Subgroup structure of finite simple groups

The study of representations of groups as permutation groups (ie. permutation representations), and the study of the subgroup structure of groups, are really the same subject, viewed from two slightly different perspectives.

Namely the indecomposable permutation representations of a group G are its transitive representations, and the equivalence classes of such representations are in 1-1 correspondence with the conjugacy classes of subgroups of G.

The modern theory of finite permutation groups proceeds by

(1) first reducing a permutation group problem to the primitive case, and then

(2) appealing to the structure of primitive permutation groups. Hopefully the problem can be solved unless the group is almost simple, so we are reduced to

(3) the study of primitive permutation representations of almost simple groups, or equivalently, maximal subgroups of almost simple groups. Then

(4) we appeal to the classification of the finite simple groups, and information about the maximal subgroups of such groups, to hopefully complete the solution of the problem.

This process has proved to be very successful over the last 25 years.

Moreover by now we have good qualitative knowledge of the maximal subgroups of almost simple groups, although very difficult technical problems remain open, particularly in the case of the classical groups.

So perhaps it is time to break some new ground and look deeper into the lattice of subgroups of finite groups, beyond the maximal subgroups, and particularly at the subgroup lattices of finite simple groups.

But what are the right questions to ask?

I don't know the answer, but over the last year or two (in collaboration with John Shareshian) I've been investigating an old question (due to Palfy and Pudlak) about the subgroup lattice of finite groups, with the hope that, in grappling with that problem, I'll be led to less specialized and more fundamental questions.

Here is the theorem which motivates the problem:

Theorem. *(Palfy-Pudlak [PP]) The following are equivalent:*

(1) Every finite lattice is isomorphic to an interval in the lattice of subgroups of some finite group.

(2) Every finite lattice is isomorphic to the lattice of congruences of some finite algebra.

Of course this leads to:

Palfy-Pudlak Question. Is each nonempty finite lattice isomorphic to a lattice $\mathcal{O}_G(H)$ of overgroups of H in G, for some finite group G and subgroup H?

The answer to the Palfy-Pudlak Question is almost certainly no.

Indeed there is a conjecture of Shareshian that, if true, what imply that relatively few finite lattices are overgroup lattices in finite groups.

But how to prove this?

Reduce to the case of almost simple groups.

I'll discuss a reduction of that flavor, aimed at showing that certain classes of lattices are not overgroup lattices.

After the reduction, one is left with two problems about almost simple groups. I will concentrate on one of those problems: Show that for certain classes $\mathcal{C}$ of lattices, there exists no almost simple group G and subgroup H of G such that $\mathcal{O}_G(H) \in \mathcal{C}$.

This focuses attention on the overgroups in almost simple groups of suitable subgroups.

I will concentrate on groups of Lie type, and usually on the classical groups.

There exist many results on overgroups in simple groups of certain subgroups.

eg. overgroups of long root subgroups and overgroups of maximal tori in groups of Lie type.

To attack the Palfy-Pudlak question, it appears to be useful to modify and extend those results.

eg. describe overgroups of short root subgroups.

Later I'll discuss existing theorems on overgroups in groups of Lie type of root subgroups and maximal tori, and some of the extensions I've obtained.

I'll also point out open problems of this sort.

Still later we'll see how to use such theory to to determine the subgroups H of a group G of Lie type, such that the lattice $\mathcal{O}_G(H)$ has one of several tight structures.

eg. the subgroups H of *depth* 2; that is those H such that the longest chain in $\mathcal{O}_G(H)$ is of length 2.

Section 1. Lattices

A *lattice* is a poset Λ such that for each $x, y \in \Lambda$, there is a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ for x, y in Λ.

Let Λ be a finite lattice.
Then Λ has a greatest member ∞ and least member 0; set $\Lambda' = \Lambda - \{0, \infty\}$.

Regard Λ' as a graph where the adjacency relation is the comparability relation.

Define the *depth* of Λ to be the maximal length of a chain in Λ.
The lattices of depth 2 are called M-*lattices* and the M-lattice with $n + 2$ elements is denoted by M_n.

