Buffon: Did He Actually Throw Sticks?

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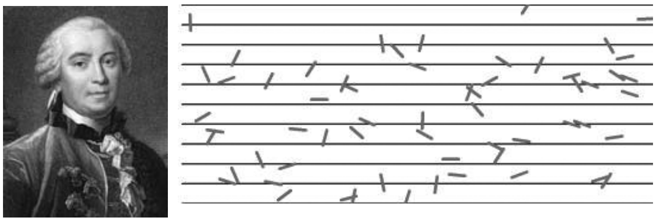
George-Louis Leclerc, Comte de Buffon (1707–1788) is famous for the following "experiment":

> Find a room with a plank floor, and denote the width of the planks by a. You will need a stick or some similarly shaped object, whose length $2r$ must be less than a. This condition guarantees that the stick will cross the edge of at most one plank when it is tossed on the floor.



Buffon and the "needle experiment."

> The probability P that such an event occurs (that is, that the stick does not end up entirely on a single plank) is then $4r/(\pi a)$. This formula contains the mathematical constant π, and we have, therefore, the opportunity of computing this number "experimentally." It will, to be sure, be necessary to throw the stick "very often," a number of times that we shall denote by n. If the stick lies across two boards k times out of the n throws, then according to the law of large numbers, k/n should be a good approximation of P, and by solving the equation $P = 4r/(\pi a)$ for π, we will have found a good approximation of π.

(Of course, instead of planks and sticks, one could simply use paper, along with needles or matches; the only requirement is that the distance between the lines on the paper be large enough.)

It is generally accepted that Buffon's experiment represents the first Monte Carlo experiment in the history of mathematics, namely an experiment in which a random method is used to solve a problem at least approximately. Such methods can be used, for example, to compute definite integrals over multidimensional domains of definition and solve numerous counting problems.

Buffon's family became wealthy through an inheritance, and so in his youth, Buffon became financially independent. Like many of his contemporaries, he became fascinated by the rapidly developing natural sciences. Indeed, he was interested in everything. His studies led to a planned fifty-volume encyclopaedia *Histoire naturelle, générale et particulière*, thirty-six volumes of which were published. From 1739 on, he served as the administrator of the royal garden in Paris ("Jardin Royal," today "Jardin des Plantes"). An echo of his service can be seen today in the fact that in the fifth arrondissement, at the southern border of the Jardin des Plantes, near the Jussieu Campus, there is to be found a Rue Buffon.

La Rue Buffon.

There were plans for this street to play an important role in November 2013 as part of a campaign for the popularization of mathematics. The rpa committee (rpa = "raising public awareness") of the European Mathematical Society (EMS) met that month in Paris, and someone came up with the idea of "reconstructing" the Buffon experiment in the Rue Buffon. The conditions could not have been more propitious, for the relevant mathematics can be easily understood by a lay audience (while at the same time being not at all trivial), and with a bit of luck and professional preparation, one could hope for good coverage in the media and to score a triumph for the popularization of mathematics.

But the project was cancelled. When some committee members got in touch with the Parisian historian of mathematics Bernard Bru to learn more about the historical details, they learned that there exists no historical evidence that Buffon saw any relationship between his theoretical deliberations and a calculation to approximate the value of π, nor whether his famous "experiment" was in fact ever carried out. We have here, then, an interesting example of the fact that historical truth and "generally accepted knowledge" need have little relationship to each other. Let us now look more closely at this discrepancy between fact and supposition.

We begin with facts that are not in dispute.

1. In 1733, Buffon submitted an article to the Royal Academy of Sciences (of which he became a member in 1734) in which he correctly calculated the probability that a randomly tossed stick of length $2r$ would intersect one of several parallel lines an equal distance a apart (provided that $2r < a$):

> *Sur un plancher qui n'est formé que de planches égales & parallèles, on jette une Baguette d'une certaine longueur, & qu'on suppose sans largeur. Quand tombera-t-elle sur une seule planche?*[1] (See [2])

[1] Freely translated: "One tosses a stick of a certain length and of negligible thickness onto a floor consisting of parallel boards of equal width. When will it fall on only a single board?

