Products of n open subsets in the space of continuous functions on $[0, 1]$

(revised version)

Ehrhard Behrends

Abstract. Let $O_1, \ldots, O_n$ be open sets in $C[0,1]$, the space of real-valued continuous functions on $[0,1]$. The product $O_1 \cdot O_2 \cdots O_n$ will in general not be open, and in order to understand when this can happen we study the following problem: given $f_1, \ldots, f_n \in C[0,1]$, when is it true that $f_1 \cdot f_2 \cdots f_n$ lies in the interior of $B_\varepsilon(f_1) \cdot B_\varepsilon(f_2) \cdots B_\varepsilon(f_n)$ for all $\varepsilon > 0$? (B_ε denotes the closed ball with radius ε and center f.)

The main result of this paper is a characterization in terms of the walk $t \mapsto \gamma(t) := (f_1(t), \ldots, f_n(t))$ in $\mathbb{R}^n$. It has to behave in a certain admissible way when approaching $\{x \in \mathbb{R}^n \mid x_1 \cdot x_2 \cdots x_n = 0\}$.

We will also show that in the case of complex valued continuous functions on $[0,1]$ products of open subsets are always open.

AMS-classification: 46B20. 46E15;

keywords: spaces of continuous functions, open sets, Banach algebra.

1. Introduction

The starting point of the investigations in [1] was the observation that the product of two open sets in the space of real-valued continuous functions is not necessarily open. However, such products always contain interior points. The results have been generalized in [4] to the space of real-valued N-times differentiable functions, and in [2] a characterization was given: what are the properties of the functions under consideration that make such a phenomenon possible? The aim of the present paper is a generalization of these results to n-fold products.

As in [2] we describe the "local obstruction": when is it true that $f_1 \cdots f_n$ lies in the interior of the product of the n balls $B_\varepsilon(f_1), \ldots, B_\varepsilon(f_n)$ for every $\varepsilon > 0$? We will characterize the families $f_1, \ldots, f_n \in C[0,1]$ where this holds.

It will turn out that the topological properties close to the zeros of $f_1 \cdots f_n$ will play a crucial role.

In order to state the main result of this paper (theorem 1.2) we need some preliminary definitions. We denote by Π the set $\{-1,+1\}^n$ and by Π^+ resp. Π^- the subset of those π where $\pi_1 \cdot \pi_2 \cdots \pi_n$ equals $+1$ resp. -1.
Each $\pi \in \Pi$ gives rise to a subset Q^π of $\mathbb{R}^n$:

$$Q^\pi := \{x \in \mathbb{R}^n \mid x_i \pi_i \geq 0 \text{ for } i = 1, \ldots, n\}.$$

Note that, e.g., the Q^π are just the quadrants in $\mathbb{R}^2$ if $n = 2$. Also it is clear that the function $H : \mathbb{R}^n \to \mathbb{R},\ x \mapsto x_1 \cdots x_n$, is ≥ 0 resp. ≤ 0 on Q^π for $\pi \in \Pi^+$ resp. $\pi \in \Pi^-$.

Now let $x \in \mathbb{R}^n$ be given. For which Q^π is it true that there are $y \in Q^\pi$ close to x such that $H(y)$ is slightly larger resp. slightly smaller than $H(x)$? More precisely we define the set $Z_x^+ \subset \Pi$ as the collection of the π such that for every $\varepsilon > 0$ there exists $y \in Q^\pi$ for which $||x - y|| \leq \varepsilon$ and $H(y) > H(x)$. ($|| \cdot ||$ will always denote the maximum norm on $\mathbb{R}^n$.) Similarly Z_x^- is defined: here $H(y) < H(x)$ has to be true.

If x is such that $H(x) \neq 0$, i.e., if x is in the interior of some Q^π, we have $Z_x^+ = Z_x^- = \{\pi\}$. For the x with $H(x) = 0$ the explicit description is as follows: π will be in Z_x^+ (resp. Z_x^-) precisely if $\pi \in \Pi^+$ (resp. $\pi \in \Pi^-$), and $\pi_i x_i > 0$ for the i with $x_i \neq 0$. In particular it follows for $\pi = (\pi_i), \tilde{\pi} = (\tilde{\pi}_i) \in Z_x^+$ that $\pi_i = \tilde{\pi}_i$ for the i with $x_i \neq 0$. (A similar result holds for $\pi = (\pi_i), \tilde{\pi} = (\tilde{\pi}_i) \in Z_x^-$.)
Note also that $Z_x^+ \cap Z_x^- = \emptyset$ for the x such that $H(x) = 0$.
As an illustration consider in $\mathbb{R}^3$ the following examples:

- $Z^+_{(1,-2,3)} = Z^-_{(1,-2,3)} = \{(+1,-1,+1)\}$;
- $Z^+_{(3,0,0)} = \{(+1,+1,+1),(+1,-1,-1)\}$, $Z^-_{(3,0,0)} = \{(+1,-1,+1),(+1,+1,-1)\}$;
- $Z^+_{(0,0,0)} = \Pi^+$, $Z^-_{(0,0,0)} = \Pi^-$.

Now we fix arbitrary $f_1, \ldots, f_n \in C\,[\,0,1\,]$, and we put $\gamma(t) := \big(f_1(t), \ldots, f_n(t)\big)$ for $0 \leq t \leq 1$. Here is the crucial definition:

Definition 1.1. We say that γ is *positive admissible* (resp. *negative admissible*) if the following holds: whenever there are given $t_1 < \cdots < t_n$ in $[\,0,1\,]$ such that $H \circ \gamma \geq 0$ (resp. ≤ 0) on $[\,t_1, t_n\,]$, then $\bigcap_i Z^+_{\gamma(t_i)} \neq \emptyset$ (resp. $\bigcap_i Z^-_{\gamma(t_i)} \neq \emptyset$).
If γ is positive admissible and negative admissible, γ is said to be *admissible*.

To illustrate this definition let us consider some *examples*:

- For $n = 1$ every γ is admissible.
- For $n = 2$ the walk γ is admissible iff it never moves directly from $Q^{(+1,+1)}$ to $Q^{(-1,-1)}$ (or vice versa) and never directly from $Q^{(+1,-1)}$ to $Q^{(-1,+1)}$ (or vice versa). In [2] this was called "γ has no positive and no negative saddle point crossings".
- Now let us consider the case $n = 3$, suppose, e.g., that γ stays in $Q^{(+1,+1,+1)}$. Then γ will be positive admissible, but it will be negative admissible only

if it does not move to three linearly independent directions on subintervals where $H \circ \gamma = 0$. For example, a walk that moves on straight lines from $(1,0,0)$ to $(0,0,0)$ to $(0,1,0)$ to $(0,0,0)$ to $(0,0,1)$ is not negative admissible.

Our main result (that generalizes the characterization in [2] for the case $n = 2$) reads as follows:

Theorem 1.2. *Let $f_1, \ldots, f_n \in C\,[0,1]$ be given. Then the following assertions are equivalent:*

(i) $f_1 \cdot f_2 \cdots f_n$ lies in the interior of $B_\varepsilon(f_1) \cdot B_\varepsilon(f_2) \cdots B_\varepsilon(f_n)$ for every $\varepsilon > 0$.

(ii) The associated walk $\gamma : t \mapsto \big(f_1(t), \ldots, f_n(t)\big)$ is admissible.

The proof will be given in *section 3* after the introduction of some further definitions and the verification of some preliminary results in *section 2*. The idea will be to show a more precise variant of the theorem by induction on n. By this variant we will be able to derive properties of γ on $[0,1]$ from properties of γ on the subintervals of appropriate partitions of $[0,1]$.

In *section 4* we prove that in the space of complex-valued functions on $[0,1]$ products of open sets are always open, and finally, in *section 5*, one finds some consequences of the main theorem and some concluding remarks.

2. Preliminaries

A translation of the problem: "walk the dog"

We fix $f_1, \ldots, f_n$, and γ is defined as before. The investigations to come are rather technical, and as in [2] it will be helpful to have an appropriate visualization.

First we note that "for every positive ε the function $f_1 \cdots f_n$ lies in the interior of $B_\varepsilon(f_1) \cdots B_\varepsilon(f_n)$" just means that for $\varepsilon > 0$ there is a $\tau_0 > 0$ such that for every $\tau \in C\,[0,1]$ with $||\tau|| \le \tau_0$ there exists a continuous $d : [0,1] \to \mathbb{R}^n$ such that $||d(t)|| \le \varepsilon$ and $H\big(\gamma(t) + d(t)\big) = H \circ \gamma(t) + \tau(t)$ for every $t \in [0,1]$: if γ is considered as your walk in $\mathbb{R}^n$, then your "dog" – its position at time t is $(\gamma + d)(t)$ – can move such that it is always ε-close to you, and its "height above sea level" $H\big((\gamma + d)(t)\big)$ relative to yours (which is $H\big(\gamma(t)\big)$) can be prescribed as $\tau(t)$ arbitrarily provided it is uniformly small.

A lemma concerning the Z_x^+ and the Z_x^-

In section 1 we have defined what it means that γ is admissible. We will need some consequences of this property.

Lemma 2.1. *Suppose that γ is admissible.*
(i) If $H\circ\gamma\ge 0$ on some subinterval $[\,a,b\,]$, then $\bigcap_{a\le t\le b} Z^+_{\gamma(t)}\ne\emptyset$.
If, in addition, there is a $t\in[\,a,b\,]$ with $H\circ\gamma(t)>0$ then $\bigcap_{a\le t\le b} Z^+_{\gamma(t)}$ is a singleton.
(ii) If $H\circ\gamma\le 0$ on some subinterval $[\,a,b\,]$, then $\bigcap_{a\le t\le b} Z^-_{\gamma(t)}\ne\emptyset$.
If, in addition, there is a $t\in[\,a,b\,]$ with $H\circ\gamma(t)<0$ then $\bigcap_{a\le t\le b} Z^-_{\gamma(t)}$ is a singleton.
(iii) If $H\circ\gamma=0$ on some subinterval $[\,a,b\,]$, then there are $\pi=(\pi_i),\tilde\pi=(\tilde\pi_i)$ such that π belongs to all $Z^+_{\gamma(t)}$ and $\tilde\pi$ belongs to all $Z^-_{\gamma(t)}$ for $t\in[\,a,b\,]$. If i is an index such that $\pi_i\ne\tilde\pi_i$ then f_i (the i'th component of γ) vanishes on $[\,a,b\,]$. Note that such i exist since $Z^+_x\cap Z^-_x=\emptyset$ whenever $H(x)=0$.