An *interval* in Λ is a sublattice of the form $[x, y] = \{z \in \Lambda : x \le y \le z\}$ for some $x \le y$ in Λ.

Example 1.1. Let G be a finite group.
Then the poset of all subgroups of G, partially ordered by inclusion, is a lattice, and for $H, K \le G$, $H \vee K = \langle H, K \rangle$ and $H \wedge K = H \cap K$.

Write $\mathcal{O}_G(H)$ for the set of all overgroups of H in G.
Thus $\mathcal{O}_G(H)$ is the interval $[H, G]$.

Define H to be of *depth* d in G if $\mathcal{O}_G(H)$ is of depth d.
Thus the maximal subgroups of G are the subgroups of depth 1.

Palfy-Pudlak Question. Is every nonempty finite lattice isomorphic to $\mathcal{O}_G(H)$ for some finite group G and some subgroup H of G?

The Palfy-Pudlak Question has remained open for the roughly ▌ 30 years since Palfy and Pudlak proved their theorem. However presumably the Question fails badly. For example there is the following conjecture of John Shareshian:

Conjecture. (Shareshian [Sh]) Let G be a finite group, $H \leq G$, and $\Delta(H,G)$ the order complex of the poset $\mathcal{O}_G(H)'$. Then $\Delta(H,G)$ has the homotopy type of a wedge of spheres.

Example 1.2. Write $\Delta(m)$ for the lattice of subsets of an m set, and define a $D\Delta$-*lattice* to be a lattice Λ such that Λ' has $r > 1$ connected components Λ_i', $1 \leq i \leq r$, and for each i, $\Lambda_i' \cong \Delta(m_i)'$ for some $m_i > 2$. If Sharesian's Conjecture holds, then there is no finite group G and subgroup H of G with $\mathcal{O}_G(H)$ a $D\Delta$-lattice.

Example 1.3. M-lattices.

Define $\mathfrak{N}_0$ to consist of the integers 1, $1+q$, $1+\frac{q^r+1}{q+1}$, $2+q$, for r an odd prime and q a prime power. Let $\mathfrak{N}$ be the set of integers n such that $M_n \cong \mathcal{O}_G(H)$ for some finite group G and subgroup H. In [BL], Baddeley and Lucchini conjecture that $\mathfrak{N}_0 = \mathfrak{N}$. Moreover they reduce the proof of this conjecture to four or five statements about finite simple groups.

Definition 1.4. Let Λ be a finite lattice.

Define Λ to be a *D-lattice* if Λ is disconnected and at least two connected components of Λ' contain edges.

For example M-lattices and $D\Delta$-lattices are disconnected, and $D\Delta$-lattices are D-lattices, but M-lattices are not D-lattices. ∎

The *dual* of Λ is the lattice obtained by reversing the partial order.

Define Λ to be a *C*-lattice* if each $x \in \Lambda'$ is of the form $x = m_1 \wedge \cdots \wedge m_n$ for some maximal members m_i of Λ'.

Further Λ is a *C-lattice* if both Λ and its dual are C^*-lattices.

Finally Λ is a *CD-lattice* if it is both a C-lattice and a D-lattice.

Observe that $D\Delta$-lattices are CD-lattices.

Section 2. Overgroup lattices

In this section G is a finite group and H is a subgroup of G. Write $\ker_H(G)$ for the largest normal subgroup of G contained in H. Thus $\ker_H(G) = 1$ precisely when G acts faithfully on the coset space G/H.

For $D \leq G$, write $\mathcal{I}_D(H)$ for the set of H-invariant subgroups of D, and set $\mathcal{V}_D(H) = \mathcal{I}_D(H) \cap \mathcal{O}_D(H \cap D)$.

Constraints on $\mathcal{O}_G(H)$ can lead to strong constraints on the structure of G.

Proposition 2.1. [A2] *Assume* $\ker_H(G) = 1$ *and* $\mathcal{O}_G(H)$ *is a* D-lattice. *Then*

(1) G has a unique minimal normal subgroup D.

