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# Groups of rotationally symmetric permutations and magic mazes

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**Abstract** A Japanese company sells a magic trick with an interesting mathematical background. Commuting families of "symmetric" permutations play a central role. Our main result states that there are essentially as many such tricks as there are abelian finite groups.

**Keywords** Finite abelian group · Mathematical magic trick · Symmetric permutation · Permutation group · Maze

**Mathematics Subject Classification** 20B · 00A08

## 1 Introduction

The Japanese company Tenyo sells a magic trick under the name "Magic Maze" that uses special cards. They look similarly to those shown in Fig. 1 (these were designed by the author).

The underlying idea of the trick is the following. A spectator has a number of such cards, he or she can put them arbitrarily together to construct an individual maze. The order of cards is free, also it is allowed to turn any card by 180 degrees. Then one follows the maze from a certain starting position on the left to the corresponding right end point. As an example consider Fig. 2.

Here one arrives, e.g., at the bottom point at the right hand side if one starts top left.

The well hidden secret with these mazes is the fact that the magician has complete control: it is known to him or her which starting position is connected with which

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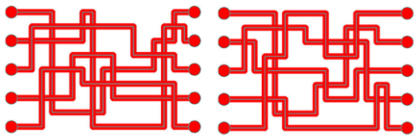

**Fig. 1** Two examples of "maze cards"

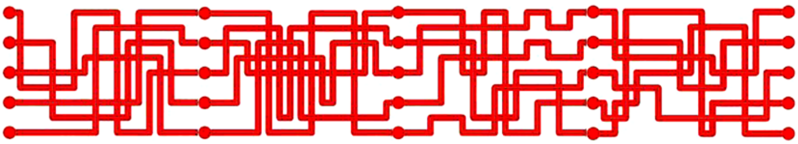

**Fig. 2** An individually designed maze

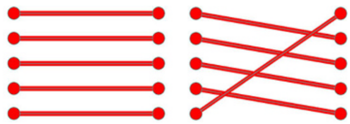

**Fig. 3** Sketches of the permutations underlying Fig. 1

final position, regardless of the individual combination of the building blocks. It is explained in the description of "Magic Maze" what should be done to transform this fact to a convincing magic trick.

The mathematical background is as follows. Each "maze card" corresponds to an element of the permutation group $S_5$ in disguise. The "raw versions" of the permutations shown in Fig. 1 are shown in Fig. 3.

They are the obvious visualizations of the permutations $\tau_0^5, \tau_1^5 \in S_5$, where $\tau_k^5$ stands for the shift $i \mapsto i + k \bmod 5$. (We consider the elements of $S_5$ as bijective mappings on $\mathbb{Z}_5$, the ring of residue classes modulo 5.)

The Tenyo cards are essentially the permutations $\tau_0^5, ..., \tau_4^5$. They have the following properties:

- They commute and their product is the identity $\tau_0^5$.
- If the visualization of a $\tau_k^5$ as in Fig. 3 is rotated by 180 degrees, one arrives at the same permutation.
- They operate transitively on $\mathbb{Z}_5$: for arbitrary $i, j \in \mathbb{Z}_5$ there is a $k$ such that $\tau_k^5(i) = j$.

This has a remarkable consequence for the magician: if he or she hands all cards with the exception of that corresponding to $\tau_{k_0}^5$ to the spectator, he or she can put them together freely and even rotate some of them. It is for sure that the result is the permutation $\tau_{-k_0}^5$, i.e., a walk will terminate (considered in cyclic order) $k_0$ positions above the starting position. In this way the magician has complete control, regardless of the seemingly free choices of the spectator.

We will study here a natural generalization of this "magic" family of permutations. First we note that the fact that "the permutation $\pi$ is the same if it is rotated by 180 degrees" just means that $\pi^* \circ \pi \circ \pi^* = \pi^{-1}$ where $\pi^* \in S_n$ is defined by $\pi^* : i \mapsto n - 1 - i$.

The collection of all such $\pi$ will be denoted by $S_n^{\text{r.s.}}$, they will be called *rotationally symmetric*. As already observed all $\tau_i^n$ lie in $S_n^{\text{r.s.}}$, but there are many other candidates, e.g. $(01)(23) \in S_4$.

