The small ball property in Banach spaces (quantitative results)

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Abstract

A metric space (M, d) is said to have *the small ball property (*sbp*)* if for every $\varepsilon_0 > 0$ there exists a sequence (r_n) of positive numbers with $r_n \leq \varepsilon_0$ for every n and $\lim r_n = 0$ such that M is the union of the closed balls $B(x_n, r_n), n = 1, 2, \ldots$ for suitable elements x_n of M.

In joint work with V. Kadets this property has been investigated systematically, some facts will be reviewed at the beginning of the present note.

The main results here concern two *quantitative versions* of this notion. We assign to M the infimum of the ε_0 with the above property. Surprisingly, for many natural subsets of Banach spaces this infimum is either zero (which corresponds to the *sbp*-case) or one. We also investigate the positive sequences (r_n) for which sequences (x_n) exist such that the union of the $B(x_n, r_n)$ covers M. The special case when this collection consists of *all* positive (r_n) is of particular interest since a long time. It is known that results into this direction can depend on the underlying set theory.

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1 Introduction

In many parts of mathematics one needs a notion to express the fact that some subset is “small” or “negligible”. It depends, of course, on the particular structure of the sets under consideration which approach is appropriate. “Small” can mean that the measure is zero, that the set is of first category, that the Hausdorff dimension is zero or something else. Interesting connections between the classical notions of smallness are discussed in [8], for more recent results we refer the reader to chapter 6 of [2].

In a joint paper ([1]) of Vladimir Kadets and the author another notion of smallness has been studied. The definition can be found in the abstract, it applies to arbitrary metric spaces. The main results of [1] are the following:

1. The unit ball of an infinite dimensional Banach space does never have the *sbp*. However, there are incomplete spaces with an *sbp* unit ball.
2. If X is the bidual of an infinite dimensional Banach space then the collection of extreme points of the unit ball of X fails to have *sbp*.
3. σ-precompactness implies *sbp*, but the converse doesn't hold.
4. If K has *sbp* then this is not necessarily true for $K \times K$.

In this note we provide *quantitative* refinements of this approach. Here are the relevant definitions:

Definition 1.1 *Let (M, d) be a metric space and A a subset of M.*

(i) *Consider the collection of all $\varepsilon_0 > 0$ such that there exist a sequence (r_n) of positive numbers with $r_n \leq \varepsilon_0$ and $\lim r_n = 0$ and suitable $x_n \in M$ such that $A \subset \bigcup_n B(x_n, r_n)$; here $B(x, r)$ denotes the closed ball with center x and radius r.*
We denote by $sbp_M(A)$ the infimum of these ε_0.

(ii) *Let $\mathcal{S}_M(A)$ be the collection of decreasing sequences (r_n) of strictly positive numbers such that $A \subset \bigcup_n B(x_n, r_n)$ for suitable $x_n \in M$. If $\mathcal{S}_M(A)$ contains all positive decreasing sequences, then A is said to have the* Borel property.

Borel has asked in [3] whether only the countable subsets of $[0, 1]$ can satisfy the previous condition. This question has attracted the attention of many mathematicians, a final answer has been given in Laver's paper [6] where one also finds a sketch of the history of the subject: The assertion “Borel's conjecture is true” is consistent, i.e., the answer depends on the underlying set theory.

The *sbp*-function will be discussed in *section 2*, in *section 3* we investigate the sets $\mathcal{S}_M(A)$ and the Borel property.

2 Properties of the function $sbp_M(\cdot)$

General properties

The collection of the admissible ε_0 in the definition of $sbp_M(A)$ is always nonempty since we deal only with bounded sets A. (This boundedness condition will also be tacitly assumed in the sequel.) The function $sbp_M(\cdot)$ is obviously monotone, also it is clear that $sbp_M(A) \leq r$ if A is contained in a ball with radius r. Note that it might happen that $sbp_M(A) = r$ when A is a ball $B(x_0, r)$: see proposition 2.3 below.

