A Surprising Magic Trick, Its Mathematical Background, and Some Generalizations

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Recently, a magic trick not known to me before was presented in our magic circle.[1] Its mathematical background is interesting, and an analysis will give rise to several generalizations.

We note that our results are formally related to Bob Hummer's tricks of type "cut and turn over two" (CATO) (see, for example, [1, Chapter 2]). In those tricks, like the one under consideration here, the magician knows more than one would expect after a spectator has produced a seemingly chaotic constellation of cards by switching and shuffling.

A Basic Version of the Trick

Here we present a slightly modified version of the trick that is better suited for explaining the idea than the original, which we shall describe later.

Step 1. The magician presents several face cards taken from a deck of playing cards, all face up, some lying on the left side of the table, others on the right; see Figure 1.[2] Anticipating what is to come, he makes special note of one card on the right-hand side of the table with a particular attribute; perhaps there is only one black card or only one heart. In the example below, he has noted that there is only one spade, the queen of spades.

Step 2. All cards are then turned over so that they are lying face down; Figure 2.

Step 3. Before turning his eyes away from the table, the magician explains what should be done once his back is turned to the table: a spectator will be chosen who may turn over any card (face down to face up or vice versa) and then move it to the other side of the table: from left to right or from right to left, repeating such an action any number of times.[3] The magician demonstrates this procedure several times.

Then he turns away, and the chosen spectator performs the permissible action as often as he or she wishes; Figure 3.

Step 4. As a final step, all cards on the right-hand side are turned over—those that are face down to face up, and conversely; Figure 4.

Although the magician has not seen which cards were involved in the spectator's actions, it turns out that he knows precisely which cards are lying face up and which are face down: the face-up cards are precisely those that were on the right-hand side of the table at the very beginning. He now uses this information to finish the trick in a surprising way.

For example, say the magician has noted, as in the example of Figure 1, that there is only one spade on the right-hand side, the queen of spades. He asks the spectator first to remove all face-down cards, then all red cards, and finally all clubs. There remains only one card, and the spectator is asked to concentrate on it. After some time the magician receives a telepathic message: the queen of spades!

(Further proposals as to how to finish the presentation can be found below. And we hasten to admit that in this "raw version," the secret of the trick might be unmasked by a clever spectator. Some refinements will also be presented below.)

The Mathematical Background

We begin our analysis with some notation. Let $n \in \mathbb{N}$ be fixed and imagine that we are given n distinguishable objects. To illustrate our definitions, we have prepared $n = 8$ numbered cards. Card j has a red j on one side and a blue j on the other. Each card has a number of properties. For example, it can lie on the right-hand or left-hand side of the table; its red or blue number can be showing. We shall call such a property a binary state. More formally, a binary state is a map $S : \{1, \ldots, n\} \to \mathbb{Z}_2$, where one has defined what $S(j) = 0$ and $S(j) = 1$ mean.

In addition, we will require that a state of an object be something that can be changed. For example, it can be moved from left to right, or it can be flipped from red to blue. Note that the state of being red or black for a playing card does not have this property.

As examples we define two binary states for our eight numbered cards: $S_1(j)$ is 0 if the card is lying on the left-hand side of the table and 1 otherwise. And $S_2(j)$ is 0 if card j has its red number facing upward, $S_2(j) = 1$ otherwise.

[1] The magician who presented it did not know who created it, and neither do I.

[2] The attentive reader will have noticed that the cards shown in the figures are from a German deck: B = Bube, the jack; D = Dame, the queen; K = König, the king.

[3] At the first step, of course, only a turn from face down to face up is possible, but later, there will be in general two choices, since some cards will be lying face up.

https://doi.org/10.1007/s00283-025-10459-1

Figure 1. The cards are placed face up on the table.

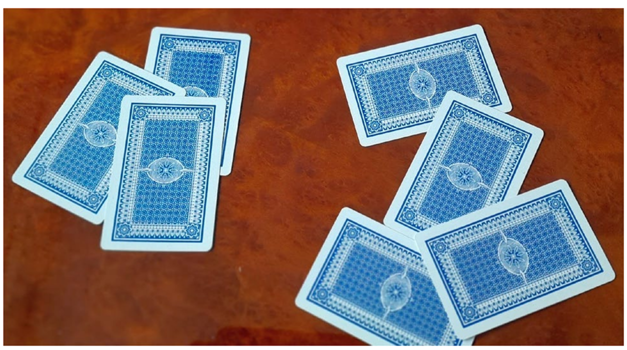

Figure 2. All the cards have been turned face down.