Proof. (i) Consider $J_i:=f_i([\,a,b\,])$ for $i=1,\dots,n$. The J_i are compact subintervals of $\mathbb{R}$, we claim that 0 is never contained as an interior point. In fact, if the product $f_i(t)f_i(t')$ would be negative for some i,t,t', we would have $Z^+_{\gamma(t)}\cap Z^+_{\gamma(t')}=\emptyset$ (since for $\pi\in\Pi^+$ the i'th component cannot be positive and negative at the same time). This would contradict the assumption that γ is positive admissible. Let Δ be the collection of i where J_i is not the interval $[\,0,0\,]$. Choose t_i for these i such that $f_i(t_i)\ne 0$.

If Δ is a proper subset of $\{1,\dots,n\}$ we are already done: we define π_i for $i\in\Delta$ such that $\pi_i f_i(t_i)>0$, and the remaining π_i are chosen in such a way that $\pi\in\Pi^+$. Then π will lie in all $Z^+_{\gamma(t)}$ for $a\le t\le b$.

Now suppose that $\Delta=\{1,\dots,n\}$. Since γ is positive admissible there is a π in $\bigcap_i Z^+_{\gamma(t_i)}$. It lies in Π^+ and it must have the property that $\pi_i f_i(t_i)$ is strictly positive for all i. Therefore $\prod f_i(t_i)>0$. Since J_i does not have 0 as an interior point it follows that $f_i(t_i)f_i(t)\ge 0$ for all $t\in[\,a,b\,]$, and this implies that a π which lies in all $Z^+_{\gamma(t_i)}$ must also lie in $Z^+_{\gamma(t)}$ for arbitrary $t\in[\,a,b\,]$.

The second part of the assertion is clear since Z^+_x contains just one element if $H(x)\ne 0$.

(ii) This can be proved in a similar way.

(iii) By (i) and (ii) it is clear that π and $\tilde\pi$ with the desired properties exist. Now let i be such that $\pi_i\ne\tilde\pi_i$. With the notation of the proof of (i) we claim that the interval J_i equals $[\,0,0\,]$. Otherwise, if J_i would contain stricly positive (resp. strictly negative) elements, π_i and similarly $\tilde\pi_i$ would both be $+1$ (resp. -1). □

Canonical positions

Suppose that someone stays during his or her walk at some time at $x\in\mathbb{R}^n$ and that one has to find a position of the dog that is close to x and that has a prescribed H-value. There will be many of them, but it will be crucial for our investigations to have a canonical one.

We start with an x such that $H(x)\ne 0$. Then x lies in the interior of some Q^π: here π is uniquely determined, and $Z^+_x=Z^-_x=\{\pi\}$. All components of x are different from zero.

We put $y_s := (1+s)x$. The function $s \mapsto H(y_s) = (1+s)^n H(x)$ is strictly monotonic on $]-1,\infty[$ (strictly increasing resp. decreasing if $\pi \in \Pi^+$ resp. $\pi \in \Pi^-$). Its range is $]0,\infty[$ or $]-\infty,0[$ and we can conclude that for $|\alpha| < |H(x)|$ there is a unique s_α such that $H(y_{s_\alpha}) = (1+s_\alpha)^n H(x) = H(x) + \alpha$. We will denote this y_{s_α} by $W(x,\pi,\alpha)$: *this* is our canonical choice.

Next we consider an x such that $H(x) = 0$ and a pair $(\pi,\tilde\pi) \in Z_x^+ \times Z_x^-$. (Note that such pairs always will exist.) Let Δ be the nonvoid set of indices i where $\pi_i \neq \tilde\pi_i$. It is obvious that the cardinality l of Δ is an odd number and that $x_i = 0$ for $i \in \Delta$.

Let $\varepsilon > 0$ be so small that $\varepsilon \le |x_i|$ for all i such that $x_i \neq 0$. Then define, for $|s| \le \varepsilon$, a vector y_s as follows. For the i such that $x_i \neq 0$ we put $(y_s)_i = x_i$, for the $i \in \Delta$ the value of $(y_s)_i$ is s, and for the remaining i (i.e., the i where $x_i = 0$ and $\pi_i = \tilde\pi_i$, if there are any) we define $(y_s)_i := \varepsilon\pi_i (= \varepsilon\tilde\pi_i)$. Then $H(y_s) = c \cdot s^l$, where c is a constant with $|c| \ge \varepsilon^{n-l}$. Since l is odd, the function $s \mapsto H(y_s),\ |s| \le \varepsilon$, is strictly monotonic, and its range contains at least the interval $[-\varepsilon^n, \varepsilon^n]$. Thus, for $|\alpha| \le \varepsilon^n$, there is a uniquely determined s such that $H(y_s) = \alpha$. This y_s will be denoted by $W_\varepsilon(x,\pi,\tilde\pi,\alpha)$. (Note that this vector will only depend on ε if there are $i \notin \Delta$ with $x_i = 0$.)

It is obvious that $W_\varepsilon(x,\pi,\tilde\pi,\alpha)$ is ε-close to x for the α under consideration. This will be – depending on $\pi,\tilde\pi$ – our canonical choice for a y such that $H(y) = H(x) + \alpha$ in the case $H(x) = 0$.

Types, admissible pairs, and pep

Let $[a,b]$ a nontrivial interval and $\phi : [a,b] \to \mathbb{R}$ a continuous function[1]. If ϕ is identically zero we will say that ϕ *is of type* $T(0)$.

If this is not the case we distinguish several cases. If $\phi(a) \neq 0$ we say that ϕ *is of left type* u. Suppose that $\phi(a) = 0$, but ϕ vanishes on no neighbourhood of a. There are three possibilities for the behaviour of ϕ:

1. There is a $\delta_0 > 0$ such that $\phi \ge 0$ on $[a, a+\delta_0]$, and for every $\delta > 0$ there exists a $t \in [a, a+\delta]$ with $\phi(t) > 0$.
2. There is a $\delta_0 > 0$ such that $\phi \le 0$ on $[a, a+\delta_0]$, and for every $\delta > 0$ there exists a $t \in [a, a+\delta]$ with $\phi(t) < 0$.
3. For every $\delta > 0$ there exist $t, t' \in [a, a+\delta]$ with $\phi(t) > 0$ and $\phi(t') < 0$.

We will say that ϕ *is of left type* $+$ *resp.* $-$ *resp.* $\pm$ if "1." resp. "2." resp. "3." holds. The *right types* u (if $\phi(b) \neq 0$) and $+, -, \pm$ (when $\phi(b) = 0$, but ϕ vanishes on no neighbourhood of b) are defined similarly.

ϕ is said to be *of type* $T(\mathcal{T}_1, \mathcal{T}_2)$ on $[a,b]$ if ϕ is of left type $\mathcal{T}_1 \in \{u,+,-,\pm\}$ and of right type $\mathcal{T}_2 \in \{u,+,-,\pm\}$. It should be clear that for every ϕ precisely one of the following 50 situation occurs:

[1] We will need the following classification only in the case when ϕ is the restriction of $H \circ \gamma$ to certain subintervals.

- ϕ is of type $T(0)$ on $[\,a,b\,]$.
- ϕ is of some type $T(\mathcal{T}_1,\mathcal{T}_2)$ on $[\,a,b\,]$.
- There is $a' \in\,]\,a,b\,[$ such that ϕ is of type $T(0)$ on $[\,a,a'\,]$ and of some type $T(\mathcal{T}_1,\mathcal{T}_2)$ on $[\,a',b\,]$. (Note that in this case $\mathcal{T}_1$ must be in the subset $\{+,-,\pm\}$. Similar restrictions apply to $\mathcal{T}_2$ in the next and to both $\mathcal{T}_1$ and $\mathcal{T}_2$ in the last case.)
- There is $b' \in\,]\,a,b\,[$ such that ϕ is of type $T(0)$ on $[\,b',b\,]$ and of some type $T(\mathcal{T}_1,\mathcal{T}_2)$ on $[\,a,b'\,]$.
- There are $a',b' \in\,]\,a,b\,[$ with $a'<b'$ such that ϕ is of type $T(0)$ on $[\,a,a'\,]$ and on $[\,b',b\,]$ and of some type $T(\mathcal{T}_1,\mathcal{T}_2)$ on $[\,a',b'\,]$.

In our case properties of the function $\phi = H\circ\gamma$ will be crucial, and depending on the type of this function we would like to be able to choose the starting and final position of the "walk of the dog", i.e., the vectors $\gamma(a)+d(a)$ and $\gamma(b)+d(b)$ in a canonical way in certain $Q^{\pi_a}\cup Q^{\tilde\pi_a}$ (at a) and in certain $Q^{\pi_b}\cup Q^{\tilde\pi_b}$ (at b), where $(\pi_a,\tilde\pi_a)\in Z^+_{\gamma(a)}\times Z^-_{\gamma(a)}$ and $(\pi_b,\tilde\pi_b)\in Z^+_{\gamma(b)}\times Z^-_{\gamma(b)}$, respectively. For example, if $\mathcal{T}_1=+$, then for a distinguished $\pi\in Z^+_{\gamma(a)}$ and *all* $\tilde\pi\in Z^-_{\gamma(a)}$ we want to choose the starting position of the dog in a canonical way in $Q^\pi\cup Q^{\tilde\pi}$.

To make this precise we need some further definitions. Let $\gamma:[\,0,1\,]\to\mathbb{R}^n$ be admissible and $[\,a,b\,]\subset[\,0,1\,]$. We suppose that $H\circ\gamma$ is of some type $T(\mathcal{T}_1,\mathcal{T}_2)$ on $[\,a,b\,]$. We define *sets of left (resp. right) admissible pairs* ($\mathcal{A}^l_{\mathcal{T}_1}$ resp. $\mathcal{A}^r_{\mathcal{T}_2}$) as follows.

- If $\mathcal{T}_1=u$, then $\mathcal{A}^l_u:=\{(\pi,\pi)\}$, where π is the unique vector such that $\gamma(a)$ lies in the interior of Q^π.
- Let $\mathcal{T}_1=+$. By definition there is a $\delta_0>0$ such that $H\circ\gamma$ is nonnegative on $[\,a,a+\delta_0\,]$, and since $H\circ\gamma$ is strictly positive at some point in $[\,a,a+\delta_0\,]$ there is a unique π_0 such that $\bigcap_{a\le t\le a+\delta_0} Z^+_{\gamma(t)}$ (cf. lemma 2.1(i)). We put $\mathcal{A}^l_+:=\{(\pi_0,\tilde\pi)\mid\tilde\pi\in Z^-_{\gamma(a)}\}$.
- Similarly, if $\mathcal{T}_1=-$, we know that $\bigcap_{a\le t\le a+\delta_0} Z^-_{\gamma(t)}=\{\tilde\pi_0\}$ for a sufficiently small δ_0 and a unique $\tilde\pi_0$. In this case we put $\mathcal{A}^l_-:=\{(\pi,\tilde\pi_0)\mid\pi\in Z^+_{\gamma(a)}\}$.
- It remains to consider the case $\mathcal{T}_1=\pm$. Then we define $\mathcal{A}^l_\pm:=Z^+_{\gamma(a)}\times Z^-_{\gamma(a)}$.
- The right admissible pairs $\mathcal{A}^r_{\mathcal{T}_2}$, are defined in a similar way.