Here are some further properties:

Lemma 2.1 *Let (M, d) be a metric space.*

(i) *For $A \subset M$ and $r > 0$ one has $sbp_M(A) \leq r$ iff one finds for any sequence $\varepsilon_1 \geq \varepsilon_2 \geq \cdots$ with $r \geq \varepsilon_1$ and $\varepsilon_n \to 0$ suitable finite sets $\Delta_n \subset M$ such that*

$$A \subset \bigcup_n \bigcup_{x \in \Delta_n} B(x, \varepsilon_n).$$

(ii) *If $A_1, \ldots, A_k$ are contained in M, then*

$$sbp_M\big(\bigcup_i A_i\big) = \sup_i sbp_M(A_i).$$

(iii) *The preceding formula also holds for countably many A_i's contained in M if there is a k_0 such that the A_i with $i > k_0$ have the small ball property. It is, however, not true in general.*

(iv) *Suppose that $A \subset M' \subset M$. Then*

$$sbp_M(A) \leq sbp_{M'}(A) \leq 2 sbp_M(A).$$

It is not true in general that $sbp_M(A) = sbp_{M'}(A)$.

Proof: (i) This is obvious.

(ii) By monotonicity we have $sbp_M(A_{i_0}) \leq sbp_M\big(\bigcup_i A_i\big)$ for every i_0 which proves that

$$sbp_M\big(\bigcup_i A_i\big) \geq \sup_i sbp_M(A_i).$$

Conversely, let $\varepsilon_0 > \sup_i sbp_M(A_i)$ be given. By assumption one finds positive sequences (r_n^i) tending to zero and bounded by ε_0 and centers x_n^i such that $A_i \subset \bigcup_n B(x_n^i, r_n^i)$ for every i. Any mixture of the r_n^i with associated centers x_n^i gives rise to sequences (r_n) and (x_n) such that $r_n \to 0$, $r_n \leq \varepsilon_0$, $\bigcup_i A_i \subset \bigcup_n B(x_n, r_n)$. This proves the other inequality.

(iii) The inequality "$\geq$" is again obvious, for the reverse inequality one has to argue a little bit more subtly. Choose, for given $\varepsilon_0 > \sup_i sbp_M(A_i)$, sequences (r_n^i) and (x_n^i) for $i = 1, \ldots, k_0$ as in the preceding proof and also, for $i > k_0$, sequences (r_n^i) and centers (x_n^i) such that $\lim_n r_n^i = 0$, $r_n^i \leq \varepsilon_0/i$ and $A_i \subset \bigcup_n B(x_n^i, r_n^i)$. Then again any mixture of the r_n^i $(i, n = 1, 2, \ldots)$ will tend to zero, it will be bounded by ε_0 and $\bigcup_{n,i} B\big(x_n^i, r_n^i\big)$ will cover $\bigcup A_i$.

In order to prove that equality does not hold in general consider any infinite dimensional separable Banach space X. If B denotes the unit ball, then – as will be shown later in proposition 2.3 – one has $sbp_X(B) = 1$. Therefore $sbp_X(C) = 2$, if C stands for the ball with radius 2.

Since the space is separable C can be covered by a sequence (B_n) of translates of B. Each of these translates satisfies $sbp_X(B_n) = 1$, and this proves our claim.

(iv) The first inequality is obvious. For the proof of the second suppose that $A \subset \bigcup B(x_n, r_n)$, with $r_n \leq \varepsilon_0$, $r_n \to 0$ and $x_n \in M$. We may assume that all $B(x_n, r_n)$ meet M' (otherwise they are not necessary for the covering of A), and therefore we can choose $y_n \in M'$ with $d(x_n, y_n) \leq r_n$. Then it follows from the triangle inequality that $A \subset \bigcup_n B(y_n, 2r_n)$, and this proves the second part.

For an example where equality does not hold we consider a metric space where there exists an uncountable set A consistig of elements of mutual distance two such that A lies in a ball of radius one. (One could choose M as the Banach space $l^1(S)$ over an uncountable index set S and A as the collection of unit vectors.) Then $sbp_M(A) \leq 1$ but $sbp_A(A) = 2$. (For a more natural example see proposition 2.4.) □

Products

As it has been already noted, the product of two *sbp*-spaces needs not be *sbp*. The counterexample $K \subset X := l^\infty(l^\infty)$ in theorem 5.3 of [1] can be used to show that also the function $sbp_X(\cdot)$ behaves "badly" if products are considered.