Figure 3. A spectator has performed the turn-over-and-change-position action several times.

Our objects may be arranged in various ways, each of which we shall call a constellation, which is natural to describe by the matrix $(S_i(j))_{i=1,2;\, j=1,\ldots,8}$, the constellation matrix C. For example, for the constellation in Figure 5, the constellation matrix is

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Figure 4. All cards on the right-hand side have been turned over.

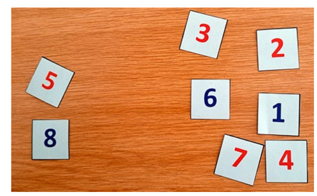

Figure 5. A constellation for the example of eight numbered cards.

Let us repeat our magic trick using these numbered cards. This time, we can distinguish them regardless of which side is turned up. The four steps can be seen in Figure 6. Certain distinguishable cards are presented, with the state S_2 equal to 1 for each card (they all show blue); now the cards are turned so that $S_2(j) = 0$ for all j; next a card is flipped and moved to the other side of the table, a process repeated several times (note that this means that for a certain j, the states S_1, S_2 are switched simultaneously); finally, for all states j with $S_1(j)$, one switches S_2; and then one has $S_2(j) = 1$ precisely for those j where one had $S_1(j) = 1$ at the beginning.

Our analysis begins with the following observations.

Observation 1. Suppose that the present constellation matrix is C and that for card j, states S_1 and S_2 are switched simultaneously. To determine the new constellation matrix one simply has to add the column vector $(1,1)^{\perp}$ to the jth column of C, where entries are added as elements of $\mathbb{Z}_2$. (Here $x^{\perp}$ denotes the transpose of a vector x.)

This is obvious. Now imagine that this switching operation has been performed several times with various j. Since $\{(0,0)^{\perp}, (1,1)^{\perp}\}$ is an additive subgroup of $(\mathbb{Z}_2)^2$, we arrive at the following.

Observation 2. If we begin with a certain constellation described by the matrix C and execute an arbitrary number of admissible switching actions, then the new constellation

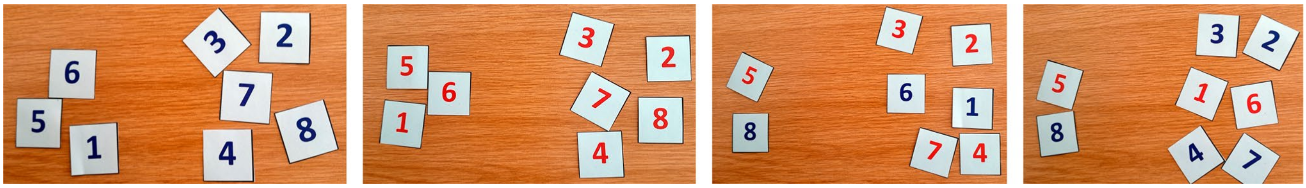

Figure 6. Magic trick redux.

matrix is $C + \mathcal{S}$, where $\mathcal{S}$ is a $2 \times n$ matrix with columns in $\{(0,0)^\perp, (1,1)^\perp\}$.

And what is the mathematics behind the final step? The spectator was asked to turn over all cards on the right-hand side of the table. This means that S_2 is switched if and only if $S_1 = 1$. This can be achieved by the operation $S_2(j) \to S_1(j) + S_2(j)$. We have thus established the following.

Observation 3. Let F be the 2×2-matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Then in the final step, the constellation matrix C has to be changed to FC.

Now only one observation is necessary to complete the mathematical analysis.

Observation 4. Let $\mathcal{S}$ be a matrix as in Observation 2. Then all entries in the second line of $F\mathcal{S}$ are zero.

This is obvious, since $(1,1)(1,1)^\perp = (1,1)(0,0)^\perp = 0$. And now it is clear why the magic trick works:

- We began with a constellation matrix C with zeros in the second row.
- The constellation matrix changed to $C + \mathcal{S}$ and later to $F(C + \mathcal{S}) = FC + F\mathcal{S}$ by the spectator's actions.
- By Observation 4, the second row of the final constellation matrix is the second row of FC. And there one finds a 1 at position j precisely if $S_1(j) = 1$.