We now turn to "*pep*" the possibility to choose *prescribed end points*, i.e. the positions at $t=a$ and at $t=b$, for the walk of the dog. As before, γ is supposed to be admissible.

Definition 2.2. Suppose that $H \circ \gamma|_{[a,b]}$ is of type $T(\mathcal{T}_1, \mathcal{T}_2)$ and $\varepsilon_0 > 0$. We will say that γ of type $T^{\varepsilon_0}_{\rm pep}(\mathcal{T}_1, \mathcal{T}_2)$ on $[a,b]$ if the following holds.

There is a positive $\varepsilon^* \le \varepsilon_0$ such that for every $\varepsilon \in \,]\,0, \varepsilon^*\,]$ one can find a $\tau_0 > 0$ such that for arbitrary $(\pi_a, \tilde\pi_a) \in \mathcal{A}^l_{\mathcal{T}_1}$ and $(\pi_b, \tilde\pi_b) \times \mathcal{A}^r_{\mathcal{T}_2}$ and for every continuous $\tau : [a,b] \to \mathbb{R}$ with $||\tau|| \le \tau_0$ there exists a continuous $d : [a,b] \to \mathbb{R}^n$ such that

- $H\big(\gamma(t) + d(t)\big) = H\big(\gamma(t)\big) + \tau(t)$ and $||d(t)|| \le \varepsilon_0$ for $t \in [a,b]$.
- At a and at b the value of $\gamma + d$ is defined in a canonical way:

$$(\gamma + d)(a) = W_\varepsilon(\gamma(a), \pi_a, \tilde\pi_a, \tau(a)) \text{ resp. } W\big((\gamma + d)(a), \pi_a, \tau(a)\big)$$

  if $\mathcal{T}_1 \in \{+, -, \pm\}$ resp. $\mathcal{T}_1 = u$, and

$$(\gamma + d)(b) = W_\varepsilon(\gamma(b), \pi_b, \tilde\pi_b, \tau(b)) \text{ resp. } W\big((\gamma + d)(b), \pi_b, \tau(b)\big)$$

  if $\mathcal{T}_2 \in \{+, -, \pm\}$ resp. $\mathcal{T}_2 = u$.

If this is true for every $\varepsilon_0 > 0$ we will say that γ is of type $T_{\rm pep}(\mathcal{T}_1, \mathcal{T}_2)$

When does $T(*,*)$ for $H \circ \gamma|_{[a,b]}$ imply $T_{\rm pep}(*,*)$ for $\gamma|_{[a,b]}$?

The answer to this question will be crucial for our investigations. We will prove three results that hold in general.

Later we will consider situations where $0 \le a < b < c < d \le 1$ are given, $H \circ \gamma$ is of some type $T_{\rm pep}$ on $[a,b]$ and on $[c,d]$ and vanishes on $[b,c]$. How can one fill the gap between b and c for the walk of the dog if there are walks on $[a,b]$ and $[c,d]$ that are provided by the *pep*-condition? The following proposition will enable us to do this.

Proposition 2.3. *Suppose that* $\gamma = (f_1, \ldots, f_n)$ *is admissible and that* $H \circ \gamma$ *is of type* $T(0)$ *on* $[a,b]$. *We assume that* $\pi = (\pi_i) \in \bigcap_{a \le t \le b} Z^+_{\gamma(t)}$ *and* $\tilde\pi = (\tilde\pi_i) \in \bigcap_{a \le t \le b} Z^-_{\gamma(t)}$ *are given*[2]. *Then, for every* $\varepsilon > 0$, *one can find a* $\tau_0 > 0$ *such that for every continuous* $\tau : [a,b] \to \mathbb{R}$ *with* $||\tau|| \le \tau_0$ *there exists a continuous* $d : [a,b] \to \mathbb{R}^n$ *such that*

$$(\gamma + d)(a) = W_\varepsilon(\gamma(a), \pi, \tilde\pi, \tau(a)), \;\; (\gamma + d)(b) = W_\varepsilon(\gamma(b), \pi, \tilde\pi, \tau(b)),$$

and $||d(t)|| \le \varepsilon$ *and* $H\big(\gamma(t) + d(t)\big) = \tau(t)$ *for all* $t \in [a,b]$.

Proof. Let $\varepsilon > 0$ be given. Let us assume that ε is smaller than all $|f_i(a)|$ where $f_i(a) \ne 0$ and also smaller than all $|f_i(b)|$ where $f_i(b) \ne 0$. (If necessary, replace ε by a smaller positive number.) Denote by Δ the collection of i where $\pi_i \ne \tilde\pi_i$. For $i \in \Delta$ the function f_i vanishes on $[a,b]$ (cf. the proof of lemma 2.1(iii)), and the number l of elements in Δ is odd.

[2]Note that by lemma 2.1(iii) such pairs exist.

The following construction makes use of the W_ε above. As a first step we pass to slight perturbations of the f_i for $i \notin \Delta$.

Let an $i \notin \Delta$ be given. Then $\pi_i = \tilde{\pi}_i$, and the function $f_i \pi_i$ is nonnegative on $[a, b]$. We define $g_i : [a, b] \to \mathbb{R}$ by $g_i(t) := f_i(t)$, if $|f_i(t)| \geq \varepsilon$ and by $g_i(t) := \varepsilon \pi_i$ otherwise. g_i is continuous and it is ε-close to f_i. We have $g_i(a) = f_i(a)$ if $f_i(a) \neq 0$ and $f_i(a) = \varepsilon \pi_i$ otherwise, and, similarly, $g_i(b) = f_i(b)$ if $f_i(b) \neq 0$ and $f_i(b) = \varepsilon \pi_i$ otherwise. For the $i \in \Delta$ we put $g_i := f_i = 0$.

Now let $\tau : [a, b] \to \mathbb{R}$ be continuous such that $||\tau|| \leq \tau_0 := \varepsilon^n$. We will define a walk that stays ε-close to γ on $[a, b]$, for which the H-value at time t is just $\tau(t)$, and for which the endpoints are the prescribed canonical points as claimed.

For $t \in [a, b]$ we put

$$P_\varepsilon(t, \pi, \tilde{\pi}, \tau) := W_\varepsilon\big(G(t), \pi, \tilde{\pi}, \tau(t))\big),$$

where $G(t) := (g_1(t), \ldots, g_n(t))$.

The map $P_\varepsilon(\cdot, \pi, \tilde{\pi}, \tau)$ has the following properties:

- It is continuous since it can explicitly described by using roots and scalar products.
- $P_\varepsilon(a, \pi, \tilde{\pi}, \tau) = W_\varepsilon(\gamma(a), \pi, \tilde{\pi}, \tau(a))$.

  This makes use of the following fact that is an immediate consequence of the definition (we use the notation in the paragraph where "canonical positions" have been introduced): Suppose that $H(x) = 0$. If $y \in \mathbb{R}^n$ is such that $x_i = y_i$ for $i \in \Delta$ and for the i with $x_i \neq 0$ and if $y_i = \varepsilon \pi_i$ for the remaining i (if there are any), then $W_\varepsilon(x, \pi, \tilde{\pi}, \alpha) = W_\varepsilon(y, \pi, \tilde{\pi}, \alpha)$ for $\alpha \in [-\varepsilon^n, \varepsilon^n]$.

- $P_\varepsilon(b, \pi, \tilde{\pi}, \tau) = W_\varepsilon(\gamma(b), \pi, \tilde{\pi}, \tau(b))$.
- $||\gamma(t) - P_\varepsilon(t, \pi, \tilde{\pi}, \tau)|| \leq \varepsilon$ for $t \in [a, b]$; this can be easily checked coordinatewise.
- $H\big(P_\varepsilon(t, \pi, \tilde{\pi}, \tau)\big) = \tau(t)$ for all t.

And this means that $d := P_\varepsilon(\cdot, \pi, \tilde{\pi}, \tau) - \gamma$ behaves as desired. □

It is possible to glue intervals together where γ is of some type T_{pep}. This is true in general and will be important later (cf. the proof of lemma 2.6 and of the main theorem in section 3). Here we will consider only a special case:

Proposition 2.4. *Let γ be admissible and $0 \leq a < b < c \leq 1$. We assume that γ is of type $T_{pep}(\mathcal{T}_1, u)$ on $[a, b]$ and of type $T_{pep}(u, \mathcal{T}_2)$ on $[b, c]$.*
Then γ is of type $T_{pep}(\mathcal{T}_1, \mathcal{T}_2)$ on $[a, c]$.

Proof. If ε_0 is a given positive number choose ε^* as the smaller of the ε^* that are appropriate for $[a,b]$ and $[b,c]$. Let $\varepsilon \in]0,\varepsilon^*]$ be given and τ_0 the smaller of the τ_0's for $[a,b]$ and $[b,c]$ for this ε.

Now let $\tau : [a,c] \to \mathbb{R}$ with $||\tau|| \leq \tau_0$ be prescribed. We find the desired walks d_1, d_2 on $[a,b]$ and on $[b,c]$ by assumption: the $\gamma + d_i$ are ε_0-close to γ, the H-value is $H\big(\gamma\big) + \tau$, and $\gamma + d_i$ have both at b the value $W(\gamma(b), \pi, \tau(b))$, where π is the unique vector with $\gamma(b) \in Q^\pi$. Thus the walks can be glued together to produce a continuous walk with the desired properties that is defined on $[a,c]$. $\square$

The next proposition concerns situations when the walk stays very close to $0 \in \mathbb{R}^n$ on $[a,b]$. Then the "dog" can move rather freely: it has not to be in the same Q^π as γ provided its position is also close to zero. The proposition is prepared with the following lemma.

Lemma 2.5. *Fix $\varepsilon_0 > 0$ and $[a,b] \subset [0,1]$. There are given x, y with $||x||, ||y|| \leq \varepsilon_0$ and a continuous function $\sigma : [a,b] \to \mathbb{R}$ with $\sigma(a) = H(x)$ and $\sigma(b) = H(y)$ and $||\sigma|| \leq \varepsilon_0^n$. We claim that in each of the following cases there is a continuous $D\,(= D_{a,b;x,y;\sigma}) : [a,b] \to \mathbb{R}^n$ such that $D(a) = x$, $D(b) = y$, and $H\big(D(t)\big) = \sigma(t)$ and $||D(t)|| \leq 2\varepsilon_0$ for every t.*

(i) There is $\pi \in \Pi$ such that x, y are in the interior of Q^π.