He points out that one can use his formula to determine the value of a for which the probability of landing on a single board is fifty percent: "Il y a donc une certaine largeur de la planche qui rendroit le pari ou le jeu égal."[2]

2. This research was published – in a more extensive version – in 1777, in his *Histoire naturelle*. It will now become even clearer that the principal motivation for his investigations was in calculating odds for gamblers:

> *Je suppose que, dans une chambre dont le parquet est simplement divisé par des points parallèles, on jette en l'air une baguette, et que l'un des joueurs parie que la baguette ne croisera aucune des parallèles du parquet, et que l'autre au contraire parie que la baguette croisera quelques-unes des ces parallèles; on demande le sort de ces deux joueurs. (On peut jouer ce jeu sur un damier avec une aiguille à coudre ou une épingle sans tête.)*[3]

3. Buffon later carried out experiments related to the St. Petersburg paradox. This involves a game in which a fair coin is tossed repeatedly until it first lands showing heads. If this occurs at the kth toss, the player wins 2^k ducats. Since the expected value of the player's winnings is infinite, it makes sense that the cost to play the game should also be infinite.

Buffon describes the game in [3], beginning on page 394, and on page 399, one learns that he has made some relevant experiments:

> *J'ai donc fait deux mille quarante-huit expériences sur cette question, c'est-à-dire j'ai joué deux mille quarante-huit fois ce jeu, en faisant jeter la pièce par un enfant.*[4]

4. Laplace took up Buffon's needle problem and stated explicitly that one could use the theoretical calculations to determine an experimental approximation of π. After determining the probability that a line would be crossed, he writes:

> *Si l'on projette un grand nombre de fois ce cylindre, le rapport du nombre de fois où le cylindre rencontrera l'une des divisions du plan au nombre total des projections sera, à très peu près, la valeur de* $4r/(a\pi)$*, ce qui fera connaître la valeur de la circonférence* 2π.[5] [9]

5. Buffon's needle problem seems to have sparked interest in actual experiments from the mid-19th century onwards. Apparently the first documented one was performed in 1850 by Rudolf Wolf [13], then a professor at the University of Bern. Augustus de Morgan refers in 1859 ([11], pp. 283–4) that a certain Ambrose Smith performed the experiment in 1855 with 3204 trials and a student of his with 600 trials.

How reasonable are such experiments? The theory says that only results of the following type are to be expected. If one tosses the stick n times and hits a line k times, using k/n in calculating an approximation to π, then with some probability p, the result that one obtains is within some value ε of π. Here one may choose a value of p (close to 1) and a value of ε (small) and one can then determine a suitable value of n, for example using Chebyshev's inequality. Unfortunately, for moderately large p and not very small ε, the required n is astronomically large and convergence exceedingly slow. Buffon's method is therefore not well suited to obtaining information about the digits of the number π.

It is worth noting in this connection the experiments reported in 1901 of one Lazzerini, who maintained that after throwing 3408 sticks, he had obtained a value of π accurate to six decimal places ([10], p. 120). Papers by Gridgeman [5] and Badger [1] refute this claim as extremely unlikely and probably due to data manipulation.

In notable contrast to the historically verified evidence, there remains the impression that Buffon explicitly had in mind a determination of an approximate value of π by means of an "experiment" and that he in fact carried it out:

1. The Buffon problem is treated regularly in textbooks on stochastic theory. In books on this subject in both English and German, we have found not a single one in which the least doubt is cast on the statement that "Buffon wished to calculate an approximation to π with his experiment". (This is true, alas, of the textbook *Elementare Stochastik* of the first author.)

2. These textbook authors find themselves in good company, for even in books on the history of the theory of probability, it is maintained, without citing any sources, that Buffon performed such experiments. Here are two examples:

- "It was originally performed with a needle" ([7], p. 75).

2 Freely translated: "There is, therefore, a certain width of board for which the wager – that is, the game – is fair."

3 Freely translated: "I assume that within a room in which the parquet is simply divided by parallel points, one tosses a stick into the air, and one player wagers that the the stick will not cross any of the parallels in the parquet, while the other player wagers that it will cross one of the parallels. One asks for the odds for each player. (One could also play this game on a checkerboard with knitting needles or headless pins.)" (See [3], p. 411ff.)