(2) $G = HD$, so G is quasiprimitive on G/H.

(3) D is the direct product of the set $\mathcal{L}$ of components of G, and H is transitive on $\mathcal{L}$.

(4) The map $\varphi : \mathcal{O}_G(H) \to \mathcal{V}_D(H)$ is an isomorphism of lattices, where $\varphi(U) = U \cap D$.

Remark 2.2. There is a similar result when H is of depth 2, except that there are many more possible structure for G, and the proof is much more complicated.

We wish to show that $\mathcal{O}_G(H)$ can not be a $D\Delta$-lattice. Since

$$\mathcal{O}_G(H) \cong \mathcal{O}_{G/ker_H(G)}(H/\ker_H(G)),$$

we may assume $\ker_H(G) = 1$, so Proposition 2.1 gives us information about G and the embedding of H in G.

To get a handle on $\mathcal{O}_G(H)$, we need the notions of "signalizer lattices" and "lower signalizer lattices".

Definition 2.3. Let L be a nonabelian finite simple group.

Define $\mathcal{T}(L)$ to be the set of triples $\tau = (H, N, I)$ such that H is a finite group, $I \trianglelefteq N \leq G$, and $F^*(N/I) \cong L$.

Assume $\tau \in \mathcal{T}(L)$ and write N_0 for the preimage in N of $F^*(N/I)$.

Define $\mathcal{W} = \mathcal{W}(\tau)$ to be the set of N-invariant subgroups W of H such that $W \cap N = I$. Define

$$\mathcal{P} = \mathcal{P}(\tau) = \{(V, K) \in \mathcal{W} \times \mathcal{O}_{N_H(V)}(VN) : N_0 V/V = F^*(K/V)\}.$$

Partially order $\mathcal{P}$ by $(V_1, K_1) \leq (V_2, K_2)$ if $V_2 \leq V_1$ and $K_2 \leq K_1$.

Let $\Lambda(\tau)$ be the poset obtained by adjoining a least element 0 to $\mathcal{P}$.

It turns out that $\Lambda(\tau)$ is a lattice called a *signalizer lattice*.

Given a normal subgroup H_0 of H, define

$$\mathcal{W}_0 = \mathcal{W}_0(\tau, H_0) = \{W \in \mathcal{W} : W \leq H_0 I\},$$

and partially order $\mathcal{W}_0$ by inclusion.

Define $\Xi = \Xi(\tau, H_0)$ to be the poset obtained by adjoining a greatest member ∞ to $\mathcal{W}_0$.

Then Ξ is a lattice called a *lower signalizer lattice*.

Proposition 2.4. [A2] *Assume $D \trianglelefteq G$ and H is a complement in G to D. Assume D is the direct product of the set $\mathcal{L}$ of components of D, H is transitive on $\mathcal{L}$, and $L \in \mathcal{L}$ is simple with $Inn(L) \leq Aut_H(L)$. Then*
(1) $\tau = (H, N_H(L), C_H(L)) \in \mathcal{T}(L)$, and
(2) $\mathcal{O}_G(H) \cong \Lambda(\tau)$.

If $\ker_H(G) = 1$ and $\mathcal{O}_G(H)$ is a D-lattice, then Proposition 2.1 comes close to supplying the hypotheses of Proposition 2.4; to close the gap, we need the extra constraints: $H \cap D = 1$ and for $L \in \mathcal{L}$, $Inn(L) \leq Aut_H(L)$.

Definition 2.5. Let Λ be a finite lattice. Write $\mathcal{G}(\Lambda)$ for the set of pairs (G, H) such that G is a finite group, $H \leq G$, and $\mathcal{O}_G(H)$ is isomorphic to Λ or its dual. Let $\mathcal{G}^*(\Lambda)$ be the the set of pairs $(G, H) \in \mathcal{G}(\Lambda)$ with $|G|$ minimal.