(ii) x resp. y lies in the interior of Q^π resp. $Q^{\tilde\pi}$, where $\pi \in \Pi^+$, $\tilde\pi \in \Pi^-$.

(iii) There is a y_0 with $H(y_0) = 0$ such that x lies in the interior of some Q^π with $\pi \in Z^+_{y_0}$, $\tilde\pi \in Z^-_{y_0}$, and $y = W_\varepsilon(y_0, \pi, \tilde\pi, \alpha)$, where $|\alpha| \leq \varepsilon_0^n$ and $0 < \varepsilon \leq \varepsilon_0$.

Proof. The translation is the following: one can move from x to y with arbitrarily prescribed H-value in these cases provided that this value is small enough.

(i) Without loss of generality we may assume that $\pi = (1, \ldots, 1)$ which implies that $H(x) > 0$. The walk will be defined by putting together three subwalks D_1, D_2 and D_3: one from x to $F := \{(\varepsilon_0, \ldots, \varepsilon_0, \alpha) \mid \alpha \in \mathbb{R},\ |\alpha| \leq \varepsilon_0\}$, one on F and a third one from F to y.

Choose $a' \in]a,b[$ such that $\sigma(t) \leq 2H(x)$ on $[a,a']$ (note that σ is continuous and $\sigma(a) = H(x) > 0$). $D_1(t)$ will be defined on this interval by

$$\big(X_1(t), \ldots, X_{n-1}(t), s_t\big),$$

where

$$X_i(t) := \big((a'-t)x_i + (t-a)\varepsilon_0\big)/(a'-a),$$

and s_t is chosen such that $H\big(D_1(t)\big) = \sigma(t)$:

$$s_t = \frac{\sigma(t)}{X_1(t) \cdots X_{n-1}(t)}.$$

It is then clear that D_1 is continuous and that $D_1(a) = x$ and

$$D_1(a') = (\varepsilon_0, \ldots, \varepsilon_0, \sigma(a')/\varepsilon_0^{n-1})$$

hold. Also note that

$$\begin{aligned} |s_t| &= \left|\sigma(t)/\big(X_1(t)\cdots X_{n-1}(t)\big)\right| \\ &\le |\sigma(t)/x_1\cdots x_{n-1}| \\ &\le |2H(x)/x_1\cdots x_{n-1}| \\ &= 2|x_n|. \end{aligned}$$

This proves that $||D_1(t)|| \le 2\varepsilon_0$ for all t.

Similarly we define a walk D_3 from F to y. It is defined on some small interval $[\,b', b\,]$ where $a' < b' < b$, the distance of the walk to zero is at most $2\varepsilon_0$, and the H-value at any time t is $\sigma(t)$.

It remains to fill the gap between a' and b'. We put

$$D_2(t) := \big(\varepsilon_0, \ldots, \varepsilon_0, \sigma(t)/\varepsilon_0^{n-1}\big).$$

This D_2 connects the first two walks in a continuous way[3], we have $H \circ D_2 = \sigma$, and the norm at every point of $[\,a', b'\,]$ is a most $2\varepsilon_0$. (In fact, it is even bounded by ε_0.)

(ii) Let Δ be the set of i where $\pi_i \ne \tilde{\pi}_i$. This set is nonempty and its cardinality is an odd number. Without loss of generality we assume that $\pi = (1, \ldots, 1)$ and that $\Delta = \{1, \ldots, l\}$ with $1 \le l \le n$.

This time we first move from x to G_1, the set where the last $n - l$ (if there are any) components equal ε_0, then to the subset $G_2 \subset G_1$ of those vectors where the first l components coincide. The walk will stay on G_2 for some time, and then it moves from G_2 to G_1 to y.

We start by choosing $a' \in \,]\,a, b\,[$ such that

$$(1+\eta)^{-1} H(x) \le \sigma(t) \le (1+\eta) H(x)$$

on $[\,a, a'\,]$; here η is a positive number that will be fixed later. Select any $a'' \in \,]\,a, a'\,[$.

Between $t = a$ and $t = a''$ we will move from x to a point in G_1. This will be done as follows. With $X_i(t) := \big((a''-t)x_i + (t-a)\varepsilon_0\big)/(a''-a)$, we select s_t such that

$$D(t) := \big(s_t x_1, \ldots, s_t x_l, X_{l+1}(t), \ldots, X_n(t)\big)$$

satisfies $H \circ D(t) = \sigma(t)$. D is continuous, it connects x with a point in G_1, and the H-value is as desired. It stays also close to zero:

$$\begin{aligned} |s_t| &= \left|\sqrt[l]{\sigma(t)/\big(x_1\cdots x_l X_{l+1}(t)\cdots X_n(t)\big)}\right| \\ &\le |\sqrt[l]{\sigma(t)/x_1\cdots x_n}| \\ &\le \sqrt[l]{1+\eta}, \end{aligned}$$

[3]Note that $D_2(a') = D_1(a')$ and $D_2(b') = D_3(b')$.

and this implies that all components of $D(t)$ are bounded by $\varepsilon_0\sqrt[l]{1+\eta}$.

Now we will move from $\hat{x} = (\hat{x}_i) := D(a'')$ to a point in G_2. We choose s_0 such that $s_0^l\varepsilon^{n-l} = \sigma(a')$, and this time we consider functions X_i that are defined by $X_i(t) := \big((a'-t)\hat{x}_i + (t-a'')s_0\big)/(a'-a'')$. D will be defined on $[\,a'', a'\,]$ by

$$D(t) := \big(s_tX_1(t), \ldots, s_tX_l(t), \varepsilon_0, \ldots, \varepsilon_0\big),$$

where s_t is the unique number such that $H \circ D(t) = \sigma(t)$. (Such an s_t exists since l is odd.)

This walk satisfies $H \circ D = \sigma$, it only remains to prove that it stays close to zero.

For the proof we observe that $\hat{x}_1 \cdots \hat{x}_l\varepsilon^{n-l} = \sigma(a'') \geq \sigma(a)/(1+\eta)$. Also $s_0^l\varepsilon^{n-l} = \sigma(a') \geq \sigma(a)/(1+\eta)$ holds so that $|X_1(t)\cdots X_l(t)\varepsilon_0^{n-l}| \geq |\sigma(a)|/(1+\eta)$. (This follows from the inequality $(1-t)c+td \leq c^{1-t}d^t$ for $c, d > 0$.) Consequently

$$\begin{aligned} |s_t| &= \Big|\sqrt[l]{\sigma(t)/\big(X_1(t)\cdots X_l(t)\varepsilon_0^{n-l}\big)}\Big| \\ &\leq \Big|\sqrt[l]{(1+\eta)\sigma(t)/\sigma(a)}\Big| \\ &\leq \sqrt[l]{(1+\eta)^2}, \end{aligned}$$

and we conclude that all components of all $D(t)$ are bounded by $\sqrt[l]{(1+\eta)^3}\varepsilon_0$. Thus it will suffice to put $\eta := \sqrt[3]{2}-1$ to guarantee that $||D(t)|| \leq 2\varepsilon_0$.

Similarly, the last part of the walk will move for t in a suitable interval $[\,b', b\,]$ from some $(s, \ldots, s, \varepsilon_0, \ldots, \varepsilon_0) \in G_2$ with $H(s, \ldots, s, \varepsilon_0, \ldots, \varepsilon_0) = \sigma(b')$ to y, and it will meet a suitable $\hat{y} \in G_1$ at some time $t = b''$ between b' and b.

The gap between a' and b' will be filled by the walk

$$D(t) := (\sqrt[l]{\sigma(t)/\varepsilon_0^{n-l}}, \ldots, \sqrt[l]{\sigma(t)/\varepsilon_0^{n-l}}, \varepsilon_0, \ldots, \varepsilon_0).$$

Note again that l is odd so that the definition applies also for the negative values of $\sigma(t)$.

The norm of $D(t)$ is as desired also on $[\,a', b'\,]$ since $|\sigma(t)/\varepsilon_0^{n-l}| \leq \varepsilon_0^l$.

(iii) As before we assume that, without loss of generality, $\pi = (1, \ldots, 1)$ and $\tilde{\pi} = (-1, \ldots, -1, 1, \ldots, 1)$, where the number $l \in \{1, \ldots, n\}$ of the entries -1 is odd. By the definition of the canonical positions y has the form $(s_y, \ldots, s_y, y_{l+1}, \ldots, y_n)$ where the $y_{l+1}, \ldots, y_n$ are positive and bounded from below by ε. We also know that $\alpha = H(y) = \sigma(b) = s_y^l y_{l+1}\cdots y_n$.

To find a continuous walk D with $H \circ D = \sigma$ we proceed as in the preceding proofs. First walk from x to some point in G_2, then the walk stays there until $t = b'$, where $b' < b$. b' is chosen close to b in the following way. We know that $|s_y| = \big|\sqrt[l]{\sigma(b)/\big(y_{l+1}\cdots y_n\big)}\big| \leq \varepsilon_0$, and we choose b' such that $\big|\sqrt[l]{\sigma(t)/\big(y_{l+1}\cdots y_n\big)}\big| \leq 2\varepsilon_0$ for $t \in [\,b', b\,]$.

At $t = b'$ the walk stays at a point $\hat{y} := (s_0, \ldots, s_0, \varepsilon_0, \ldots, \varepsilon_0)$ with $s_0^l\varepsilon_0^{n-l} = \sigma(b')$ and $|s_0| \leq \varepsilon_0$. It remains to move to y.

We put $Y_i(t) := \big((b-t)\varepsilon_0 + (t-b')y_i\big)/(b-b')$ for $i = l+1, \ldots, n$: the Y_i are continuous and they lie between y_i and ε_0. We define

$$D(t) := (s_t, \ldots, s_t, Y_{l+1}(t), \ldots, Y_n(t)),$$

where s_t is such that $H \circ D(t) = \sigma(t)$.
Then D is continuous, it connects $\hat{y}$ with y, and the norm is small:

$$\begin{aligned} |s_t| &= \left|\sqrt[l]{\sigma(t)/\big(Y_{l+1}(t)\cdots Y_n(t)\big)}\right| \\ &\leq \left|\sqrt[l]{\sigma(t)/\big(y_{l+1}(t)\cdots y_n(t)\big)}\right| \\ &\leq 2\varepsilon_0. \end{aligned}$$

□

With these preparations it is now possible to show that the *pep* property can be guaranteed for walks that stay close to the origin.

Proposition 2.6. *Let $\varepsilon_0 > 0$ be given and suppose that $||\gamma(t)|| < \varepsilon_0$ for $t \in [\,a, b\,]$ and that $H \circ \gamma$ has type $T(\mathcal{T}_1, \mathcal{T}_2)$ on this subinterval. As before we assume that γ is admissible.*
We claim that γ is of type $T_{pep}^{3\varepsilon_0}(\mathcal{T}_1, \mathcal{T}_2)$ on $[\,a, b\,]$.