Proposition 2.2 *The space K of [1], theorem 5.3, has the following properties:*

(i) *K lies in a ball with radius $1/2$.*

(ii) *$sbp_X(K) = 0$, i.e., K has the small ball property.*

(iii) *If $X \times X$ is provided with the maximum distance, then*

$$sbp_{X\times X}(K \times K) = 1/2,$$

i.e., this number is as large as possible.

Proof: (i) K lies in the ball $B(x, 1/2)$, where

$$x = \big((1/2, 1/2, 1/2, \ldots), (0, 0, 0, \ldots), (0, 0, 0, \ldots), \ldots\big).$$

(ii) This has been shown in [1], theorem 5.2.

(iii) It is clear that $sbp(K \times K) \le 1/2$ since $K \times K$ lies in a ball with radius $1/2$. Let $\varepsilon_0 < 1/2$ be given. We have to prove that $K \times K$ cannot be covered by a sequence of balls in $X \times X$ for which the radii (r_n) tend to zero and are bounded by ε_0.

Let such a sequence (r_n) and centers $c_1, c_2, \ldots \in X \times X$ be given, we argue as in [1]. With the notation of this paper we have $K \times K = \bigcup_{i,j} K_i \times K_j$, where the $K_i \times K_j$ have mutual distance one. Thus every ball $B_n := B(c_n, r_n)$ will meet at most one $K_i \times K_j$. Choose m_1 such that $r_n \le \varepsilon_0/2$ for $n \ge m_1$ and then an a_1 with the property $\big(K_{a_1} \times K_1\big) \cap \big(\bigcup_{n=1}^{m_1} B_n\big) = \emptyset$.

We write $K_{a_1} \times K_1$ as the disjoint union of the sets $K_{a_1} \times K_{1n}$. The mutual distance of these buildung blocks is $1/2$ so that every B_n with $n > m_1$ meets at most one of these. Select an $m_2 > m_1$ so that $r_n \le \varepsilon_0/2^{a_1}$ for $n > m_2$ and then a $K_{a_1} \times K_{1a_2}$ which is disjoint to the $B_1, \ldots, B_{m_2}$. If this construction is continued, one gets an

$$x = \big(\Phi(a_1 a_3 a_5 \ldots), \Phi(a_2 a_4 a_6 \ldots)\big)$$

such that $x \in K \times K$ but $x \notin \bigcup_n B_n$. □

Balls and spheres in Banach spaces

Unit balls in Banach spaces are always “big”:

Proposition 2.3 *Let X be an infinite dimensional Banach space and B_X the closed unit ball of X. Then $sbp_X(B_X) = 1$, i.e. $sbp_X(B_X)$ is as large as possible.*

Remark: It has been kindly pointed out to us by the referee that this result is also an immediate consequence of theorem 1 in [4].

Proof: It is clear that all $\varepsilon_0 \geq 1$ are admissible so that $sbp_X(B) \leq 1$. Now let a number $\varepsilon_0 < 1$ be given. We have to show that, whenever (r_n) is a sequences of positive numbers tending to zero and bounded by ε_0, it is impossible to find centers $x_1, x_2, \ldots$ in X such that $B_X \subset \bigcup_n B(x_n, r_n)$.

Let such a sequence (r_n) and any $x_1, x_2, \ldots \in X$ be given. We choose a positive δ such that $\varepsilon_0 + 2\delta < 1$ and then an index n_1 with the property that $r_n \leq \delta/10$ for $n \geq n_1$.

The Riesz lemma allows us to find a y_1 in B_X such that $\|y_1 - x_n\| \geq 1 - \delta$ for every $n \leq n_1$; in particular we have

$$B(y_1, \delta) \cap B(x_n, r_n) = \emptyset$$

for these n.