4 Freely translated: I have carried out 2048 experiments with respect to this question, that is, I have played this game 2048 times, making use of a child to toss the coin."

5 Freely translated: If one tosses this cylinder with great frequency, then the quotient of the total number of throws for which the stick lands on one of the divisions of the plane and the total number of throws will have approximately the value $4r = (a\pi)$ which will suffice to determine the value of the circumference 2π."

- Many investigators (including Buffon) used this result for the experimental determination of π" ([10], p. 120).

(Other history books at least leave open the connection of the approximation of π.) According to Bernard Bru, this situation is the result of a misunderstanding: the experiments on the St. Petersburg paradox were at some time or other extrapolated to encompass experiments on π and then the "facts" were simply repeated without verification from original sources.

3. The internet is not a big help. At the website Mac Tutor History of Mathematics (http://turnbull.mcs.st-and.ac.uk/history/), which we visit frequently and value highly, the following is said about Buffon: "His most notable contribution to mathematics was a probability experiment which he carried out calculating π by throwing sticks over his shoulder onto a tiled floor and counting the number of times the sticks fell across the lines between the tiles." (Many years ago, there was even more nonsense served up. Instead of "sticks" being tossed, it was "white loaves of bread". One does not have to look far to see how that error arose. The word "baguettes" in Buffon's original text (see above) was mistranslated as the familiar stick-shaped loaf of French bread (*la baguette de pain*). But "la baguette" has a number of meanings in French, including simply "stick", which makes considerably more sense in this context. At the time, the first author notified the manager of the website to check the translation and soon thereafter the offending word was changed.

It is also worth noting that Buffon's "experiment" lends itself to a number of interesting generalisations. Here are a few examples:

- What happens if the stick is longer than the distance between the planks? (This question was answered by Laplace. For more recent treatments, see [4] by P. Diaconis.)
- Can one replace sticks with some two-dimensional surface, such as a coaster? (One can, of course, obtain formulas for the probability but it depends on the shape of the surface whether the number π will appear in the formula. Thus, for example, square coasters are suitable for calculating approximations to π, while circular ones are not.)
- How does the situation change if one replaces the stick with a curved segment in the plane? (See in this regard "Buffon's Noodles" [12].)

Buffon's needle problem is the first in the then unknown territory of geometric probability, and opened up a whole new area of mathematical thinking. Klain and Rota state that it is "(…) the theorem leading into the heart of Geometric Probability" ([8], p. 3).

For this reason alone Buffon would rightfully deserve a place in the history of mathematics. Regarding the questions raised here, on the other hand, it is unlikely that new documents will surface that will provide conclusive information. Therefore, we recommend to all authors of future books on stochastic or probability theory not to involve Monsieur Buffon in any throwing of sticks, needles, loaves of bread or similar articles.

In closing, we would like to thank Bernard Bru (Paris) and Eberhard Knobloch (Berlin) for their help in elucidating the problems described in this work.

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The present paper is an extended version of an article that was originally published in German by the first author in the proceedings of the German Mathematical Society (March 2014; this version was translated to English by David Kramer).

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ICMI Column

Mariolina Bartolini Bussi (Università di Modena e Reggio Emilia, Italy) and Jean-Luc Dorier (Université de Genève, Switzerland)

The 23rd ICMI Study: Primary Mathematics Study on Whole Numbers
Co-Chairs: Mariolina Bartolini Bussi & Sun Xuhua

A new study will be conducted by the International Commission on Mathematical Instruction. This study, the 23rd led by the ICMI, addresses for the first time mathematics teaching and learning in primary school (and pre-school), taking into account inclusive international perspectives, socio-cultural diversity and institutional constraints. The broad area of *Whole Number Arithmetic* (*WNA*), including operations and relations and arithmetic word problems, forms the core content of all primary mathematics curricula. The study of this core content area is often regarded as foundational for later mathematics learning. However, the principles and main goals of instruction in the foundational concepts and skills in WNA are far from universally agreed upon, and practice varies substantially from country to country. An ICMI study that provides a meta-level analysis and synthesis of what is known about WNA would provide a useful base from which to gauge gaps and silences and an opportunity to learn from the practice of different countries and contexts.