**Definition 1** Let $n$ be an integer and $\mathcal{F} = \{\pi_0, ..., \pi_{n-1}\}$ a subset of the permutation group $S_n$

(i) $\mathcal{F}$ is called a (*)-*family*, if the elements of $\mathcal{F}$ commute and if they operate transitively on $\{0, 1, ..., n-1\}$.
(ii) If, in addition, all $\pi \in \mathcal{F}$ are rotationally symmetric, then $\mathcal{F}$ will be called a (**)-*family*.
(iii) We say that a $(*)$- or $(**)$-family $\mathcal{F} = \{\pi_0, ..., \pi_{n-1}\}$ is *normalized*, if $\pi_i(0) = i$ for all $i$.

$(*)$- and $(**)$-families $\mathcal{F}$ can be used similarly as the Tenyo cards to perform a magic trick:

- Use the $\pi \in \mathcal{F}$ to prepare cards with correponding more or less complicated mazes.
- Hand out all but one of these cards, say that correponding to $\pi' \in \mathcal{F}$, to a spectator. He or she may use them in any order to produce his or her personal maze. In the case of $(**)$-families it is also admissible to rotate the cards.
- Then the magician knows that this maze corresponds to $(\pi')^{-1} \circ \Pi_{\pi \neq \pi'} \pi$. If $\pi'$ runs through $\mathcal{F}$, the $(\pi')^{-1} \circ \Pi_{\pi \neq \pi'} \pi$ operate transitively so that – depending on the choice of $\pi'$ – the spectator will produce a maze for which the properties are known in advance.

Our main results concerning the possible $(*)$- and $(**)$-famillies are Proposition 1 ($(*)$-families are groups), Proposition 3 (isomorphic $(*)$-families are conjugate) and Proposition 4 (every finite group occurs as a $(**)$-family).

We note that Proposition 1 was already published in the article [2], and the explanation for the German magicians of the idea underlying the Tenyo trick could be found in [1].

## 2 $(*)$-families

First we show that, rather surprisingly, the families considered here are groups:

**Proposition 1** *Let $\mathcal{F} = \{\pi_0, ..., \pi_{n-1}\}$ be a $(*)$-family, w.l.o.g we may assume that it is normalized. Then $\mathcal{F}$ is an abelian subgroup of $S_n$, and $\pi_0$ is the identical permutation Id.*

***Proof*** We have $\pi_0(i) = \pi_0 \circ \pi_i(0) = \pi_i \circ \pi_0(0) = \pi_i(0) = i$, and this proves that $\pi_0 = \text{Id}$. Similarly one shows that $\pi_i(j) = \pi_j(i)$ for all $i, j$.

Now let $i, j$ be arbitrary and $k := \pi_i(j) = \pi_j(i)$. We claim that $\pi_i \circ \pi_j = \pi_k$, i.e., $\pi_i \circ \pi_j(l) = \pi_k(l)$ for all $l$. For $l = 0$ this is true by the definition of $k$, and for general $l$ we argue as follows:

$$\pi_i \circ \pi_j(l) = \pi_i \circ \pi_j \circ \pi_l(0) = \pi_l \circ \pi_i \circ \pi_j(0) = \pi_l(k) = \pi_k(l)$$

Therefore the product of two elements of $\mathcal{F}$ lies in $\mathcal{F}$, and this implies the group property. □

It is easy to construct new $(*)$-families:

**Proposition 2** *Let $\mathcal{F} \subset S_n$ be a $(*)$-family and $\nu \in S_n$. Then the conjugated group*

$$\mathcal{F}_\nu := \{\nu \circ \pi \circ \nu^{-1} \mid \pi \in \mathcal{F}\}$$

*is also a $(*)$-family.*

(The straightforward proof is omitted.)

