $p^s + 1$ have this property in the case $\Delta = (\mathbb{Z}_p, +)^d$. The crucial lemma is the assertion that the top color is

$$\binom{n-1}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{D-2}}{i_{D-1}}x$$

if the first layer has x at the position $(i_1, \dots, i_{D-1})$ and the other entries are zero.

More general ϕ

So far we have assumed that ϕ is given by the sum of the input colors. This approach has the advantage that the definition of ϕ is "commutative": all permutations of the entries lead to the same value. This makes it easier to translate the rules to an audience of nonmathematicians, but it is in fact not necessary.

Consider the simplest example, where $D = 2$ and $\Delta = (\mathbb{Z}_p, +)$. In [1] we worked with the $\phi : \Delta^2 \to \Delta$ defined by $(c_1, c_2) \mapsto c_1 + c_2$ and $(c_1, c_2) \mapsto -c_1 - c_2$. Now fix any nonzero $\alpha, \beta \in \mathbb{Z}_p$ and define $\phi_{\alpha,\beta} : \Delta^2 \to \Delta$ by $(x, y) \to \alpha x + \beta y$. Here we consider Δ as a vector space over $\mathbb{Z}_p$, and we note that $\phi_{\alpha,\beta}$ is the most general group homomorphism from Δ^2 to Δ; the preceding cases correspond to the choice $\alpha = \beta = 1$ and $\alpha = \beta = -1$, respectively.

Fix an n and a family $(x_i)_{i=0,\ldots,n-1} \in \Delta^n$. We will consider these x_i as in [1] as the colors of the first row of colored squares of a triangle, and with the help of $\phi_{\alpha,\beta}$ we build a second row, then a third one, and so on: finally a single square, the bottom square of the triangle, will be found. Surprisingly we can predict this color as before for the integers of the form $p^s + 1$, the color will be $\phi_{\alpha,\beta}(x_0, x_{n-1})$.

For the proof we have to combine the following facts:

- Consider a starting pattern $(x_i)_{i=0,\ldots,n-1}$ where there is color x at position k and all other x_i are zero. Then the top color is $\binom{n-1}{k}\alpha^{n-1-k}\beta^k x$. This can be proved by induction.
- The "little Fermat theorem": one has $\gamma^p = \gamma$ for all $\gamma \in \mathbb{Z}_p$.

It is then possible to show the claim with the same techniques that we have used previously. Similarly we can pass in the case of 3 dimensions from our map ϕ of the first section to

$$\phi_{\alpha,\beta,\gamma,\delta}(c_1, c_2, c_3, c_4) := \alpha c_1 + \beta c_2 + \gamma c_3 + \delta c_4,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_p$ are fixed nonzero elements. The same results will hold: the n of the form $p^s + 1$ are always "good" numbers, and if all α, β, γ, δ are different from zero there are no others.

Here is an example: a pyramid with a square base where $n = 4$. We work with $\Delta = \mathbb{Z}_3$ and the usual translation: 0, 1, 2 correspond to r, g, b, and we consider ϕ defined by $\phi : (c_1, c_2, c_3, c_4) \mapsto -c_1 - c_2 - c_3 - c_4$. It is no surprise that the top color must be blue:

A pyramid: square base, $n = 4, \Delta = \mathbb{Z}_3$ and $\alpha = \beta = \gamma = \delta = -1$.

The next step would be to treat similar generalizations for D dimensions, but we omit the clumsy technical details here.

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