Now we need the following consequence of the Riesz lemma: Whenever K is a ball with radius r there are $z_1, z_2, \ldots$ in K such that

- $B(z_n, r/10) \subset B$ for all n, and
- $\|z_n - z_m\| > 3r/10$ for $n \neq m$.

We apply this to the ball $B_1 := B(y_1, \delta)$. Since $r_n \leq \delta/10$ for $n \geq n_1$ we know that $B(x_n, r_n)$ will meet at most one ball $B(z_i, \delta/10)$, $i = i(n)$, for these n. Choose $n_2 > n_1$ such that $r_n \leq \delta/10^2$ for $n \geq n_2$. By the preceding observation there must be a ball[1] $B_2 := B(z_i, \delta/10)$ which has an empty intersection with the $B(x_n, r_n)$ for $n = n_1 + 1, \ldots, n_2$. One even has $B_2 \cap B(x_n, r_n) = \emptyset$ for all $n \leq n_2$ since $B(x_n, r_n) \cap B_1 = \emptyset$ and $B_2 \subset B_1$ hold.

It should be clear how to continue. We repeat the construction with B_2 instead of B_1 to get a B_3 of radius $\delta/10^2$ in B_2 such that

$$B_3 \cap \left(\bigcup_{n=1}^{n_3} B(x_n, r_n) \right) = \emptyset,$$

where n_3 is such that $r_n \leq \delta/10^3$ for $n \geq n_3$.

In this way we obtain a sequence $B_X \supset B_1 \supset B_2 \supset \cdots$ of closed balls the radii of which tend to zero such that

$$B_k \cap \left(\bigcup_{n=1}^{n_k} B(x_n, r_n) \right) = \emptyset,$$

[1] There are even infinitely many such balls.

where $n_1 < n_2 < \cdots$. By completeness there exists an x_0 in $\bigcap B_k \subset B_X$, and this x_0 cannot be contained in $\bigcup_n B(x_n, r_n)$. This completes the proof. $\square$

With a similar proof one can provide a more natural example to the assertion made in lemma 2.1:

Proposition 2.4 *Let X be the Banach space l^1 and S the unit sphere of X. Then $sbp_X(S) = 1$ and $sbp_S(S) = 2$.*

Proof: Let $\varepsilon_0 < 2$ be given, (r_n) a sequence tending to zero with $r_n \leq \varepsilon_0$ and $x_1, x_2, \ldots$ any points in S. One has to find an $x \in S$ with $x \notin \bigcup_n B(x_n, r_n)$. Choose first a $\delta > 0$ with $\varepsilon_0 + 2\delta < 2$ and then an n_1 such that $r_n \leq \delta/10$ for $n \geq n_1$. The special structure of l^1 allows us to find y_1 in the unit sphere such that $\|y_1 - x_n\| \geq 2 - \delta$ for $n \leq n_1$; the vector y_1 could be chosen as a unit vector associated with a "large" index. Then continue as in the preceding proof, this time working in the sphere. $\square$

Remark: More generally it can similarly be shown that $sbp_S(S) = 2^{1/p}$ for the unit sphere S in l^p, where $1 \leq p \leq \infty$.

Boundaries

We recall that a *boundary* of a Banach space X is a subset B of the dual unit ball such that for every $x \in X$ there exists $x' \in B$ such that $x'(x) = \|x\|$. It is an elementary exercise to show that the collection of extreme functionals is a boundary.

It depends on the geometry of X how large boundaries must be. Consider first the space $X = c_0$. Then $X' = l^1$, a space with countably many extreme points: this shows that boundaries can have the small ball property. On the other hand it is known ([7]) that the collection of extreme points in infinite dimensional reflexive Banach spaces is always uncountable. Theorem 4.2 of [1] explains this different behaviour: boundaries for infinite dimensional reflexive Banach spaces never have the small ball property, in particular they have to be uncountable.