We note that all groups $\mathcal{F}_\nu$ are isomorphic, and a reverse implication is also true:

**Proposition 3** *Let $\mathcal{G}$ and $\mathcal{H}$ be (not necessarily commutative) subgroups of $S_n$. We suppose that both contain n elements, that they operate transitively on $\{0, ..., n-1\}$ and that they are isomporphic. Then they are conjugate.*

***Proof*** We write $\mathcal{G} = \{g_0, g_1, ..., g_{n-1}\}$ and $\mathcal{H} = \{h_0, h_1, ..., h_{n-1}\}$, and w.l.o.g. we may assume that $g_i(0) = i$ and $h_j(0) = j$ for all $i, j$. Note that this implies that $g(0) = h(0) = \text{Id}$ since otherwise the family could not operate transitively.

Let $\phi : \mathcal{G} \to \mathcal{H}$ be a group isomorphism. We define $\nu : \mathbb{Z}_n \to \mathbb{Z}_n$ by $\nu(i) := \phi(g_i)(0)$, i.e., $\nu(i)$ is that $j$ that satisfies $\phi(g_i) = h_j$.

Surely $\nu \in S_n$, and we claim that $\nu \circ g_i = \phi(g_i) \circ \nu$ holds for all $i$, i.e., $(\nu \circ g_i)(k) = \phi(g_i)(\nu(k))$ for arbitrary $k$.

For $i = 0$ or $k = 0$ this is true by the definition of $\nu$, since $g_0$ and $h_0$ necessarily are the identity.

If $k$ is arbitrary, define $l$ by $g_i \circ g_k = g_l$. Then

$$\begin{aligned}(\nu \circ g_i)(k) &= (\nu \circ g_i)(g_k(0)) \\ &= \nu(g_i \circ g_k(0)) \\ &= \nu(g_l(0)) \\ &= \nu(l) \\ &= \phi(g_l)(0) \\ &= \phi(g_i) \circ \phi(g_k)(0) \\ &= \phi(g_i)(\nu(k)).\end{aligned}$$

This proves the claim. □

Thus $(*)$-families in $S_n$ correspond to one-to-one representations of commutative groups with $n$ elements in $S_n$, and two of them are isomorphic iff they are conjugate.

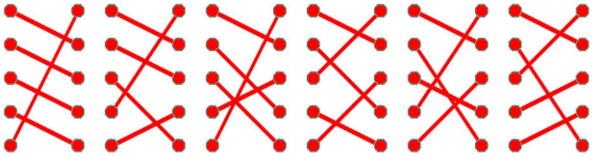

**Fig. 4** All $\pi \in S_5$ with period 5 and $\pi(0) = 1$

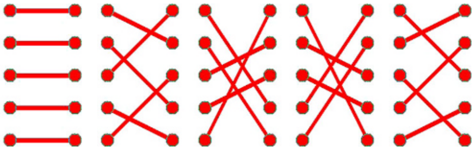

**Fig. 5** The $(**)$-family generated by $\pi = (01342)$

Particularly simple is the situation if $n$ is prime. Then every group with $n$ elements is cyclic so that it is conjugate to $\mathcal{T}_n = \{\tau_0^n, ..., \tau_{n-1}^n\} \subset S_n$, the subgroup of cyclic translations in $S_n$.

We stress, however, that this does not mean that the associated magic tricks use cards that look very similarly. As an example consider the case $n = 5$. There are 6 permutations $\pi$ with period 5 for which $\pi(0) = 1$, they generate different $(*)$-families (see Fig. 4).

Two of them (permutations 1 and 4) are rotationally symmetric so that they generate even a $(**)$-family. The first permutation gives rise to the group of cyclic translations, in the second case the elements of the generated group are shown in Fig. 5.

## 3 $(**)$-families

Does every commutative group give rise to a $(**)$-family? The example $\mathcal{T}_n$ shows that this is true for cyclic groups, and it remains to glue these examples together by considering tensor products.

We start with a number $n$ that is written as $n = k \cdot l$ with $k, l > 1$. Numbers $i \in \{0, ..., n-1\}$ can be uniquely written as $i = a \cdot l + b$, with $0 \le a < k$ and $0 \le b < l$; note that in the case $k = l$ this is the $k$-adic representation of $i$. We will use the notation $i = [a, b]$,

Now let $\nu \in S_k$ and $\mu \in S_l$ be permutations. We define a permutation in $S_n$ by $i = [a, b]_{k,l} \mapsto [\nu(a), \mu(b)]_{k,l}$. It will be denoted by $\nu \otimes \mu$.