Generalizations

More Than Two Binary States

It is not hard to deal similarly with s binary states, where $s \geq 2$ is arbitrary. We then have s maps $S_i : \{1, \ldots, n\} \to \mathbb{Z}_2$, and a constellation is described by the $s \times n$ matrix $C = (S_i(j))_{i=1,\ldots,s;\, j=1,\ldots,n}$. This time, we allow the following action: choose a j and switch two of the states simultaneously.

If this is done as often as one wishes, the constellation is now described by $C + \mathcal{S}$, where all columns of $\mathcal{S}$ are in the subgroup of $(\mathbb{Z}_2)^s$

$$\left\{ (x_1, \ldots, x_s)^\perp \mid \sum x_i = 0 \right\},$$

which is the same as $\{(x_1, \ldots, x_s)^\perp \mid \#\{i \mid x_i = 1 \text{ is even }\}\}$.

And in the final step, the constellation matrix is multiplied on the left by the $s \times s$ matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}. \qquad (1)$$

What is the effect of this multiplication? It doesn't change the first $s-1$ states, and $S_s(j)$ is switched if and only if the number of 1's in the jth column of $C + \mathcal{S}$ is odd. And this happens if and only if this is true for C, since the last row of $F\mathcal{S}$ is zero.

Thus we we have shown that if we start with a constellation C in which $S_s(j) = 0$ for every j, we will finally have $S_s(j) = 1$ if and only if an odd number of the $(S_i(j))_{i=1,\ldots,s-1}$ are equal to 1.

Here is an example to illustrate this generalization. As above, we use the eight numeric cards, each with a number printed blue on one side and red on the other. But this time the table will be divided into four quadrants determined by left and right halves and upper and lower halves.

And here are the states we will work with:

- $S_1(j)$ is 0 if card j is lying on the left half of the table. Otherwise, it is 1.
- $S_2(j)$ is 0 if card j is lying on the lower half of the table. Otherwise, it is 1.
- $S_3(j)$ is 0 if the red number is visible. Otherwise, it is 1.

We have three ways of manipulating a card. Our choice depends on the two states.

- States 1 and 2. We transfer a card (without turning it over) from the upper left half to the lower right half or vice versa. Or from the upper right half to the lower left half or vice versa.
- States 1 and 3. Take a card from the upper left half, turn it over, and move it to the upper right half. Or upper right half / turn / upper left half. Or lower right half / turn / lower left half. Or lower left half / turn / lower right half. This means that we are changing the card's position in the horizontal direction and turning it over.
- States 2 and 3. This time we change the position of the card in the vertical direction and turn it over.

And the final action here reads as follows: if the card is situated such that (S_1, S_2) is (0, 1) (i.e., in the upper left quadrant) or (1, 0) (i.e., the lower right quadrant), then turn it over.

Here is an example (See Figure 7, left). We begin with the constellation

$$\begin{pmatrix} 1&0&0&1&0&1&1&1 \\ 0&0&1&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Then some actions take place resulting in the constellation of Figure 7, middle, and finally, the cards in the upper left and the lower right quadrants are turned over. And as expected, it is precisely those cards that at the beginning were in the upper left or lower right quadrant that have their blue side turned up.

From Binary States to More General Ones

In the preceding analysis we considered only binary states, i.e., states with range $\mathbb{Z}_2$. Now we shall discuss more general states. Again we will deal with n distinguishable objects, but this time the states $S_1, \ldots, S_s$ are maps from $\{1, \ldots, n\}$ to $\mathbb{Z}_m$ for a fixed m. As before, we assume that it is easy to change the states of any j such that $(S_1(j), \ldots, S_s(j))$ is an arbitrarily described element of $(\mathbb{Z}_m)^s$. Admissible states might be possible positions on the table or directions of arrows, as in the examples below.

We adopt the ideas that worked successfully in the preceding cases. We shall call a particular choice of the states for all n elements a constellation, and it is described by the $\mathbb{Z}_m$-valued $s \times n$ constellation matrix $C = (S_i(j))_{i=1,\ldots,s;\, j=1,\ldots,n}$. Again it will be useful to consider the subgroup $\mathbb{S} := \{(x_1, \ldots, x_s)^\perp \mid \sum x_i = 0\}$ of $(Z_m)^s$ and to declare as admissible actions all transformations $C \mapsto C + \mathcal{S}$, where all columns of $\mathcal{S}$ lie in $\mathbb{S}$.