Let Λ be a $D\Delta$-lattice. We need to show that the set $\mathcal{G}^*(\Lambda)$ of minimal counter examples is empty. The follow theorem begins our reduction to the almost simple case:

Theorem 2.6. [A2] *Assume Λ is a CD-lattice and $(G, H) \in \mathcal{G}^*(\Lambda)$. Then either G is almost simple or the following hold:*
(1) $D = F^*(G)$ is the direct product of the set $\mathcal{L}$ of components of G, the components are simple, and H is transitive on $\mathcal{L}$.
(2) H is a complement to D in G.
(3) Let $L \in \mathcal{L}$. Then $Inn(L) \leq Aut_H(L)$, $\tau = (H, N_H(L), C_H(L)) \in \mathcal{T}(L)$, and $\mathcal{O}_G(H) \cong \Lambda(\tau)$.

Now we must analyze signalizer lattices.

Theorem 2.7. [A3] *Assume L is a nonabelian finite simple group, Λ is a CD-lattice, $\tau = (H, N, I) \in \mathcal{T}(L)$ with $\Lambda(\tau)$ isomorphic to Λ or its dual, and $|H|$ is minimal subject to this constraint. Then $F^*(H)$ is the direct product of nonabelian simple groups permuted transitively by H.*

Finally to complete our reduction, we have:

Theorem 2.8. [A3] *Assume Λ is a $D\Delta$-lattice which is isomorphic to an overgroup lattice in some finite group. Then there exists an almost simple group G such that either*
(1) $\Lambda \cong \mathcal{O}_G(H)$ for some subgroup H of G, or
(2) there exists a nonabelian finite simple group L and $\tau = (G, N, I) \in \mathcal{T}(L)$ such that $G = F^*(G)N = \langle \mathcal{W}_0(\tau, F^*(G)), N \rangle$, and $\Lambda \cong \Xi(\tau, F^*(G))$.

Remark 2.9. We wish to show that Pudley-Pudlak Question has a negative answer, by showing no $D\Delta$-lattice is an overgroup lattice. Theorem 2.8 reduces that problem to two problems about almost simple groups G:

Problem 1. There exists no subgroup H of G such that $\mathcal{O}_H(G)$ is a $D\Delta$-lattice.

Problem 2. There exists no simple group L and $\tau = (G, N, I) \in \mathcal{T}(L)$ such that the lower signalizer lattice $\Xi(\tau, F^*(G))$ is a $D\Delta$-lattice.

In [ASh] and [A4], these problems are solved when G is an alternating or symmetric group. We will concentrate on the case where $F^*(G)$ is a classical group.

Remark 2.10. There exists an analogous reduction for M-lattices.

Namely to prove the Baddeley-Lucchini conjecture that $\mathfrak{N} = \mathfrak{N}_0$, it (essentially) suffices to prove that, under the hypotheses of cases (1) and (2) of Theorem 2.9, when Λ is an M_n-lattice, then $n \in \mathfrak{N}_0$.

Further in [B], A. Basile solves problem 1 when G is alternating or symmetric.

However I fear the M-lattice problem might be very difficult when G is classical, whereas the problem for $D\Delta$-lattices should be tractable.

This is because we have relatively little control over nearly simple maximal subgroups of classical groups, whereas the work of Liebeck-Praeger-Saxl supplies relatively good control over almost simple primitive subgroups of symmetric groups.

Thus we can't expect that Basile's success with M-lattices in symmetric groups will translate into an analogous result for the classical groups.

But we can hope that the extra structure possessed by $D\Delta$-lattices, will make it possible to overcome such difficulties in treating those lattices.

Note that in some respects, the lower signalize lattices are easier to work with.

Namely if $W \in \mathcal{W}_0$ is a nontrivial lower signalizer, then WN is *not* almost simple, so the difficulties arising from almost simple subgroups are not encountered in this case.

Section 3. The finite classical groups

In this section we assume the following hypothesis:

Hypothesis 3.1. (The (V, f)-setup) F is a finite field of characteristic p, V is an n-dimensional vector space over F, $q = p^e$ is a prime power, and $f : V \times V \to F$ is a form on V satisfying one of the following:

(I) f is trivial and $F = \mathbf{F}_q$.