**Lemma 1**

*(i)* $\nu \otimes \mu$ *is an element of* $S_n$ *.*
*(ii)* $(\nu_1 \circ \nu_2) \otimes (\mu_1 \circ \mu_2) = (\nu_1 \otimes \mu_1) \circ (\nu_2 \otimes \mu_2)$ *for* $\nu_1, \nu_2 \in S_k$ *and* $\mu_1, \mu_2 \in S_l$.
*(iii) With* $\nu, \mu$ *also* $\nu \otimes \mu$ *is rotationally symmetric.*

***Proof*** (i) and (ii) are obvious.

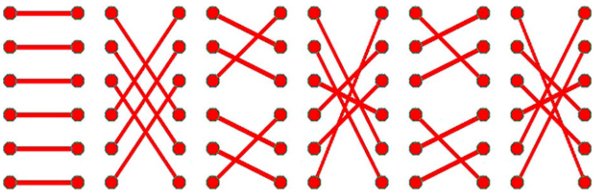

**Fig. 6** The $(**)$-family $\mathcal{T}_2 \otimes \mathcal{T}_3$ in $S_6$

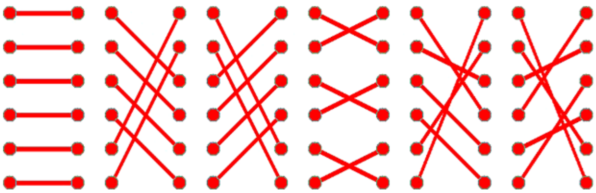

**Fig. 7** The $(**)$-family $\mathcal{T}_3 \otimes \mathcal{T}_2$ in $S_6$

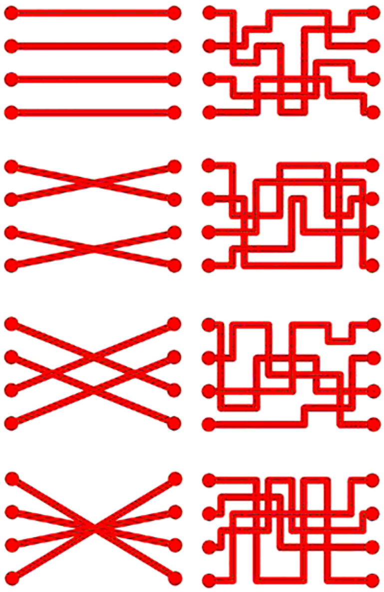

**Fig. 8** Klein's group in $S_4$ together with proposals of associated mazes

(iii) Denote the $\pi^*$-permutations in $S_n$, $S_k$ and $S_l$ by $\pi_n^*, \pi_k^*$ and $\pi_l^*$, respectively. The equations $(\nu \otimes \mu)^{-1} = \nu^{-1} \otimes \mu^{-1}$ and $\pi_n^*([a,b]_{k,l}) = [\pi_k^*(a), \pi_l^*(b)]_{k,l}$ are easy to check. For rotationally symmetric $\nu, \mu$ we then have

$$\begin{aligned}(\nu \otimes \mu) \circ \pi_n^*([a,b]_{k,l}) &= \nu \otimes \mu([\pi_k^*(a), \pi_l^*(b)]_{k,l}) \\ &= [\nu \circ \pi_k^*(a), \mu \circ \pi_l^*(b)]_{k,l} \\ &= [\pi_k^* \circ \nu^{-1}(a), \pi_l^* \circ \mu^{-1}(b)]_{k,l} \\ &= \pi_n^* \circ (\nu \otimes \mu)^{-1}([a,b]_{k,l}),\end{aligned}$$

