Locality and Dyson Schwinger equations from Hochschild cohomology of renormalization Hopf algebras

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(joint work with Dirk Kreimer)

In quantum field theory, Feynman graphs and the corresponding analytic expressions – Feynman integrals – constitute the building blocks of perturbative expansions. In realistic quantum field theories, Feynman integrals typically diverge when their graphs contain cycles. The process of assigning a sensible value to these divergent integrals is called renormalization. In simple cases it suffices, for example, to subtract off the first terms of the Taylor series with respect to the external momentum. In more general cases, that is when already subgraphs contain cycles and thus already subintegrals diverge, the renormalization process is described by the Bogoliubov recursion. It was Kreimer's discovery that the solution of the Bogoliubov recursion is essentially given by the antipode map of a Hopf algebra $(\mathcal{H}, m, \Delta, 1, \epsilon)$ on rooted trees [4] where the rooted trees keep track of nested and disjoint subdivergences. Similarly, Hopf algebras based directly on Feynman graphs can be considered [2]. As an algebra, $\mathcal{H}$ is free commutative, and the coproduct Δ cuts rooted trees into pieces. We describe the first Hochschild coalgebra cohomology of $\mathcal{H}$ [3] and show how the fact that certain linear endomorphisms B_+ satisfy the 1-cocycle condition

$$\Delta B_+ = (id \otimes B_+)\Delta + B_+ \otimes 1 \tag{1}$$

translates into physics: A proof that the renormalization procedure provides a finite result using local counterterms is easily afforded by (1). This is illustrated using a two-loop example in dimensional regularization with minimal subtraction and on-shell scheme, respectively.

On the nonperturbative side, the same Hochschild 1-cocycles provide (combinatorial) Dyson-Schwinger equations, for example

$$X = 1 + \alpha B_+(X^2) + \alpha^2 B_+(X^3)$$

with solution $X \in \mathcal{H}[[\alpha]]$. An important consequence of B_+ satisfying the cocycle condition is that if X is decomposed as $X = \sum X_n \alpha^n$, the X_n generate a Hopf subalgebra of $\mathcal{H}$. Indeed,

$$\Delta X_n = \sum_{i=0}^{n} \left(\sum_{l_1+\ldots+l_{i+1}=n-i} X_{l_1} \ldots X_{l_{i+1}} \right) \otimes X_i, \tag{2}$$

which holds for various other Dyson-Schwinger equations as well [1]. See Kreimer's talk for physical implications.

The talk is based on the recent review paper [1] which also contains a proof of (2).

References

[1] C. Bergbauer, D. Kreimer, *Hopf algebras in renormalization theory: Locality and Dyson–Schwinger equations from Hochschild cohomology*, Proc. 75ième Renc. Phys. & Math., V. Turaev, L. Nyssen, eds., Eur. Math. Soc., in press, hep-th/0506190.

[2] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I and II*, Commun. Math. Phys. **210** (2000) 249, hep-th/9912092 and **216** (2001) 215, hep-th/0003188

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