Whole numbers are part of everyday language in most cultures but there are different views on the most appropriate age at which to introduce whole numbers in the school context. Whole numbers, in some countries, are introduced in pre-school, where the majority of children attend before the age of 6. In some countries, primary schooling comprises Grades 1–6; in others it comprises Grades 1–5. Thus the entrance age of students for primary school may vary from country to country. For these reasons, this study addresses teaching and learning WNA from the early grades, i.e. the periods in which WNA is systematically approached in formal schooling – in some contexts this includes pre-school.

Primary schooling is compulsory in most countries (in all Western countries), although there is considerable variation in the facilities, resources and opportunities for students. This is the uneven context where mathematics teaching and learning takes place. Mathematics is a central feature of early education and the content, quality and delivery of the curriculum is of critical importance in view of the kinds of citizens each country seeks to produce.

In the international literature, there are many contributions about primary school mathematics. In many cases, especially in the West, early processes of mathematical thinking, often observed in early childhood (i.e. 3–8 year-old children), are also investigated by cognitive and developmental psychologists. They sometimes study the emergence of these processes in clinical settings, where children are stimulated by suitable models so as to observe the emergence of aspects such as one-to-one correspondences, counting, measuring and other processes. In several countries, Piaget's theory has been very influential despite criticism. Neuroscientists have also been studying for some years the emergence of "number sense". However, recent perspectives highlight that what is still needed is serious and deep interdisciplinary work with experts in mathematics education.

The ICMI has acknowledged that it is timely to launch, for the first time in its history, an international study that specifically focuses on early mathematics education that is both basic and fundamental. When foundational processes are concerned, a strong epistemological basis is needed. This is where the involvement of the ICMI adds value with respect to analyses carried out in other fields. Such epistemological analysis was part of the classical works of professional mathematicians (e.g. Klein, Smith and Freudenthal) who played a major role in the history of the ICMI and considered mathematics teaching as a whole.

The ICMI study will be organised around five themes that provide complementary perspectives on approaches to early WNA in mathematics teaching and learning. Contributions to the separate themes will be distinguished by the theme's specific foci and questions, although it is expected that interconnections between themes will emerge and warrant attention.

The five themes are:

1. The why and what of WNA.
2. Whole number thinking, learning and development.
3. Aspects that affect whole number learning.
4. How to teach and assess WNA.
5. Whole numbers and connections with other parts of mathematics.

Themes 1 and 2 address foundational aspects from the cultural-historic-epistemological perspective and from the (neuro)cognitive perspective. What is especially needed are reports about the impact of foundational aspects on practices (both at the micro-level of students and classrooms and at the macro-level of curricular choices).

Themes 3 and 4 address learning and teaching, respectively, although it is quite difficult, sometimes, to separate the two aspects; for example, in some languages and cultures (e.g. Chinese, Japanese and Russian) the two words collapse into only one.

Theme 5 addresses the usefulness (or the need) to consider WNA in connection with (or as the basis for) the transition to other kinds of numbers (e.g. rational numbers) or with other areas of mathematics traditionally separated from arithmetic (e.g. algebra, geometry and modelling).

ICMI Study 23 is designed to enable teachers, teacher educators, researchers and policymakers around the world to share research, practices, projects and analyses. Although reports will form part of the programme, substantial time will also be allocated for collective work on significant problems in the field, which will eventually form part of the study volume. As in every ICMI study, ICMI Study 23 is built around an international conference and directed towards the preparation of a published volume. The study conference will take place in Macau, China, and will be hosted by the University of Macau (3–7 June 2015).

As is usual practice for ICMI studies, participation in the study conference will be by invitation only for the authors of submitted contributions that are accepted. Contributions have to be submitted before 15 September 2014; they will be reviewed and a selection will be made according to the quality of the work and the potential to contribute to the advancement of the study, with explicit links to the themes contained in the discussion document and the need to ensure diversity among the perspectives. The number of invited participants will be limited to approximately 100 people.