Here we will show more: boundaries in infinite dimensional reflexive spaces are always "as big as possible":

Proposition 2.5 *Let X be an infinite dimensional reflexive Banach space and $B \subset X'$ a boundary. Then $sbp_{X'}(B) = 1$.*

The proof will depend on the following lemma which is a refinement of lemma 4.1 in [1]. (In [1] it was only necessary to deal with centers in $B_{X'}$, now they might be anywhere in X'.)

Lemma 2.6 *Let X be a Banach space and $Y \subset X$ an infinite dimensional closed affine subspace. Further let Δ be a finite subset of X' and a and ε numbers in $]\,0,1\,[$ such that there exists $y_0 \in Y$ with $\|y_0\| < a$. Then there is an infinite dimensional closed affine subspace W of Y such that*

(i) *there exists $w_0 \in W$ with $\|w_0\| < a + 2\varepsilon$;*

(ii) *for every $y' \in \bigcup_{x' \in \Delta} B(x', \varepsilon)$ with $\|y'\| \le 1$ and every $y \in B_X \cap W$ one has $y'(y) < \|y\|$, i.e. none of these y' recognizes $\|y\|$.*

Proof: Define Z as

$$Z := \{y \mid y \in Y,\ x'(y) = x'(y_0) \text{ for every } x' \in \Delta\}.$$

This is a closed infinite dimensional affine subspace of Y which contains y_0. The next steps are to select any $w_0 \in Z$ such that

$$\|y_0\| + 2\varepsilon < \|w_0\| < a + 2\varepsilon$$

and an $x'_0 \in X'$ with $\|x'_0\| = 1$ and $x'_0(w_0) = \|w_0\|$. Then we define

$$W := \{y \mid y \in Z,\ x'_0(y) = x'_0(w_0)\};$$

we claim that W has the desired properties.

W is obviously a closed infinite dimensional affine subspace of Y, and $w_0 \in W$ has the claimed property. Now let $x' \in \Delta$ and z' with $\|z'\| \le \varepsilon$ and $\|x' + z'\| \le 1$ be given. We have to show that $x' + z'$ does not assume its norm at any $y \in W \cap B_X$.

Select any such y. Then, on the one hand, we have

$$\|y\| \ge |x'_0(y)| = |x'_0(w_0)| = \|w_0\| > \|y_0\| + 2\varepsilon.$$

On the other hand we can obtain the following inequality:

$$\begin{aligned}(x' + z')(y) &= x'(y) + z'(y)\\ &= x'(y_0) + z'(y)\\ &= (x' + z')(y_0) + z'(y) - z'(y_0)\\ &\le \|y_0\| + 2\varepsilon.\end{aligned}$$

This completes the proof of the lemma. □

We now turn to the *proof of proposition 2.5*. Let $B \subset B_{X'}$ be a boundary for X. In order to show that $sbp_{X'}(B) = 1$ we will use lemma 2.1(i): for given $\varepsilon_0 < 1$ we have to provide a sequence (ε_n) with $\varepsilon_n \to 0$ and $\varepsilon_0 \geq \varepsilon_n$ such that for no choice of finite subsets $\Delta_n \subset X'$ one can have

$$B \subset \bigcup_n \bigcup_{x' \in \Delta_n} B(x, \varepsilon_n).$$

Choose for such an ε_0 a sequence (ε_n) with $\varepsilon_0 + 2\varepsilon_1 + 2\varepsilon_2 + \cdots < 1$. Suppose that $\Delta_1, \Delta_2, \ldots$ are any finite subsets of X'. First we apply the preceding lemma with $Y = Y_1 := X$, $a = \varepsilon_0$, $\varepsilon = \varepsilon_1$ and $\Delta := \Delta_1$. Let $Y_2 \subset Y_1$ be the space W of this lemma, we note that $Y_2 \cap B_X \neq \emptyset$ since $\|w_0\| < \varepsilon_0 + 2\varepsilon_1 < 1$.

We continue with another application of the lemma, this time with $Y := Y_2$, $a := \varepsilon_0 + 2\varepsilon_1$ and $\varepsilon := \varepsilon_2$. In this way we get a sequence $Y_1 \supset Y_2 \supset \cdots$ of closed infinite dimensional affine subspaces of X with the following properties:

- The sets $Y_n \cap B_X$ are nonempty and decreasing.
- No $y' \in B_{X'} \cap \left(\bigcup_{x' \in \Delta_n} B(x', \varepsilon_n)\right)$ assumes its norm on $Y_n \cap B_X$.