(II) f is symplectic and $F = \mathbf{F}_q$.

(III) f is orthogonal with quadratic form Q, such that for each $x, y \in V$, $Q(x + y) = Q(x) + Q(y) + f(x, y)$, and $F = \mathbf{F}_q$.

(IV) f is unitary and $F = \mathbf{F}_{q^2}$.

Remark 3.2. Recall in case (III) when p is odd that for $x \in V$, $Q(x) = f(x, x)/2$, so that f and Q determine each other. On the other hand when $p = 2$, there are many quadratic forms Q associated to f. Also odd dimensional orthogonal spaces exist only when q is odd.

Notation 3.3. Let $\Gamma L(V)$ be the group of semilinear maps g on V; that g preserves addition, and for some $\sigma(g) \in Aut(F)$ and all $a \in F$ and $v \in V$, $(av)g = a^{\sigma(g)}v$.

Let $\Gamma = \Gamma(V, f)$ (or $\Gamma(V, Q)$ in III) consist of those $g \in \Gamma L(V)$ preserving f (or Q); that is for some $\tau(g) \in F^{\#}$, and for all $u, v \in V$, $f(vg, ug) = \tau(g)f(u, v)^{\sigma(g)}$ (or $Q(vg) = \tau(g)Q(v)^{\sigma(g)}$ in III).

Let $\Delta = \Delta(V, f)$ (or $\Delta(V, Q)$ in III) be the group of *similarities* of (V, g); that is the subgroup of Γ consisting of those g with $\sigma(g) = 1$.

Let $O = O(V, f)$ (or $O(V, Q)$ in III) be the group of *isometries* of (V, f); that is the subgroup of those $g \in \Delta$ with $\tau(g) = 1$.

Define **R** to be the set of root subgroups of O and write $\Omega = \Omega(V, f)$ (or $\Omega(V, Q)$ in III) for the subgroup $\langle \mathbf{R} \rangle$ of O.

Let $\hat{Z} = Z(\Delta)$ be the group of scalar maps on V, and $\hat{G} = \Omega\hat{Z}$.

Let $PG(V)$ be the *projective geometry* of V. Thus $PG(V)$ is the poset of nonzero proper subspaces of V, partially ordered by inclusion. We have a representation $P : \Gamma \to Aut(PG(V))$, where for $g \in \Gamma$ and $U \in PG(V)$, $P(g) : U \mapsto Ug$, and of course $\hat{Z} = \ker(P)$ is the kernel of P. Write $P\Gamma$, $P\Delta$, etc. for the image of the corresponding group under P.

Recall *points* of V (or really of $PG(V)$) are 1-dimensional subspaces, *lines* are 2-dimensional subspaces, and *hyperplanes* are subspaces of codimension 1.

For our purposes, a *classical group* over F is a group G such that $\hat{G} \leq G \leq \Gamma$, or the image PG of such a group in $Aut(PG(V))$.

Definition 3.4. Let $U \leq V$. Recall

$$U^{\perp} = \{v \in V : f(u, v) = 0 \text{ for all } u \in U\}.$$

The form f (or Q in III) restricts to a form on U, which we also write as f.

The *f-radical* of U is $Rad_f(U) = U \cap U^{\perp}$, and in III when q is even we also have the *Q-radical* $Rad_Q(U) = \{v \in Rad_f(U) : Q(v) = 0\}$.

The *radical* $Rad(U)$ of U is defined to be $Rad_f(U)$, unless III holds with q even, where $Rad(U) = Rad_Q(U)$.

Further U is *nondegenerate* if $Rad_f(U) = 0$ and U is *totally singular* if $Rad(U) = U$.

Recall that if U is nondegenerate, then (U, f) (or (U, Q) in III) also satisfies Hypothesis 3.1.