Proof. Suppose that our assertion has been shown for the following cases:

1. $T(u, u)$, and $H \circ \gamma \geq 0$ on $[\,a, b\,]$.
2. $T(u, u)$, and $H \circ \gamma(a) > 0 > H \circ \gamma(b)$.
3. $T(u, +)$, $H \circ \gamma(a) > 0$, and $H \circ \gamma \geq 0$ on $[\,a, b\,]$.
4. $T(u, \pm)$, and $H \circ \gamma(a) > 0$.

We claim that then we are done. In fact, if one considers $\tilde{\gamma} = (-f_1, f_2, \ldots, f_n)$, then the assertions 1., 2., 3, 4, applied to $\tilde{\gamma}$ yield four new assertions for γ. For example, "4." for $\tilde{\gamma}$ covers the case "$T(u, \pm)$, and $H \circ \gamma(a) < 0$" for γ. Similarly a proof for $T(u, +)$ implies one for $T(+, u)$: simply pass from γ to $t \mapsto \gamma(1-t)$. One only has to note that with γ also $\tilde{\gamma}$ and $t \mapsto \gamma(1-t)$ are admissible.

However, there remain some other situations to be treated. Consider for example the case $T(u, u)$ where $H \circ \gamma(a)$ and $H \circ \gamma(b)$ are strictly positive, but $H \circ \gamma$ is negative at some point a' of $[\,a, b\,]$. We consider $[\,a, a'\,]$ and $[\,a', b\,]$ separately. There $H \circ \gamma$ is of type $T(u, u)$ for which the *pep* property is already known, and it remains to glue together the walks on these subintervals with the help of proposition 2.4. Similarly the cases $T(+, +), T(+, -), \ldots$ can be reduced to the above assertions 1., 2. , 3., 4. by choosing a suitable $a' \in [\,a, b\,]$ and discussing the intervals $[\,a, a'\,]$ and $[\,a', b\,]$ separately: the smaller of the ε^* associated with these subintervals will work for $[\,a, b\,]$ in definition 2.2.

ad 1.: We have to find a positive $\varepsilon^* \le 3\varepsilon_0$ with the properties described in definition 2.2. We claim that it suffices to choose ε^* such that $||\gamma(t)|| + \varepsilon^* \le \varepsilon_0$ for all $t \in [\,a, b\,]$.

To show that this choice is appropriate let a positive ε with $\varepsilon \le \varepsilon^*$ be given. We put $\tau_0 := \varepsilon_0^n - (\varepsilon_0 - \varepsilon)^n$. For a continuous $\tau : [\,a, b\,] \to \mathbb{R}$ with $||\tau|| \le \tau_0$ we have to find a continuous walk d such that $||d|| \le 3\varepsilon_0$, $H \circ (\gamma + d) = H \circ \gamma + \tau$, and $\gamma + d$ connects $W\big(\gamma(a), \pi, \tau(a)\big)$ with $W\big(\gamma(b), \pi, \tau(b)\big)$; here π is the unique vector such that $\gamma(a), \gamma(b)$ is in the interior of Q^π. (It is a consequence of lemma 2.1(i) that it is the same π for $\gamma(a)$ and for $\gamma(b)$.)
We define $\sigma := H \circ \gamma + \tau$ on $[\,a, b\,]$, $x := W\big(\gamma(a), \pi, \tau(a)\big)$ and $y := W\big(\gamma(b), \pi, \tau(b)\big)$. Since

$$|\sigma(t)| \le |H\big(\gamma(t)\big)| + |\tau(t)| \le (\varepsilon_0 - \varepsilon)^n + \tau_0 \le \varepsilon_0^n$$

the assumptions of lemma 2.5(i) are satisfied. We find $D = D_{a,b,x,y;\sigma}$ as in this lemma, and then it is clear that $d := D - \gamma$ has the desired properties.

ad 2.: The proof is similar, this time lemma 2.5(ii) comes into play.

ad 3.: Here lemma 2.5(iii) will be used. Note first that $Z^+_{\gamma(a)}$ is a singleton $\{\pi\}$ so that $\mathcal{A}^l_{\mathcal{I}_1} = \{(\pi, \pi)\}$. Since we assume that $H \circ \gamma \ge 0$ we have $\mathcal{A}^r_{\mathcal{I}_2} = \{(\pi, \tilde{\pi}) \mid \tilde{\pi} \in Z^-_{\gamma(b)}\}$.

We define ε^* as in the proof of "1.", and additionally we assume that ε^* is so small that the absolute value of all components of $\gamma(b)$ that are non-zero are bounded from below by ε^*. The claim is that this choice of ε^* is appropriate.

Let $\varepsilon \in \,]\,0, \varepsilon^*\,]$ be given, we define $\tau_0 := \varepsilon^n$; note that then also $\varepsilon_0^n - (\varepsilon_0 - \varepsilon)^n \ge \tau_0$. Suppose that $\tau : [\,a, b\,] \to \mathbb{R}$ is continuous with $||\tau|| \le \tau_0$. We put $\sigma := H \circ \gamma + \tau$ (defined on $[\,a, b\,]$), $x := W\big(\gamma(a), \pi, \tau(a)\big)$, $y_0 := \gamma(b)$ and $y := W_\varepsilon(y_0, \pi, \tilde{\pi}, \tau(b))$. We note that this is possible since $|\tau(b)| \le \varepsilon^n$. Lemma 2.5(iii) provides a path $D_{a,b;x,y;\sigma}$ from x to y with $||D|| \le 2\varepsilon_0$, and it is easy to check that $d := D - \gamma$ satisfies $||d|| \le 3\varepsilon_0$, $H \circ \big(\gamma + d\big) = H \circ \gamma + \tau$, and at a and b the walk $\gamma + d$ touches the canonical positions.

ad 4: Here a little trick will be necessary. We define ε and τ_0 as in the proof of „1.“, and we suppose, e.g., that $H \circ \gamma(a) > 0$. Let τ, π_1, π_2 and $\tilde{\pi}$ be given, where

- τ is continuous and $||\tau|| \le \tau_0$.
- π_1 is such that $\gamma(a) \in Q^{\pi_1}$.
- $\pi_2 \in Z^+_{\gamma(b)}$, $\tilde{\pi} \in Z^-_{\gamma(b)}$.

And we have to produce a walk $\gamma + d$ that starts at $x := W\big(\gamma(a), \pi_1, \tau(a)\big)$, that ends at $y := W_\varepsilon\big(\gamma(b), \pi_2, \tilde{\pi}, \tau(b)\big)$, and that satisfies $H \circ (\gamma + d) = H \circ \gamma + \tau$.

The problem is that π_1 might be different from π_2. But $H \circ \gamma$ is of type "$\pm$" at b so that we may choose a', b' with $a < a' < b' < b$ such that $H \circ \gamma(a') < 0 < H \circ \gamma(b')$. We decrease τ_0 (if necessary) such that $H \circ \gamma(a') + \tau_0 < 0 < H \circ \gamma(b') - \tau_0$. This guarantees that the function $\sigma := H \circ \gamma + \tau$ is strictly negative at a' and strictly positive at b'.

Next we choose an $x' \in Q^{\tilde\pi}$ such that $H(x') = \sigma(a')$ and $||x'|| \le \varepsilon_0$. This is possible since $|\sigma(a')| \le \varepsilon_0^n$. Also we select $y' \in Q^{\pi_2}$ with $H(y') = \sigma(b')$ and $||y'|| \le \varepsilon_0$.

It remains to apply lemma 2.5 to find $D : [\,a, b\,] \to \mathbb{R}^n$ such that $||D(t)|| \le 2\varepsilon_0$ and $H \circ D(t) = \sigma(t)$ for all t: First move from x to x' according to the walk described in lemma 2.5(ii), continue (again using the construction in 2.5(ii)) to y', and the final part of the walk is as described in 2.5(iii), where $y_0 = \gamma(b)$. With $d := \gamma - D$ we have found a function with the desired properties. □

3. Proof of the main result

After the preceding preparations we are now able to prove our main result, theorem 1.2. The structure will be as follows:

- Proof of $(i) \Rightarrow (ii)$; this will be rather simple.
- Definition of a more refined variant of "$(ii) \Rightarrow (i)$".
- Proof by induction that the refined variant holds for all n.
- A summary.

Proof of (i)⇒(ii)

We start with an observation concerning the definition of Z_x^+ for $x = (x_i) \in \mathbb{R}^n$. Let $\varepsilon > 0$ be such that $|x_i| > \varepsilon$ for all x_i with $x_i \ne 0$. It then follows immediately from the definition of Z_x^+ that $x + y$ will lie in some Q^π with $\pi \in Z_x^+$ whenever $H(x + y) > H(x)$ and $||y|| \le \varepsilon$.

Now we prove by contradiction that (i) implies (ii). We assume that γ is not admissible, and we will show that then (i) cannot be true. Suppose that, e.g., γ is not positive admissible. Then there are $t_1 < \cdots < t_n$ such that $H \circ \gamma \ge 0$ on $[\,t_1, t_n\,]$, and $\bigcap_i Z^+_{\gamma(t_i)} = \emptyset$. Choose $\varepsilon > 0$ such that it satisfies the condition of the preceding paragraph for all $x = \gamma(t_i)$, $i = 1, \ldots, n$. If (i) would be true we could find a continuous $d : [\,0, 1\,] \to \mathbb{R}^n$ such that $||d(t)|| \le \varepsilon$ for all t and $H \circ (\gamma + d) > H \circ \gamma$. In particular $H \circ (\gamma + d)$ would be strictly positive on $[\,t_1, t_n\,]$.

Now we apply the preceding observation. Each $(\gamma + d)(t_i)$ will lie in some Q^π with $\pi \in Z^+_{\gamma(t_i)}$. But there is no π that lies in *all* $Z^+_{\gamma(t_i)}$ so that there must exist i, j such that $(\gamma + d)(t_i)$ resp. $(\gamma + d)(t_j)$ lie in Q^π resp. $Q^{\tilde\pi}$ with $\pi \ne \tilde\pi$. But every continuous path from a point of Q^π to one in $Q^{\tilde\pi}$ has to pass through $\{H = 0\}$ so that we find a t between t_i and t_j with $H\big((\gamma + d)(t)\big) = 0$. This is a contradiction, since $H \circ (\gamma + d)$ was assumed to be strictly positive on $[\,t_1, t_n\,]$.