Now the reflexivity of X comes into play. Since the $Y_n \cap B_X$ are weakly compact one finds a y in the intersection of these sets. Since no y' in

$$B_{X'} \cap \Big(\bigcup_n \bigcup_{x' \in \Delta_n} B(x', \varepsilon_n)\Big)$$

assumes its norm at y, the set $\bigcup_n \bigcup_{x' \in \Delta_n} B(x', \varepsilon_n)$ cannot cover a boundary. This completes the proof of proposition 2.5. □

Remark: With a similar proof as in in the discussion of the small ball property in section 4 of [1] even more can be shown: boundaries B in infinite dimensional biduals satisfy $sbp(B) = 1$. The main technical difficulty is to guarantee that the space W of lemma 2.6 can be chosen to be weak*-closed if X is a dual space. This enables one to use the Alaoglu-Bourbaki theorem in order to show that the intersection of the $Y_n \cap B_X$ is nonempty.

3 The sets $\mathcal{S}_M(A)$ and the Borel property

General properties

To illustrate the definition we start with two simple *examples*.

1) First we consider $A = [\,0,1\,]$ as a subset of the metric space $M = \mathbb{R}$. Let (r_n) be in $\mathcal{S}_M(A)$, i.e. $[\,0,1\,] \subset \bigcup_n [\,x_n - r_n, x_n + r_n\,]$ for suitable $x_1, x_2, \ldots$ in $\mathbb{R}$. It follows that

$$1 = \lambda(A) \leq \sum_n \lambda\big([\,x_n - r_n, x_n + r_n\,]\big) = 2\sum_n r_n.$$

On the other hand, if positive r_n satisfy $1 = 2\sum_n r_n$, we may put

$$x_1 := 1 - r_1, x_k := r_2 + \cdots + r_k \text{ for } k \geq 2 \;;$$

then $[\,0,1\,] = \bigcup_n [\,x_n - r_n, x_n + r_n\,]$, and it follows that

$$\mathcal{S}_M(A) = \{(r_n) \mid \ (r_n) \text{ is decreasing, and } \sum r_n \geq 1/2\}.$$

2) Let $C := \{0,1\}^{\mathbb{N}}$ be the *Cantor set*. As usual C is provided with the following product metric: $d\big((x_n),(y_n)\big)$ is zero if the sequences (x_n) and (y_n) are identical and 2^{1-k} otherwise, where k is the first index n with $x_n \neq y_n$.

On this metric space we consider the product measure μ associated with the uniform distribution on $\{0,1\}$. This μ is a probability measure on C for which a ball $B = B(x, 2^{-k})$ satisfies $\mu(B) = 2^{-k}$ $(k = 0,1,\ldots)$. More generally $\mu\big(B(x,r)\big) \leq r$ is true, and this implies that $\sum_n r_n \geq 1$ whenever $C \subset \bigcup B(x_n, r_n)$ for suitable $x_1, x_2, \ldots$.

We denote, for $r > 0$, by $\alpha(r)$ the largest number 2^{-k} with $k \in \mathbb{N}_0$ and $2^{-k} \leq r$; this is a useful definition here since one has $B(x,r) = B\big(x, \alpha(r)\big)$ for all x and $r > 0$. With this notation we may rephrase the preceding observation as

$$\mathcal{S}_C(C) \subset \{(r_n) \mid \sum_n \alpha(r_n) \geq 1\}.$$

Suppose that $\sum \alpha(r_n) > 1$. Then there exists an m such that $\sum_{n=1}^m \alpha(r_n) \geq 1$ and it is easy to find centers $z_1, \ldots, z_m$ such that

$$C \subset \bigcup_{n=1}^m B\big(z_n, \alpha(r_n)\big) \subset \bigcup_{n=1}^m B(z_n, r_n).$$

If, e.g., $r_1 = 1/2$, $r_2 = r_3 = 1/4$ one could work with $z_1 = (0,0,0,\ldots)$, $z_2 = (1,1,1,\ldots)$ and $z_3 = (1,0,0,\ldots)$.