i.e., $\nu \otimes \mu$ is also rotationally symmetric. □

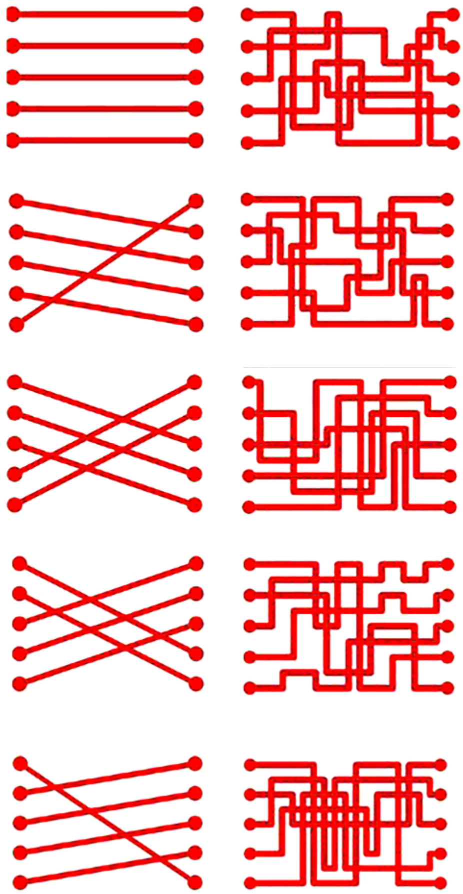

**Fig. 9** The $\tau_0^5, ..., \tau_4^5 \in S_5$ in disguise

**Corollary 1** *Let* $\mathcal{F} = \{\nu_0, ..., \nu_{k-1}\} \subset S_k$ *and* $\mathcal{G} = \{\mu_0, ..., \mu_{l-1}\} \subset S_l$ *be* $(**)$-*families. Then* $\mathcal{F} \otimes \mathcal{G} := \{\nu_i \otimes \mu_j \mid i = 0, ..., k-1,\ j = 0, ..., l-1\} \subset S_{k \cdot l}$ *is also a* $(**)$-*family.*

***Proof*** One only has to note that the $\nu_i \otimes \mu_j$ operate transitively if the $\nu_i$ and the $\mu_j$ have this property. □

As a simple example we consider the case $n = 6 = 2 \cdot 3$, and we consider in $S_2$ resp. $S_3$ the $(**)$-families $\mathcal{T}_2$ resp. $\mathcal{T}_3$ of cyclic translations. The $\tau_i^2 \otimes \tau_j^3$ can be visualized as in Fig. 6:

The $(**)$-family associated with the choice $k = 3, l = 2$ looks differently (Fig. 7), but the families are conjugate due to Proposition 3.

We also note that the example $\mathcal{F} = \mathcal{G} = \mathcal{T}_2 \subset S_2$ leads to Klein's well-known non-cyclic group with 4 elements

Our results also yield the answer to the question from the beginning of this section:

**Proposition 4** *For every commutative group $G$ with $n$ elements there is a* $(**)$-*family $\mathcal{G} \subset S_n$ such that $G$ and $\mathcal{G}$ are isomorophic.*

***Proof*** One only has to combine the following facts:

$\mathcal{F} \otimes \mathcal{G}$ is isomorphic to the product of $\mathcal{F}$ with $\mathcal{G}$ (this follows from Lemma 1 (ii)).

The assertion is true for cyclic groups.

Every commutative finite group is the product of cyclic groups. □

In fact we can generate "many" examples when $n$ has "many" divisors, but for a fixed $G$ all of them are conjugate as a consequence of Proposition 3.

Suppose that $\mathcal{F} \subset S_n$ is a $(**)$-family and that $\nu \in S_n$. Then the conjugated family $\mathcal{F}_\nu$ is a $(*)$-family, but in general it will not be a $(**)$-family since it is not generally true that with $\mu$ also $\nu \circ \mu \circ \nu^{-1}$ lies in $S_n^{\text{r.s.}}$. It is easy to see that this holds if $\nu$ commutes with $\pi^*$ but a characterization of the admissible $\nu$ for special $\pi$ seems to be difficult. Consequently it is likely that there is no simple way to describe *all* $(**)$-families for a given $n$.

We close this article with the invitation to copy the mazes in Fig. 8 or in Fig. 9 (where you find proposals to disguise the $\tau_i^5$) and to present a magic trick as described above.

## References

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