Definition of a refined variant of "(ii) ⇒ (i)"

Definition 3.1. By $(\mathbb{A})_n$ we mean the following assertion: Whenever functions $f_1, \ldots, f_n \in C\,[0,1]$ are given such that the associated walk γ is admissible, then the following holds: If $H \circ \gamma$ is of type $T(\mathcal{T}_1, \mathcal{T}_2)$ on $[0,1]$, then γ is of type $T_{\rm pep}(\mathcal{T}_1, \mathcal{T}_2)$.

Admittedly this looks much more clumsy than the statement "$(ii) \Rightarrow (i)$". In fact it is a sharper assertion:

Proposition 3.2. *Suppose that $(\mathbb{A})_n$ holds. Then it is true that "$(ii) \Rightarrow (i)$" is valid in theorem 1.2.*

Proof. First we note that $(\mathbb{A})_n$ implies that one may replace $[0,1]$ in the definition of $(\mathbb{A})_n$ by any subinterval $[a,b]$. (Simply consider the walk $t \mapsto \gamma\big(a+t(b-a)\big)$ instead of γ; this map is also admissible.)

Now let an admissible γ and an $\varepsilon_0 > 0$ be given. We have to provide – for "sufficiently small" functions τ – a continuous d with $||d|| \le \varepsilon_0$ such that $H \circ (\gamma + d) = H \circ \gamma + \tau$.

Suppose first that $H \circ \gamma$ vanishes identically. That in this case $H \circ \gamma$ is an interior point of the product of the balls $B_{\varepsilon_0}(f_i)$ (i.e., the assertion (i) of the theorem holds) is an immediate consequence of proposition 2.3.

So let us assume that there is an $a \in\,]\,0,1\,[$ with $H \circ \gamma(a) \ne 0$. If $H \circ \gamma$ is of some type $T(\mathcal{T}_1, \mathcal{T}_2)$ on $[0,1]$ it is of type $T_{\rm pep}(\mathcal{T}_1, \mathcal{T}_2)$ by assumption, and (i) of the theorem follows again immediately (cf. definition 2.2 where the *pep*-property was introduced).

It remains to deal with situations where there are possibly a', b with $0 < a' < a < b < 1$ such that $H \circ \gamma$ vanishes identically on $[0, a']$ and/or on $[b, 1]$. We may suppose that $[0, a']$ is a maximal interval where $H \circ \gamma$ vanishes so that $H \circ \gamma$ is of some type $T(\mathcal{T}_1, u)$ on $[a', a]$ with $\mathcal{T}_1 \in \{+, -, \pm\}$. Now proposition 2.3 comes again into play, the argument will depend on $\mathcal{T}_1$.

Consider first the case $\mathcal{T}_1 = +$. Then $H \circ \gamma$ is nonnegative on a suitable interval $[0, a' + \delta_0]$, and there are t where this function is strictly positive. Consequently there is (by lemma 2.1(i)) a unique $\pi \in \Pi^+$ such that $\bigcap_{t \in [0, a'+\delta_0]} Z^+_{\gamma(t)} = \{\pi\}$. Also (by lemma 2.1(ii)) there is a $\tilde{\pi} \in \bigcap_{t \in [0,a']} Z^-_{\gamma(t)}$.

Choose ε^* for ε_0 according to the *pep*-condition on $[a', a]$, put $\varepsilon := \varepsilon^*$ and select then a τ_0 for this ε^*. We may suppose that τ_0 is so small that it satisfies the conditions of proposition 2.3. Now let a continuous $\tau : [0, a] \to \mathbb{R}$ with $||\tau|| \le \tau_0$ be given. Proposition 2.3 and the *pep*-condition provide continuous walks of the dog d_1 and d_2 on $[0, a']$ and on $[a', a]$ respectively such that the norm is bounded by ε_0, $H \circ (\gamma + d_1) = H \circ \gamma + \tau$ on $[0, a']$ and $H \circ (\gamma + d_2) = H \circ \gamma + \tau$ on $[a', a]$. At a' the functions $\gamma + d_1$ and $\gamma + d_2$ coincide, both have the value $W_\varepsilon\big(\gamma(a'), \pi, \tilde{\pi}, \tau(a')\big)$ so that d_1 and d_2 can be glued together in a continuous way. This gives rise to a walk d on $[0, a]$ with the desired properties.

If $\mathcal{T}_1 = -$ one argues similarly. Finally suppose that $\mathcal{T}_1 = \pm$. Choose any $\pi \in \bigcap_{t \in [0,a']} Z^+_{\gamma(t)}$, $\tilde{\pi} \in \bigcap_{t \in [0,a']} Z^-_{\gamma(t)}$ (which is possible by lemma 2.1(iii)) and apply as before proposition 2.3 and the *pep*-condition with these $\pi, \tilde{\pi}$.

In this way we have produced an admissible walk on $[0,a]$. The interval $[a,1]$ can be treated in the same way, and it only remains to glue the walks together at a. Since the positions at a for both walks $\gamma + d$ (on $[0,a]$ and on $[a,1]$) are the canonical vector $W\big(\gamma(a),\pi,\tau(a)\big)$ (where $Z^+_{\gamma(a)} = \{\pi\}$) this construction gives rise to a continuous walk on all of $[0,1]$ with the desired properties. ☐

Proof by induction that the refined variant holds for all n

It remains to show that $(\mathbb{A})_n$ holds for every n. The case $n = 1$ is rather simple, one can always work with $d(t) = \tau(t)$. Let us suppose that $(\mathbb{A})_n$ has been verified for some n, and we will prove that $(\mathbb{A})_{n+1}$ also holds.

To this end let $f_0,\ldots,f_n \in C[0,1]$ be given such that the associated walk $\gamma : t \mapsto (f_0(t),\ldots,f_n(t))$ is admissible and $H\circ\gamma$ is of some type $T(\mathcal{T}_1,\mathcal{T}_2)$ on $[0,1]$. We have to show that γ is of type $T_{pep}(\mathcal{T}_1,\mathcal{T}_2)$.

The *idea of the proof* will be to partition $[0,1]$ into finitely many subintervals such that on each of these subintervals one of the following conditions is satisfied:

- At least one component of γ is bounded away from zero, or
- all components of γ are close to zero.

In the first case $T_{\rm pep}$ will follow from $(\mathbb{A})_n$, and in the second with the help of the constructions at the end of the last section. It then will only be necessary to glue the parts together as in the proof of lemma 2.4.

This will now be made precise. We suppose that $H\circ\gamma$ is of some type $T(\mathcal{T}_1',\mathcal{T}_2')$ on $[0,1]$ and that $\varepsilon_0 > 0$. We will show that γ is of type $T^{3\varepsilon_0}_{\rm pep}(\mathcal{T}_1',\mathcal{T}_2')$.

Lemma 3.3. *There is a partition $0 = a_0 < a_1 < \cdots < a_k = 1$ such that the intervals $I_j := [a_j,a_{j+1}]$ have the following property:*

1. *On each I_j $H\circ\gamma$ is either of type $T(0)$ or of some type $T(\mathcal{T}_1,\mathcal{T}_2)$.*
2. *If $H\circ\gamma$ is of some type $T(\mathcal{T}_1,\mathcal{T}_2)$ on I_j, then (at least) one of the following statements is true: $||\gamma(t)|| < \varepsilon_0$ for all $t \in I_j$, or there is an $i \in \{0,\ldots,n\}$ such that $|f_i(t)| \geq \varepsilon_0/2$ (all $t \in I_j$).*
3. *No intervals of type $T(0)$ and no intervals where $H\circ\gamma$ has some type $T(\mathcal{T}_1,\mathcal{T}_2)$ are adjacent.*

Proof. This is simple. In a first step one finds the I_j such that “2.” holds. Split each I_j further (if necessary) into intervals for which $H\circ\gamma$ has some type $T(\mathcal{T}_1,\mathcal{T}_2)$ and others with type $T(0)$. Finally pass to unions of adjacent intervals with type $T(0)$ and to unions of adjacent intervals where $H\circ\gamma$ has some type $T(\mathcal{T}_1,\mathcal{T}_2)$. ☐

Lemma 3.4. *Suppose that an interval of the preceding partition has some type $T(\mathcal{T}_1,\mathcal{T}_2)$. Then it has type $T^{3\varepsilon_0}_{pep}(\mathcal{T}_1,\mathcal{T}_2)$.*

Proof. For the I_j where $||\gamma(t)|| < \varepsilon_0$ this is just the assertion of proposition 2.6. Suppose that one component of γ is bounded away from zero. Without loss of generality we may assume that $f_0 \geq \varepsilon_0/2$ on I_j. In order to apply the induction hypothesis we consider the walk $\tilde{\gamma} : t \mapsto \big(f_1(t), \ldots, f_n(t)\big)$ for $t \in I_j$. It is straightforward to show that $\tilde{\gamma}$ is admissible. The elementary argument starts with the observation that a $(\pi_0, \ldots, \pi_n)$ belongs to $Z^+_{\gamma(t)}$ iff $\pi_0 = 1$ and $(\pi_1, \ldots, \pi_n) \in Z^+_{\tilde{\gamma}(t)}$ (for $t \in I_j$). It is also easy to verify that $H \circ \tilde{\gamma}$ has type $T(\mathcal{T}_1, \mathcal{T}_2)$ on I_j if $H \circ \gamma$ has this type[4]. Only very elementary facts come into play: If $x_0 \geq \varepsilon/2$ and $x_0 \cdots x_n > 0$ then $x_1 \cdots x_n > 0$ etc.

By assumption $\tilde{\gamma}$ has $T_{\rm pep}(\mathcal{T}_1, \mathcal{T}_2)$ on I_j. We choose ε^* as in definition 2.2 for ε_0 and we select any $\varepsilon \in \,]\,0, \varepsilon^*\,]$ and the associated τ_0. Put $\tau_0' := \varepsilon_0\tau_0/2$ and consider a continuous $\tau : I_j \to \mathbb{R}$ with $||\tau|| \leq \tau_0'$. Then $\tilde{\tau} := (\tau/f_0)|_{I_j}$ satisfies $||\tilde{\tau}|| \leq \tau_0$, and therefore there is a continuous $\tilde{d} : I_j \to \mathbb{R}^n$ such that $||\tilde{d}|| \leq \varepsilon_0$, $H \circ (\tilde{\gamma} + \tilde{d}) = H \circ \tilde{\gamma} + \tilde{\tau}$, and at the end points of I_j the walk $\tilde{\gamma} + \tilde{d}$ is at the canonical positions[5].

We claim that $d(t) := (0, \tilde{d}_1, \ldots, \tilde{d}_n)$ has (essentially) the desired properties. In fact, it is continuous, the norm is bounded by ε_0 and $H \circ \gamma + \tau = H \circ (\gamma + d)$. It remains to check whether the walk starts at the canonical points. This is true whenever the left and right type is in $\{+, -, \pm\}$, this follows from the definition 2.2 of the canonical positions. But it is not true in the case $\mathcal{T} = u$.