Remark 3.5. When f is nontrivial, the form f adds extra combinatorial structure to the projective geometry of V. Indeed the building of the classical group is essentially the subposet of $PG(V)$ consisting of the totally singular subspace of V.

We will be investigating the subgroup structure of a classical group G using its linear representation on (V, f), and the projective representation on the building. We begin by recalling a theorem which gives a qualitative description of the maximal subgroups of G. I will give only an imprecise statement of the result.

Theorem 3.6. [A1] *For each subgroup H of G, either*

(1) H stabilizes one of a number of natural structures on V, or

(2) PH is almost simple, and H is absolutely irreducible and primitive on V, tensor indecomposable, etc.

The "natural structures" consist of certain subspaces, direct sum decompositions, extension field structures, subfield structures, tensor product structures, etc. Theorem 3.6 says that if M is a maximal subgroup of G, then either M is the stabilizer in G of one of the structures, or PM is almost simple, and the representation of M has various properties, resulting from the constraint that M stabilizes none of the structures.

The question remains, which of these candidates for maximal subgroups is actually maximal?

The answer is supplied for the stabilizers of structures in a theorem of Kleidman and Liebeck in [KL].

More precisely, [KL] tells us which stabilizers are maximal when $n = \dim(V) \geq 13$.

Old papers list the maximal subgroups of certain very small dimensional classical groups, and I believe Kleidman's thesis, and some of his published work, treats some of the other small cases.

One corollary to some of the work on overgroups we will be discussing, is an alternate approach to determining when certain stabilizers are maximal, which works in all dimensions.

Definition 3.7. The *Witt index* of (V, f) is the maximal dimension of a totally singular subspace.

In I the Witt index is n, while in II and IV, and in III when n is odd, it is the greatest integer less than or equal to $n/2$.

In case III when n is even, the Witt index is $n/2$ or $(n/2) - 1$; define the *sign* of (V, Q) to be $+1$ or -1 in the respective case, and write $\mathrm{sgn}(V)$ for the sign.

Section 4. Finite groups of Lie type

In this section G is a finite group of Lie type over a finite field of characteristic p.

I won't be precise as to what that means, and my use of the term will be more inclusive than is usually the case. But what ever it means, it includes the following facts:

Associated to G is a *root system* Φ, and a choice Φ^+ of positive roots. Further associated to each root $\alpha \in \Phi$ is a subgroup U_α of G called the *root subgroup* of α. Write Ω for the subgroup of G generated by the root subgroups U_α, $\alpha \in \Phi$, and write

$$U = \prod_{\alpha \in \Phi+} U_\alpha$$

for the product of the root subgroups determined by the positive roots. Then $U \in Syl_p(\Omega)$, and Φ determines a certain complement T to U in $B = N_G(U)$. The conjugates of B are the *Borel subgroups* of G, and the conjugates of T are the *Cartan subgroups* of G.

If G is untwisted, then each root subgroup is isomorphic to the additive group of a certain finite field F of order q and characteristic p, but for the twisted groups the situation is a bit more complicated.

The *root subgroups* of G are the conjugates of the subgroups U_α, $\alpha \in \Phi$. The roots have one or two lengths. In the first case all root groups are *long*, and in the latter, the long root subgroups are conjugates of the U_α, with α a long root.

Example 4.1. The classical groups are groups of Lie type.

In I and II, and in III when n is odd, or when n is even and $\mathrm{sgn}(V) = +1$, the groups are untwisted and the defining field F is the field of 3.1.

In the remaining cases, G is twisted.

Recall a *transvection* on V is a nontrivial element $t \in GL(V)$ such that $[V, t]$ is a point (called the *center* of V), $C_V(t)$ is a hyperplane (called the *axis* of t), and $[V, t] \leq C_V(t)$.

The *root group* of t is the set of transvections with the same axis and center as t (together with 1).

In I, there is one root length, and the root subgroups of G, in the Lie theoretic sense, are the root subgroups of transvections.

In II and IV, for each transvection t, the center $[V, t]$ of t is a singular point, and $[V, t]^{\perp}$ is the axis of t.