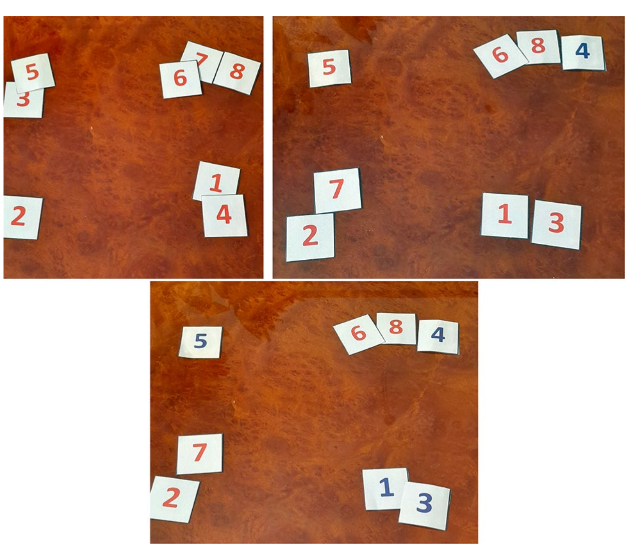

Figure 7. The magic trick with three binary states.

Then one can proceed essentially as before:

- Begin with a constellation C in which the last state S_s is zero for all j.
- Pass from C to $C + \mathcal{S}$ with an admissible $\mathcal{S}$.
- In a final step, replace the last state by $S_1(j) + \cdots + S_s(j)$, an operation that is nothing but multiplying $C + \mathcal{S}$ from the left by the $s \times s$ matrix F defined above in equation (1).

This produces a constellation whose last state S_s has value $a \in \mathbb{Z}_m$ for a certain j if and only if at the very beginning, we had $S_1(j) + \cdots + S_{s-1}(j) = a$.

Let us illustrate this situation by an example with $s = 2$ and $m = 3$. We have prepared nine small squares, three on each of three larger cards. The squares are designated by two numbers from $\mathbb{Z}_3 = \{0, 1, 2\}$. For example, square 1.2 is the third square on card 2 (the card with a 5 in Figure 8, left). Our eight little cards lie on these little squares, all showing their blue number (the color will not be important here). If the little card j lies on square $a.b$, we define $S_1(j) := a$ and $S_2 := b$. And as before, every constellation is described by a matrix. For example, the constellation depicted in Figure 8 (left) is associated with

$$\begin{pmatrix} 1&0&0&0&1&2&0&2 \\ 1&0&1&1&2&0&2&1 \end{pmatrix}.$$

Then our procedure beginning with this constellation is very similar to the preceding ones:

- Change the constellation such that $S_2(j) = 0$ for all j (Figure 8, right). All cards move to the left little square.
- Add $(1, 2)^\perp$ or $(2, 1)^\perp$ to arbitrary columns as often as one wishes.[4] See Figure 9 (left). For example, the first column

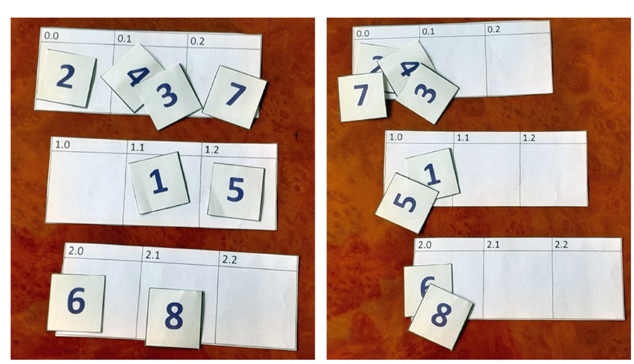

Figure 8. Original constellation (left) and preparation of the trick (right).

[4]For nonmathematicians, this of course has to be translated. For example, instead of saying, "add $(1, 2)^\perp$," one says, "move the card horizontally to the right one square and vertically down two squares (cyclically if necessary)."