The problem is the following. Suppose, e.g., that the left type is u. Then the walk that we have constructed starts at some point

$$x := \big(f_0(a_j), f_1(a_j) + s, \ldots, f_1(a_j) + s\big)$$

with a suitable small number s, but it *should* start at the $(n+1)$-dimensional

$$W\big(\gamma(a_j), \pi, \tau(a_j)\big) = y := \big(f_0(a_j) + s', \ldots, f_n(a_j) + s'\big)$$

where both vectors have the same H-value. This can be overcome by using the same techniques as in the proof of lemma 2.5: Choose $a' > a_j$ that is sufficiently close to a_j and apply the preceding argument to the interval $[\,a', a_{j+1}\,]$: The walk will now start at some $x' := \big(f_0(a'), f_1(a') + s, \ldots, f_1(a') + s\big)$. And the interval $[\,a, a'\,]$ will be used for a walk from y to x' that stays close to γ and for which $H \circ (\gamma + d) = H \circ \gamma + \tau$. This can be done without much effort since we are in a situation where all functions are nonzero and – if $a' - a_j$ is small – nearly constant. □

We now *complete the induction proof*. $[\,0, 1\,]$ is partitioned into intervals $I_0, \ldots, I_{k-1}$ as in lemma 3.3, and on intervals where $H \circ \gamma$ has some type we know that γ has the corresponding $T^{3\varepsilon_0}_{\rm pep}$-type. We will show that this will suffice

[4]For the sake of simplicity we use the same symbol H for the functions $(x_0, \ldots, x_n) \mapsto x_0 \cdots x_n$ and $(x_1, \ldots, x_n) \mapsto x_1 \cdots x_n$.

[5]More precisely: $(\tilde{\gamma} + \tilde{d})(a_j)$ equals $W(\tilde{\gamma}(a_j), \pi, \tilde{\tau}(a_j))$ if $H(\tilde{\gamma}(a_j)) \neq 0$ and $W_\varepsilon\big(\tilde{\gamma}(a_j), \pi, \tilde{\pi}, \tilde{\tau}(a_j)\big)$ otherwise; here $(\pi, \tilde{\pi})$ can be prescribed as any left-admissible pair. Similar conditions are satisfied at a_{j+1}.

to prove that γ has type $T^{3\varepsilon_0}_{\rm pep}(\mathcal{T}_1', \mathcal{T}_2')$ on $[\,0,1\,]$. The idea how to do this is not new, we have used it in the proofs of proposition 2.4 and proposition 3.2. We will construct the desired walk of the dog d with prescribed $H\circ(\gamma+d) = H\circ\gamma+\tau$ by glueing together the walks on $I_0,\ldots,I_{k-1}$. To achieve this one has to check whether the possible boundary conditions fit.

As a *first example* consider a situation where $H\circ\gamma$ is of type $T(\mathcal{T}_1,+)$ on I_j, of type $T(0)$ on I_{j+1} and of type $T(+,\mathcal{T}_2)$ on I_{j+2}. The claim is that the *pep*-condition on I_j and I_{j+2} implies that γ has type $T^{3\varepsilon_0}_{\rm pep}(\mathcal{T}_1,\mathcal{T}_2)$ on $I_j\cup I_{j+1}\cup I_{j+2}$.

We know that at a_{j+1}, the right end point of I_j, one may prescribe an endpoint in $Q^\pi\cup Q^{\tilde\pi}$, where π is the unique element of $\bigcap_{t\in[\,a_{j+1}-\delta_0,a_{j+1}\,]} Z^+_{\gamma(t)}$ for some positive δ_0 and $\tilde\pi\in Z^-_{\gamma(a_{j+1})}$ is arbitrary. A similar fact is known for the starting point in I_{j+2}, it lies in $Q^{\pi'}\cup Q^{\tilde\pi'}$, where $\bigcap_{t\in[\,a_{j+2},a_{j+2}+\delta_0\,]} Z^+_{\gamma(t)} = \{\pi'\}$ and $\tilde\pi'\in Z^-_{\gamma(a_{j+2})}$ is arbitrary.

With the help of proposition 2.3 the gap between $t=a_{j+1}$ and $t=a_{j+2}$ could be filled if we knew that $\pi=\pi'$. Fortunately this is true: One simply has to apply lemma 2.1(i) to the interval $[\,a_j-\delta_0, a_{j+2}+\delta_0\,]$. $H\circ\gamma$ is nonnegative there and sometimes strictly positive so that just one element lies in the intersection of the $Z^+_{\gamma(t)}$, $t\in[\,a_j-\delta_0,a_{j+2}+\delta_0\,]$.

The rest is routine. Choose ε and τ_0 so small that they are appropriate for I_j, I_{j+1} and I_{j+2}, any left admissible pair $(\pi_a,\tilde\pi_a)$ at a_j and any right admissible pair $(\pi_b,\tilde\pi_b)$ at a_{j+3}. Then, if $\tau:[\,a_j,a_{j+3}\,]\to\mathbb{R}$ is continuous with $||\tau||\le\tau_0$ we can find walks d_j,d_{j+1},d_{j+3} on I_j,I_{j+1},I_{j+2}, respectively, with small norm such that at a_{j+1} (resp. at a_{j+2}) d_1 and d_2 (resp. d_2 and d_3) occupy the same position. Therefore they can be glued together to give rise to a walk an $[\,a_j,a_{j+3}\,]$.

The preceding example shows that the essential part of the argument is to guarantee that the admissible end point conditions fit. Here is a *second example* where $H\circ\gamma$ is of type $T(\mathcal{T}_1,+)$ on I_j, of type $T(0)$ on I_{j+1} and of type $T(\pm,\mathcal{T}_2)$ on I_{j+2}. This is even simpler, because then we can choose again $\pi\in\bigcap_{t\in[\,a_{j+1}-\delta_0,a_{j+2}\,]} Z^+_{\gamma(t)}$ as in the first example and any $\tilde\pi\in Z^-_{\gamma(a_{j+1})}$. Then $(\pi,\tilde\pi)$ is a right-admissible pair for I_j and a left-admissible pair for I_{j+2}, and the rest of the proof is similar.

As a *third example* we consider a situation where $H\circ\gamma$ is of type $T(\mathcal{T}_1,+)$ on I_j, of type $T(0)$ on I_{j+1} and of type $T(-,\mathcal{T}_2)$ on I_{j+2}. Note that there is a unique $\pi\in\bigcap_{t\in[\,a_{j+1}-\delta_0,a_{j+2}\,]}$ and a unique $\tilde\pi\in\bigcap_{t\in[\,a_{j+1},a_{j+2}+\delta_0\,]}$ for a sufficiently small positive δ_0 (by lemma 1.1(i) and (ii)). $(\pi,\tilde\pi)$ is a right-admissible pair for I_j and a left-admissible pair for I_{j+1} and we can continue as in the first example.

All other possibilities can be treated in a similar way, and after applying this procedure several times we finally arrive at a walk of the dog that is defined for all $t\in[\,0,1\,]$. Thus the proof of theorem 1.2 is complete.

A summary

It has to be admitted that the proof is technically rather involved. The main ingredients are:

- Treat intervals where $\gamma(t)$ is small separately.
- Use induction where some component of γ is bounded away from zero.
- Use a general result on intervals where $H \circ \gamma$ vanishes.

Needless to say that it is not easy to provide a concrete positive τ_0 for given ε_0, since in any of the finitely many construction steps it might be necessary to pass to a smaller τ_0. Our use of canonical end points has made it possible to glue together walks in a continuous way that are defined on adjacent subintervals, and lemma 2.1 was important to guarantee that the conditions when coming from the right resp. from the left are compatible.

4. THE CASE OF COMPLEX SCALARS

We will now consider the case of complex valued continuous functions on $[0,1]$. It will be shown in the next proposition that there the product of open sets is always open. The proof is prepared by three lemmas.

Lemma 4.1. *For every $r > 0$ there is a $\delta > 0$ with the following property: There exists a continuous function*

$$\phi : \{a \in \mathbb{C} \mid |a| \leq r\} \times \{d \in \mathbb{C} \mid |d| \leq \delta\} \to \mathbb{C}$$

such that $z_0 := \phi(a,d)$ solves the equation $z_0 + az_0^2 = d$, and $\phi(0,d) = d$ for all d with $|d| \leq \delta$.

Proof. Let $\delta > 0$ be such that $4r\delta < 1$. Put $\phi(0,d) := d$ and $\phi(a,d) :=$ "the root of $z + az^2 = d$ that is closer to zero" for $a \neq 0$. This mapping ϕ is well-defined, and it has the desired properties. □

Lemma 4.2. *Suppose that $0 < \varepsilon < r$ are given. There is a $\delta > 0$ such that there exist continuous functions*

$$\psi_1, \psi_2 : \{(a,b,d) \in \mathbb{C}^3 \mid \varepsilon \leq |a|^2 + |b|^2 \leq r, |d| \leq \delta\}$$

with the following property: the numbers $z = \psi_1(a,b,d)$, $w = \psi_2(a,b,d)$ solve the equation

$$az + bw + zw = d,$$

and $\psi_1(a,b,0) = \psi_2(a,b,0) = 0$

Proof. ψ_1 and ψ_2 will be defined with the help of lemma 4.1. We will put

$$\psi_1(a,b,d) := \frac{\overline{a} \cdot d}{|a|^2 + |b|^2} + \overline{a} \cdot z_0, \psi_2(a,b,d) := \frac{\overline{b} \cdot d}{|a|^2 + |b|^2} + \overline{b} \cdot z_0$$

with a "small" z_0. If d is sufficiently small the equation $az + bw + zw = d$ (where $z = \psi_1(a,b,d)$, $w = \psi_2(a,b,d)$) leads precisely to an equation as in lemma 4.1 so that z_0 as a continuous function of the parameters can be found. □

Lemma 4.3. *Let $\varepsilon_0 > 0$ be given. We suppose that z_0, w_0, z_1, w_1 are complex numbers with absolute value at most ε_0 and that $\sigma : [\,a, b\,] \to \mathbb{C}$ is a continuous function such that*

$$\sigma(a) = z_0 w_0,\ \sigma(b) = z_1 w_1$$

and $|\sigma(t)| \leq \varepsilon_0^2$ for all t. Then there are continuous functions $z, w : [\,a, b\,] \to \mathbb{C}$ such that

$$z_0 = z(a),\ w_0 = w(a),\ z_1 = z(b),\ w_1 = w(b),$$

and $|z(t)|, |w(t)| \leq \varepsilon_0$ and $z(t)w(t) = \sigma(t)$ hold for all t.