So far we have proved that $\mathcal{S}_C(C)$ contains the positive decreasing (r_n) with $\sum \alpha(r_n) > 1$ and is contained in the collection of the (r_n) with $\sum \alpha(r_n) \geq 1$.

Even more can be said. If $\sum_{n=1}^m \alpha(r_n) < 1$ for every m and $z_1, z_2, \ldots$ are abitrary, then $\big(\bigcup_{n=1}^m B(z_n, \alpha(r_n))\big)_{m=1,2,\ldots}$ is an increasing sequence of open proper subsets of C so that – by compactness – $\bigcup_n B(z_n, \alpha(r_n))$ cannot be all of C. In this way we arrive at a characterization of $\mathcal{S}_C(C)$: A decreasing sequence (r_n) of strictly positive numbers belongs to this set iff $\sum \alpha(r_n) > 1$.

In the definition of $\mathcal{S}_M(A)$ we have restricted our attention to *decreasing* strictly positive sequences. This is a reasonable restriction since positive sequences tending to zero have a decreasing rearrangemant. The following statements are obviously true:

Lemma 3.1 *Let (M,d) be a metric space and A, B bounded subsets of M such that $A \subset B$.*

(i) $\mathcal{S}_M(B) \subset \mathcal{S}_M(A)$.

(ii) *The collection $\mathcal{S}_M(A)$ has, as a subset of the space s of all sequences, the following properties:*

- *Let $(x_n) \in s$ be strictly positive and decreasing. If (x_n) contains a subsequence which is in $\mathcal{S}_M(A)$, then $(x_n) \in \mathcal{S}_M(A)$ also holds.*
- *$(x_n) \in \mathcal{S}_M(A)$ and $y_n \geq x_n$ (all n) imply that $(y_n) \in \mathcal{S}_M(A)$.*

Remark: It would be interesting to know which properties characterize the sets $\mathcal{S}_M(A)$. More precisely: Suppose $\Delta \subset s$ is a collection of positive decreasing sequences with the properties described in (ii). Is it true – and if not, which additional properties are necessary – that there exist (M,d) and $A \subset M$ such that $\mathcal{S}_M(A) = \Delta$?

The Borel property

In order to study the "size" of a set A by means of the collection $\mathcal{S}_M(A)$ it is necessary to know that it contains some nontrivial information. Clearly one can decide whether A has the small ball property and one can also calculate $sbp_M(A)$ if $\mathcal{S}_M(A)$ is known. In this subsection we deal with a very modest question: Does $\mathcal{S}_M(A)$ recognize whether A is countable? This is the Borel conjecture, the answer is rather complicated. Note that

$\mathcal{S}_M(A)$ is the collection $\mathcal{S}$ of *all* decreasing strictly positive sequences if A is countable so that the problem can be rephrased by asking whether $\mathcal{S}_M(A)$ is a proper subset of $\mathcal{S}$ for uncountable A.

A canonical generalization of the ideas which have been used in the discussion of $[0,1]$ and the Cantor set $\{0,1\}^{\mathbb{N}}$ immediately leads to the following

Lemma 3.2 *Suppose that there exists a probability measure μ on M with the following properties:*

(i) $\mu(A) > 0$.

(ii) *There is a function $K :]\,0, +\infty\,[\,\to\,]\,0, +\infty\,[$ with $\lim_{r\to 0} K(r) = 0$ such that $\mu\big(B(x,r)\big) \le K(r)$ for arbitrary x and r.*

Then $\mathcal{S}_M(A)$ is a proper subset of $\mathcal{S}$.