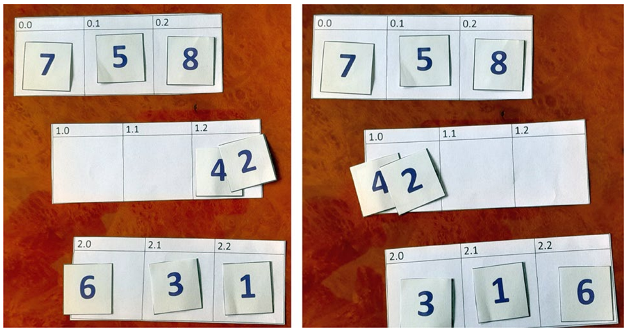

Figure 9. The constellation after the manipulations (left) and after the final step (right).

Figure 10. Three $\mathbb{Z}_3$-states.

changed from $(1,0)^\perp$ to $(2,2)^\perp$ after the addition of $(1,2)^\perp$. The constellation matrix is now

$$\begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 2 & 0 & 0 \\ 2 & 2 & 1 & 2 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

- In a final step, change only S_2: card j stays where it is if $S_1(j) = 0$, and it moves one or two squares cyclically in the horizontal direction if it lies respectively in the second or third row (Figure 9, right); the final constellation matrix is

$$\begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \end{pmatrix}.$$

And one knows that the little cards originally in row i are now somewhere in column i. In our example, one can check this by comparing Figure 8 (left) with Figure 9 (right).

One could consider further $\mathbb{Z}_3$-valued states in our example. We might simply add little arrows to our numbers, and the direction of the arrows is variable. If on card j the arrow points to the east, southwest, or northwest, this is defined to mean that $S_3(j)$ equals respectively 0, 1, or 2. Thus the constellation matrix for a constellation, an extract of which is shown in Figure 10, would begin with the columns $(0,2,2)^\perp$, $(0,1,1)^\perp$, $(0,0,0)^\perp$. No new ideas are necessary to deal with this generalization, and we omit the details.

Mixed Families of States

In the preceding investigations, there were three essential features:

- There is a given family of states $S_1, \ldots, S_s : \{1, \ldots, n\} \to \mathbb{Z}_m$.
- A nontrivial group homomorphism $\phi : (\mathbb{Z}_m)^s \to \mathbb{Z}_m$ was chosen. We worked with $\phi(x_1, \ldots, x_s) := x_1 + \cdots + x_s$, and it was admissible to add elements from the kernel of ϕ to the columns of the constellation matrix.
- In a final step, state s was modified from $S_s(j)$ to $\phi(S_1(j), \ldots, S_s(j))$ for every j.

There are various ways to choose such an additive ϕ. Our choice is the most natural one.

What has to be done if we consider mixed families of states, e.g., $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}_4$-valued states at the same time? Surely such situations might be interesting for magic tricks, since $\mathbb{Z}_m$-valued states with $m > 2$ lead to new examples, but binary states are particularly simple to realize (left/right, even/odd, face up/down).

Thus we arrive at the following problem.

Problem 5. Let $m_1, \ldots, m_s \geq 2$ be given. Are there suitable additive maps $\phi : \prod_{i=1}^{s} \mathbb{Z}_{m_i} \to \mathbb{Z}_{m_s}$?

Surely, as a first step, it will be helpful to know the additive maps from $\mathbb{Z}_{m_i}$ to $\mathbb{Z}_{m_s}$. The answer to this can be found in the following elementary lemma.

Lemma 6.

1. *If $\phi : \mathbb{Z}_m \to \mathbb{Z}_{m'}$ is additive, then there exists $a \in \mathbb{Z}_{m'}$ with $ma = 0$ such that $\phi(x) = xa$ for all $x \in \mathbb{Z}_m$.*
2. *Conversely, let $a \in \mathbb{Z}_m$ with $ma = 0$ be given. Then $x \mapsto xa$ is a well-defined additive map.*
3. *Suppose that m' divides m. Then $x \mapsto x \mod m'$ is a well-defined additive map from $\mathbb{Z}_m$ to $\mathbb{Z}_{m'}$.*
4. *Suppose that m_s divides $m_1, \ldots, m_{s-1}$. Then*

$$\phi : \prod_{i=1}^{s} \mathbb{Z}_{m_i} \to \mathbb{Z}_{m_s}, (x_1, \ldots, x_s) \mapsto (x_1 + \cdots + x_s) \mod m_s$$

is a well-defined additive map.