Again the long root group U_α is the root group of some transvection t, except in IV when n is odd, where U_α is non-abelian of order q^3 and $Z(U_\alpha)$ is the root group of the transvection t.

Finally in III, the long root groups are in 1-1 correspondence with totally singular lines l, with the corresponding root group $R = C_G(l^{\perp})$.

Parabolic Subgroups.

Recall the *parabolic subgroups* of G are the overgroups of Borel subgroups. Further if l is the number of simple roots in Φ then the lattice $\mathcal{O}_G(B)$ is isomorphic to $\Delta(l)$. The integer l is the *Lie rank* of G. For each proper parabolic over P, $F^*(P) = O_p(P)$ is the *unipotent radical $Rad(P)$* of P, and there is a distinguished class of complements to $Rad(P)$ in P called the *Levi factors* of P.

Note that each maximal parabolic is a maximal subgroup of G.

Example 4.2. Let G be a classical group. Then (essentially) the maximal parabolics are the stabilizers in G of the totally singular subspaces W of V. Further if $P = N_G(W)$ is a maximal parabolic then $Rad(P)$ is the subgroup of G centralizing each factor in the chain $0 < W \leq W^\perp \leq V$, and the Levi factors are the subgroups $N_G(W) \cap N_G(W')$, where in I, W' ranges over the complements to W in V, while in the remaining cases W' ranges over the complements to $W^\perp$ in V. The parabolics $N_G(W')$ are called *opposites* to the parabolic $N_G(W)$.

Theorem 4.3. *(Timmesfeld [T2]) Let G be a group of Lie type, and Ω the subgroup generated by the root subgroups of G. Assume Ω has a connected Dynkin diagram, and let R be the radical of some proper parabolic. Then the maximal overgroups of R in G, which do not contain Ω, are maximal parabolics.*

An undergraduate at Caltech (Po-Ling Loh) first proved this result for classical groups. Timmesfeld also has stronger versions of the theorem which gives information about the overgroups of suitable subgroups of R.

Theorem 4.4. *(Borel-Tits) Assume G is generated by its root subgroups. Then for each nontrivial p-subgroup S of G:*

(1) There exists a parabolic subgroup P of G such that $S \leq Rad(P)$ and $N_G(S) \leq P$.

(2) $S = O_p(N_G(S))$ iff $N_G(S)$ is a parabolic and $S = Rad(N_G(S))$.

Example 4.5. Assume $\hat{G}$ is a finite group of Lie type such that $\hat{G} = \Omega\hat{Z}$, where Ω is the subgroup generated by the root subgroups of G, and $\hat{Z} = Z(\hat{G})$.

Assume $\hat{G} \trianglelefteq G$ with $C_G(\hat{G}) = \hat{Z}$. By a Frattini argument, $G = \Omega N_G(U)$.

Further $B = N_{\hat{G}}(U) \trianglelefteq N_G(U)$, so $N_G(U) = N_G(B)$, and hence $N_G(U)$ permutes the set $\mathcal{O}_{\hat{G}}(B)$ of parabolics of $\hat{G}$ over B.

Now G is a group of Lie type with Borel subgroup $N_G(U)$ and parabolics the conjugates of the members of $\mathcal{O}_G(N_G(U))$.

The Dynkin diagram D of Ω has l nodes, which index the set $\mathcal{P}^*$ of maximal parabolics of $\hat{G}$ over B.

The action of $N_G(U)$ on $\mathcal{P}^*$ induces an equivalent action on D, with the set $D/N_G(U)$ of orbits indexing the maximal members of $\mathcal{O}_G(N_G(U)) - \{G\}$, which we will call the *maximal G-parabolics*.

In particular, we say G is *trivial* on D if all orbits of $N_G(U)$ on D are of length one.

In that event the G-parabolics over $N_G(U)$ are the subgroups $N_G(P)$, for $P \in \mathcal{O}_{\hat{G}}(B)$ a parabolic of $\hat{G}$.

Moreover for each such P, $G = \Omega N_G(P)$.