Proof. Note that this lemma has no analogue in the case of real scalars. It is true since the boundary of the complex ball with radius ε_0 is connected. Consider as a typical example the case $[\,a, b\,] = [\,0, 1\,]$, $z_0 = w_0 = \varepsilon_0 = 1$, $z_1, w_1 = -1$ and $\sigma(t) = 1$. Solutions with real $z(\cdot), w(\cdot)$ do not exist, but $z(t) := e^{it\pi}, w(t) := e^{-it\pi}$ have the desired properties.

The general case can be treated by applying the same idea. Suppose, e.g., that $\sigma(a), \sigma(b) \neq 0$ and that $|z_0| \geq |w_0|$ and $|z_1| \geq |w_1|$. Choose $z(\cdot)$ as a continuous path from z_0 to z_1 such that z is nowhere zero, bounded by ε_0 and that $|z(t)| \geq \sqrt{|\sigma(t)|}$ holds for all t. Put $w(t) := \sigma(t)/z(t)$. The other possible cases can be treated similarly. □

Proposition 4.4. *Let $O_1, \ldots, O_n$ be open subsets of the Banach space $C_{\mathbb{C}}\,[\,0, 1\,]$ of continuous complex-valued functions on $[\,0, 1\,]$, provided with the supremum norm. Then $O_1 \cdots O_n$ is also open.*

Proof. It will suffice to prove the proposition for $n = 2$. The assertion will follow easily from the following

Claim: Let $f_1, f_2 : [\,0, 1\,] \to \mathbb{C}$ be continuous and $\varepsilon > 0$. Then one can find a $\tau_0 > 0$ with the following property: whenever $\tau : [\,0, 1\,] \to \mathbb{C}$ is continuous with $||\tau|| \leq \tau_0$ there exist continuous $d_1, d_2 : [\,0, 1\,] \to \mathbb{C}$ such that $||d_1||, ||d_2|| \leq 5\varepsilon$, and

$$\big(f_1(t) + d_1(t)\big)\big(f_2(t) + d_2(t)\big) = f_1(t)f_2(t) + \tau(t) \tag{4.1}$$

for all t.

Proof of the claim: Let $\varepsilon > 0$ be given. We partition $[\,0, 1\,]$ into intervals $I_i = [\,a_i, a_{i+1}\,]$ $(i = 0, \ldots, k-1)$ such that for each i one of the following conditions is satisfied:

1. $|f_1(t)|^2 + |f_2(t)|^2 \geq \varepsilon$ for all $t \in I_i$; or
2. $|f_1(t)|^2 + |f_2(t)|^2 \leq 2\varepsilon$ for all $t \in I_i$.

We assume that no subintervals of type "1" and no subintervals of type "2" are adjacent.

Let a continuous $\tau : [\,0, 1\,] \to \mathbb{C}$ with "sufficiently small" $||\tau||$ be given (the maximal size of τ will be made precise in the following proof). First we define

d_1, d_2 on the I_i with type "1". Choose r such that $|f_1(t)|^2 + |f_2(t)|^2 \le r$ on $[0,1]$. Then we put

$$d_1(t) := \psi_2(f_1(t), f_2(t), \tau(t)),\ d_2(t) := \psi_1(f_1(t), f_2(t), \tau(t)),$$

where ψ_1, ψ_2 are as in lemma 4.2: this can be done if (with the notation of this lemma) $||\tau|| \le \delta$. We then know that d_1, d_2 are continuous, that the norm of these functions will be bounded by 5ε if $||\tau||$ is sufficiently small and that

$$f_1(t)d_2(t) + f_2(t)d_1(t) + d_1(t)d_2(t) = \tau(t);$$

this is precisely the equation 4.1.

It remains to extend the definition of d_1, d_2 to the I_i of type "2". Let I_i be such an interval. Then I_{i-1} and I_{i+1} are of type "1" (the obvious modifications of the proof when $i = 0$ or $i = k-1$ are left to the reader). The functions d_1, d_2 are already defined on I_{i-1} and I_{i+1}, and we put

$$z_0 := (f_1 + d_1)(a_i), w_0 := (f_2 + d_2)(a_i),$$

$$z_1 := (f_1 + d_1)(a_{i+1}), w_1 := (f_2 + d_2)(a_{i+1}).$$

Now lemma 4.3 comes into play, where the function $\sigma : I_i \to \mathbb{C}$ will be defined by $z \mapsto f_1(t)f_2(t)+\tau(t)$. With $\varepsilon_0 := 3\varepsilon$ the conditions of the lemma are satisfied. Let $z(\cdot), w(\cdot)$ be as in this lemma. We define

$$d_1(t) := z(t) - f_1(t),\ d_2(t) := w(t) - f_2(t).$$

Lemma 4.3 guarantees that the equation 4.1 holds, and

$$|d_1(t)| \le |z(t)| + |f_1(t)| \le 5\varepsilon,\ |d_2(t)| \le |w(t)| + |f_2(t)| \le 5\varepsilon.$$

The definitions of the d_i on the various I_i can be glued together to give rise to continuous functions since at the endpoints a_i the values coincide. □

5. Consequences of the main theorem, concluding remarks

We have characterized the fact that $f_1 \cdots f_n$ is an interior point of the set $B_\varepsilon(f_1) \cdots B_\varepsilon(f_n)$ for all $\varepsilon > 0$ by a geometric-topological condition. This implies an easy-to-check criterion:

Proposition 5.1. *Let $f_1, \ldots, f_n \in C[0,1]$ be such that there are no common zeros, i.e., the sets $\{f_i = 0\}$ are pairwise disjoint. Then $f_1 \cdots f_n$ is an interior point of $B_\varepsilon(f_1) \cdots B_\varepsilon(f_n)$ for all $\varepsilon > 0$.*

Proof. We will show that $\gamma := (f_1, \ldots, f_n)$ is positive admissible. That γ is negative admissible follows by a similar argument (or by an application of the first part to $(-f_1, f_2, \ldots, f_n)$) so that the assertion is a consequence of our theorem 1.2.

Let $[a, b] \subset [0, 1]$ be an interval such that $H \circ \gamma|_{[a,b]} \geq 0$. Then no f_i changes its sign on $[a, b]$: this follows easily from the fact that the $\{f_i = 0\}$ are pairwise disjoint. Therefore we may choose $\pi_i \in \{-1, +1\}$ such that $\pi_i f_i|_{[a,b]} \geq 0$, and with $\pi := (\pi_i)$ we have found a π that lies in all $Z^+_{\gamma(t)}$. This proves that γ is positive admissible. □

Corollary 5.2. *Let $f_1, \ldots, f_n \in C[0, 1]$ be arbitrary and $\varepsilon > 0$. Then $B_\varepsilon(f_1) \cdots B_\varepsilon(f_n)$ contains interior points.*

Proof. Choose polynomials $g_1, \ldots, g_n$ such that g_i is $(\varepsilon/4)$-close to f_i. We may choose sufficiently small $\delta_i > 0$ such that the functions $\hat{g}(t) := g_i(t - \delta_i)$ have no common zeros and $\hat{g}_i$ is $(\varepsilon/2)$-close to f_i for every i. It follows from the preceding proposition that $\hat{g}_1 \cdots \hat{g}_n$ is an interior point of $B_{\varepsilon/2}(\hat{g}_1) \cdots B_{\varepsilon/2}(\hat{g}_n)$, and this set is contained in $B_\varepsilon(f_1) \cdots B_\varepsilon(f_n)$. □

Here is a *natural generalization* of the problem that we have discussed in this paper:

> Let A be a Banach algebra. How can one characterize the n-tupels $(x_1, \ldots, x_n) \in A^n$ such that $x_1 \cdots x_n$ is an interior point of $B_\varepsilon(x_1) \cdots B_\varepsilon(x_n)$ for every $\varepsilon > 0$?

In view of the rather involved investigations that were necessary here in the case $A = C_{\mathbb{R}}[0, 1]$ it is unlikely that a characterization in the general case is possible. Up to now only partial results are known, e.g. it is true for arbitrary $x_1, \ldots, x_n$ in $A = l^\infty$ that $x_1 \cdots x_n$ is always an interior point of $B_\varepsilon(x_1) \cdots B_\varepsilon(x_n)$. (A similar result holds, more generally, for arbitrary $f_1, \ldots, f_n$ in CK whenever K is a zero-dimensional compact Hausdorff space; see [3].)

We have proved that for $C[0, 1]$ the behaviour is different for real and complex scalars. In the following example of an operator algebra both cases can be treated simultaneously[6]:

Example: Let X be real or complex Banach space such that there exist an isometry $T : X \to X$ together with a unit vector e such that

$$||e + Tx|| = \max\{||e||, ||Tx||\} \, (= \max\{1, ||x||\})$$

holds for every x. (Consider, e.g., $X = l^\infty$, $T(x_1, x_2, \ldots) := (x_1, 0, x_2, 0, x_3)$ and $e = (0, 1, 0, 1, \ldots)$.)

Then, in the Banach algebra A of operators on X, the zero operator 0 is not an interior point of $B_1 \circ B_2$, where B_1 is the open ball with radius one and center T and B_2 is the open unit ball.

Proof. Let U be an operator on X such that $||U|| < 1$. We will show that $T + U$ is not surjective. Then $(T+U) \circ V$ will not be surjective for $V \in B_2$, in particular the operators $\varepsilon\,\mathrm{Id}$ will not be in $B_1 \circ B_2$ which would prove our claim.

[6] The example is a generalization of an example due to V. Kadets.

We will show that e is not in the range of $T + U$. If this were the case we could write $e = Tx + Ux$, and this would imply

$$||x|| > ||Ux|| = ||e - Tx|| = \max\{1, ||x||\}$$

which is absurd. □

ACKNOWLEDGEMENT: The author wishes to express his gratitude to the unknown referee for many helpful remarks that gave rise to a number of modifications. It is to be hoped that this helped to clarify the rather technical investigations of this paper.

REFERENCES

[1] M. BALCERZAK, W. WACHOWICZ, W. WILCZYŃSKI. *Multiplying balls in the space of continuous functions on* $[0,1]$. Studia Math. **170(2)** (2005), 203–209.

[2] E. BEHRENDS. *Walk the dog, or: products of open balls in the space of continuous functions.* Functiones et approximatio (to appear 2011).

[3] A. KOMISARSKI. *A connection between multiplication in* CX *and the dimension of* X. Fund. Math. **189(2)** (2006), 149–154.

[4] A. WACHOWICZ. *Multiplying balls in* $C^{(N)}[0,1]$. Real Analysis Exchange **34(2)** (2008/2009), 445–450.

Mathematisches Institut, Freie Universität Berlin, Arnimallee 6,
D-14 195 Berlin, Germany; e-mail: behrends@math.fu-berlin.de