This enables us to treat the case of mixed states similarly to how they were treated previously if certain divisibility conditions are satisfied:

- Let s states for our n objects be given: $S_i : \{1, \ldots, n\} \to \mathbb{Z}_{m_i}$ for $i = 1, \ldots, s$. We assume that m_s divides $m_1, \ldots, m_{s-1}$.
- Begin with an arbitrary constellation matrix $(S_i(j))_{i=1,\ldots,s;\, j=1,\ldots,n}$.
- Modify the matrix in such a way that all $S_s(j)$ are zero (the starting constellation).
- Allow modifications: to any column $(S_1(j), \ldots, S_s(j))^\perp$ one may add any $(x_1, \ldots, x_s)^\perp$ such that $(x_1 + \cdots + x_s) \mod m_s = 0$. This can be done arbitrarily often.
- In a final step, for every j, the last entry of the jth column is replaced by

$$(S_1(j) + \cdots + S_s(j)) \mod m_s.$$

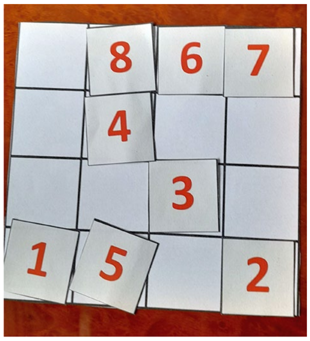

Figure 11. A starting constellation for three mixed states.

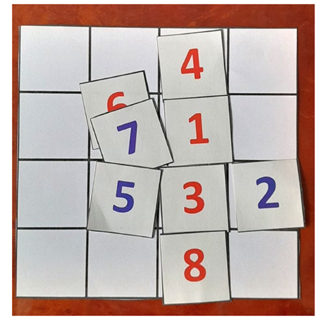

Figure 12. The constellation after some manipulations.

Then one knows for sure that for all $a \in \mathbb{Z}_{m_s}$, the j that satisfy $S_s(j) = a$ are precisely those for which in the starting constellation, the condition $(S_1(j) + \cdots + S_{s-1}(j)) \bmod m_s = a$ was satisfied.

Let us illustrate this assertion by an example with $s = 3$, $m_1 = m_2 = 4$, $m_3 = 2$. As before, we use our little cards:

- $S_1, S_2 : \{1, \ldots, 8\} \to \mathbb{Z}_4$ describes the position of the card in a 4×4 grid (i.e., S_1 denotes the row and S_2 the column).
- $S_3 = 0$ and $S_3 = 1$ mean that one sees red and blue respectively.

Figure 11 shows a constellation for which the associated matrix is

$$\begin{pmatrix} 3 & 3 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is a typical starting position, since $S_3(j) = 0$ for all j. The following manipulation is admissible: One may add to a column, as often as one wishes, a vector $(x, y, z)^{\perp} \in \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, provided that $(x + y + z) \bmod 2 = 0$. Suppose that we end up with

$$\begin{pmatrix} 1 & 2 & 2 & 0 & 2 & 1 & 1 & 3 \\ 2 & 3 & 2 & 2 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

the matrix of the constellation in Figure 12.

And now the final step. The last entry of each column $(x, y, z)^{\perp}$ has to be changed to $(x, y, (x + y + z) \bmod 2)^{\perp}$ (Figure 13):

$$\begin{pmatrix} 1 & 2 & 2 & 0 & 2 & 1 & 1 & 3 \\ 2 & 3 & 2 & 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

And as expected, it is precisely those cards that were on a square (i, j) with odd $i + j$ that show blue. These squares are marked by ■ in the following diagram:

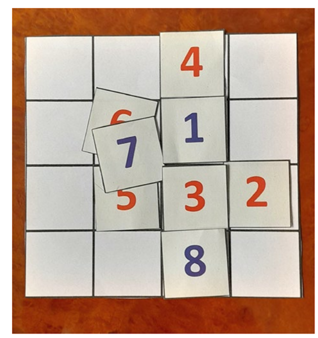

Figure 13. The final constellation.

	■		■
■		■	
	■		■
■		■	

Mathematical Magic

Our ability to transform the preceding results into magic tricks relies on the following two facts:

- Someone has shuffled the cards, and now everyone in the audience is convinced that the magician has absolutely no information about the cards' states.
- But the magician knows precisely for which objects j the condition $S_2(j) = 1$ is satisfied.[5]

We begin by describing the original version of the magic trick of which a simplified version was presented in the introduction. In the original version, five cards are lying face up in a row (Figure 14).