The *first product* of ICMI Study 23 is an electronic volume of the proceedings, to be made available first on the conference website and later on the ICMI website. It will contain all the accepted papers as reviewed papers in the conference proceedings (with an ISBN number).

The *second product* is a gallery of commented videoclips about practices in WNA, to be hosted on the conference website and later, possibly, on the ICMI website.

The *third product* is the ICMI study volume. The volume will be informed by the papers, the video-clips and the discussions at the study conference as well as its outcomes.

The International Programme Committee for ICMI Study 23 invites submissions of contributions of several kinds: theoretical or cultural-historic-epistemological-essays (with deep connections to classroom practice, curricula or teacher education programmes); position papers discussing policy and practice issues; discussion papers related to curriculum issues; reports on empirical studies; and video-clips on explicit classroom or teacher education practice. To ensure rich and varied discussion, participation from countries with different economic capacity or with different cultural heritage and practices is encouraged.

The ICMI Study 23 website is open at the address:

http://www.umac.mo/fed/ICMI23/.

The website contains a longer version of this discussion document and the detailed information about deadlines for submission; it will be regularly updated and used for sharing the contributions of those invited to the conference in the form of conference pre-proceedings. Further information may be requested at the following address:

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The members of the International Programme Committee are: Maria G. Bartolini Bussi (University of Modena and Reggio Emilia, Italy), Xuhua Sun (University of Macau, China), Berinderjeet Kaur (National Institute of Education, Singapore), Hamsa Venkatakrishnan (University of the Witwatersrand, Johannesburg, South Africa), Joanne Mulligan (Macquarie University, Sydney, Australia), Jarmila Novotna (Charles University, Prague, Czech Republic), Lieven Verschaffel (KU Leuven University, Belgium), Maitree Inpasitha (Khon Kaen University, Thailand), Sarah Gonzalez de Lora (PUC Madre y Maestra, Dominican Republic), Sybilla Beckmann (University of Georgia, Athens, GA USA), Roger E. Howe, ICMI Liaison (Yale University, New Haven, CT, USA), Abraham Arcavi, ex-officio, ICMI Secretary General (The Weizman Institute of Science, Rehovot, Israel) and Ferdinando Arzarello, ex-officio, President of the ICMI (University of Turin, Italy).

The ICMI Emma Castelnuovo Award for Excellence in the Practice of Mathematics Education

The International Commission on Mathematical Instruction (ICMI) is committed to the "development of mathematical education at all levels" and its aims are "to promote the reflection, collaboration, exchange and dissemination of ideas on the teaching and learning of mathematics, from primary to university level. The work of the ICMI stimulates the growth, synthesis and dissemination of new knowledge (research) and of resources for instruction (curricular materials, pedagogical methods, uses of technology, etc.)".

The ICMI has decided in the past to create two awards to recognise outstanding achievements in mathematics education research: the Felix Klein Award, honouring a lifetime achievement, and the Hans Freudenthal Award, recognising a major cumulative programme of research.

In order to reflect a main aspect of the ICMI (as stated above) not yet recognised in the form of an award, the ICMI has decided to create a third award to recognise outstanding achievements in the practice of mathematics education. This award will be named after Emma Castelnuovo, an Italian mathematics educator born in 1913, to celebrate her 100th birthday and honour her pioneering work. Further details are available at http://www.mathunion.org/icmi/activities/awards/emma-castelnuovo-award/.

While preparing this column, we received the very sad news that, on 13 April, Emma Castelnuovo passed away in her sleep. In the discussion lists of Italian mathematics teachers, dozens of condolence and memory messages have been posted. An interview with Emma can be downloaded from http://www.icmihistory.unito.it/clips.php.

EMF2015, Tipaza (Alger), Algeria, 10–15 October 2015

The scientific meetings of the *Espace Mathématique Francophone* have been organised every third year since 2000 and are acknowledged as a regional conference by the ICMI. The next one will take place in Tipaza (Algeria) in 2015 (10–15 October). The theme will be: '*Cultural plurality and universality of mathematics: challenges and prospects for their teaching and learning*'. A very short excerpt from the general presentation follows.