Proof: This is easy: $A \subset \bigcup B(x_n, r_n)$ implies $0 < \mu(A) \le \sum_n K(r_n)$, and since $\lim K(r)=0$ this cannot be true for all positive sequences (r_n). □

There remain many cases where this idea cannot be applied directly. For example, what about the Cantor set when considered as a subset of $[0,1]$ with the usual metric? Since the Lebesgue measure is zero one has to argue differently. In fact, the preceding idea can be used to treat the case of arbitrary uncountable compact sets:

Proposition 3.3 *Let K be an uncountable compact metric space. Then $\mathcal{S}_M(K)$ cannot be all of $\mathcal{S}$.*

Proof: First we will prove the following

Claim: If L is an uncountable compact metric space then there exist an $\varepsilon > 0$ and subsets L_1, L_2 of L with the following properties:

- Both L_1 and L_2 are uncountable and compact and the diameter is bounded by ε.
- $d(x,y) \ge \varepsilon$ for $x \in L_1$ and $y \in L_2$.

Proof of the claim: Suppose that the assertion is false. We will prove that L is countable.

Given $\varepsilon > 0$ we may cover L with compact subsets $L_1, \ldots, L_k$ of diameter at most ε. Suppose the first l of these sets are uncountable. Then they must lie in some ball with center in L_1 and radius 3ε since otherwise the claim were true. This shows that for every ε our space is countable up to a possible subset of diameter 3ε. Iterating this this procedure we know that L is a countable union of countable sets plus the intersection of possibly uncountable subsets the diameters of which tend to zero. Then L must be countable.

We now are ready for the proof of the proposition. By the claim one finds positive $\varepsilon_1 > \varepsilon_2 > \cdots$ and uncountable compact subsets $L_{i_1 i_2 \ldots i_k}$ of K for $k = 1, 2, \ldots$ and $i_\kappa \in \{0, 1\}$ with the following properties:

- The mutual distance of L_0 and L_1 is at least ε_1, and the mutual distance of $L_{i_1 i_2 \ldots i_{k-1} 0}$ and $L_{i_1 i_2 \ldots i_{k-1} 1}$ is at least ε_k for arbitrary $k \geq 2$ and $i_1, \ldots, i_{k-1}$.
- The diameter of $L_{i_1 i_2 \ldots i_k}$ is at most ε_k (all k and i_κ).
- $\lim \varepsilon_n = 0$.

The proof can now be completed easily. We note that for any $(i_n) \in C$ the intersection of the decreasing family $(L_{i_1 i_2 \ldots i_k})_{k=1,2,\ldots}$ contains precisely one point which we will call $x_{i_1 i_2 \ldots}$; here it is again essential that K is compact.

Define $\phi : C \to K$ by $(i_n) \mapsto x_{i_1 i_2 \ldots}$. It is then routine to show that ϕ is one-to-one and continuous. Denote by ν the image measure of the measure μ on C which we have defined above. By construction, a ball of radius ε_n has measure 2^{-n}. Therefore the assertion follows from lemma 3.2. □

The Borel property obviously passes to subsets. Thus it follows from the preceding proposition that $\mathcal{S}_M(A)$ will be different from $\mathcal{S}$ whenever A contains an uncountable compact subset. This happens for "nearly all" uncountable sets. To make this precise it is necessary to repeat some notions from set theory.

Denote by $\mathcal{N}$ the space of sequences in $\mathbb{N}$, i.e. $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$; we provide $\mathcal{N}$ with the product topology. Call a subset A of a Polish space *analytic* if it is the continuous image of $\mathcal{N}$. This is a very general class of sets, it contains not only all Borel sets in arbitrary Polish spaces but even continuous images of such sets (Lemma 39.2 in [5]). It can be shown that every uncountable analytic set contains a homeomorphic image of the Cantor set C: cf. theorem 94 and lemma 4.2 in [5]. Therefore, by proposition 3.3, there are only countable sets which have the Borel property and are at the same time analytic.

To phrase it differently: For all analytic sets A one may hope to find essential properties of A encoded in $\mathcal{S}_M(A)$. Recall, however, that we have already noted that for sets which are not necessarily analytic this can not be guaranteed since the assertion "Every set with the Borel property is countable" might be true or false depending on the model of set theory under consideration (see [6]).

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