The cards lying at an even position (positions 2 and 4) are the interesting ones for the magician: the king of clubs and the seven of hearts. All cards are turned face down,

[5]In the generalizations, which j satisfy $S_s(j) = a$ (where $a \in \mathbb{Z}_{m_s}$ is arbitrary)?

and the magician explains what can be done with the cards when he has turned his back to the table: any pair of adjacent cards may be chosen, then both turned over (face down to face up and vice versa), interchanged, and then each placed at the other's position in the row.[6] He demonstrates this procedure several times.

Then he turns away, and a spectator repeats this action arbitrarily often. The situation might then appear as in Figure 15, left.

As a final step, the spectator is asked to turn over the cards lying at positions 2 and 4. The magician knows that the only face-up cards are the king of clubs and the seven of hearts (Figure 15, right). It is a matter of taste how to use this information. Here are some possibilities.

The prediction. Before turning away, the magician writes "the seven of hearts will remain" on a piece of paper, which is placed in an envelope. After the final step, he tells the spectator, "Remove all face-down cards and all black cards!" Then the prediction in the envelope is checked: it is correct.

Telepathy. After the request "Remove all face-down cards and all black cards!" the spectator is invited to concentrate intensively on the card (or cards) remaining. After some time and seemingly strenuous mental effort, the magician announces that he sees only one card: the seven of hearts.

Note that this version is a special case of the trick at the beginning of this paper. In our general approach, the binary state was "left half / right half of the table," while the original trick worked with "odd/even position." Also, only a small part of the possible actions were admitted here: it would have been admissible to take any card at any even (respectively odd) position and place it at an odd (respectively even) position after turning it over.

It is not hard to generalize this idea and to invent further examples.

(a) One could begin with a row of any number of cards. The final action is now to turn over all cards at an even position.
(b) If the magician uses only cards with values 2, 3, … , 10 (no jacks, queens, kings, or aces), it is easy to arrange things such that the sum over the cards with an even position number has a special value, such as the age of the person at whose house the presentation is taking place. The final step then is, of course, "Calculate the sum of the face-up cards!"
(c) Another way to disguise a binary state is the following. Someone is asked to produce with eight cards a 3×3 square such that the middle of the square is omitted (Figure 16, left).

The instruction is now, "Interchange and turn any corner card with an adjacent one arbitrarily often." And finally, "Turn over the corner cards." A possible result can be found in Figure 16, right: It is precisely the corner cards of the starting position that are now lying face up. And as before, this fact can be used for a surprising closure: identification of a particular card by "telepathy," prediction of the sum of the values of the face-up cards, etc. Readers are invited to transform our more ambitious investigations (more than two states, nonbinary states, mixed states) into attractive magic tricks.

We close with the remark that using playing cards for a presentation is a natural choice, but other objects, such as postcards or photos, could be taken as well.

Further Reading

The author has published, in German, three books (two for the general public and one for mathematicians) and several articles on mathematical magic. One of the books has been translated into English [1].

A detailed analysis of Hummer's CATO tricks can be found in [2, Chapter 1].

Figure 14. The original trick, step 1.

Figure 15. The original trick, steps 3 and 4.

[6]For example, the cards that lie at positions 1 and 2 are turned over and moved to positions 2 and 1.

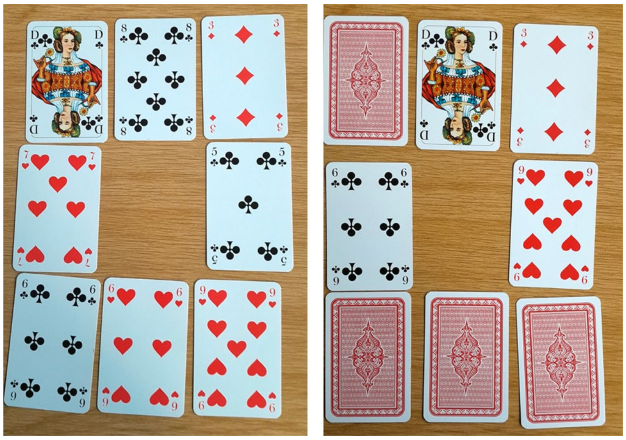

Figure 16. The trick with a 3×3 square.

Funding Open Access funding enabled and organized by Projekt DEAL.

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