Buildings

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September 2, 2009

Introduction

All non-abelian finite simple groups are either

- alternating OR
- sporadic OR
- automorphism groups of spherical buildings.

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Generalized polygons

Definition

A generalized n-gon is a

- bipartite graph
- of diameter n such that
- the length of a shortest circuit is $2n$.

Generalized polygons

Definition

A generalized n-gon is thick if each vertex has at least three neighbors.

Definition

A generalized n-gon is thin if each vertex has at exactly two neighbors.

Examples

- generalized 2-gons = complete bipartite graphs
- generalized 3-gons = projective planes

Generalized polygons

We always assume that

- Γ is thick.
- $n \geq 3$.

Definitions

A root is a path of length n.

An apartment is a circuit of length $2n$.

- Every path of length $n+1$ lies on a unique apartment.

The Moufang property

Definition

Let

$$\alpha = (x_0, x_1, x_2, \ldots, x_{n-1}, x_n)$$

be a root. The root group U_α is the pointwise stabilizer of

$$\Gamma_{x_1} \cup \Gamma_{x_2} \cup \cdots \cup \Gamma_{x_{n-1}}.$$

Definition

Γ is Moufang if for every root α, the root group U_α acts transitively on the set of apartments containing α.

Root group sequences

Let Σ be an apartment. We number its vertices consecutively

$$x_0, x_1, x_2, \ldots$$

(with indices modulo $2n$).

- Let U_i denote the root group $U_{(x_i, x_{i+1}, \ldots, x_{i+n})}$.
- $U_1, U_2, \ldots, U_n$ fix the vertices x_n and x_{n+1}.
- Let $U_+ = \langle U_1, U_2, \ldots, U_n \rangle$.

Uniqueness

Definition

The sequence

$$(U_+, U_1, U_2, \ldots, U_n)$$

is called the root group sequence of Γ.

Theorem (Uniqueness)

Γ is uniquely determined by its root group sequence.

Properties of root groups

Let

$$U_{[k,s]} = U_k U_{k+1} \cdots U_s$$

for all k, s with $1 \leq k \leq s \leq n$ and $U_{[k,s]} = 1$ if $s < k$.

- $[U_i, U_j] \subset U_{[i+1, j-1]}$ for all i, j with $1 \leq i < j \leq n$.
- $[U_i, U_{i+1}] = 1$.

Thus $U_+ = U_1 U_2 \cdots U_n$.

- The product map from $U_1 \times U_2 \times \cdots \times U_n$ to U_+ is a bijection.

Key observation

The structure of $U_+ = \langle U_1, U_2, \ldots, U_n \rangle$ is uniquely determined by the individual U_i and the commutator relations of the form

$$[u_i, u_j] = u_{i+1} \cdots u_{j-1},$$

where $u_k \in U_k$ for all k.

$n = 3$

- K is a field.
- $x_i \colon K \to U_i$ is an isomorphism for $i = 1, 2, 3$:

$$x_i(s)x_i(t) = x_i(s + t) \text{ for all } s, t \in K.$$

- $[x_1(s), x_3(t)] = x_2(st)$.

This construction works also if K is a skew field or an octonion division algebra. The Moufang triangles we obtain are

- *algebraic* if K is finite dimensional over its center
- *classical* if K is a skew field.
- *exceptional* if K is octonion.

$n = 4$: Quadratic form type

Let (K, L, q) be an anisotropic quadratic space:

- K is a field.
- L is a vector space over K.
- $q \colon L \to K$

such that

- $f(a, b) = q(a + b) - q(a) - q(b)$ is bilinear.
- $q(ta) = t^2 q(a)$.
- $q(a) = 0$ if and only if $a = 0$.

Let $x_i \colon K \to U_i$ for $i = 1$ and 3 and $x_i \colon L \to U_i$ for $i = 2$ and 4 and

$$[x_1(t), x_4(a)] = x_2(ta)x_3(tq(a)) \text{ and } [x_2(a), x_4(b)] = x_3(f(a, b)).$$

Anisotropic quadratic forms

Examples

- The norm of a quadratic extension.
- If K is finite, then $\dim_K L \leq 2$.

If $\mathrm{char}(K) \neq 2$, then $q(a) = f(a, a)/2$.

$n = 4$: Involutory type

Let K be a field or skew field and let σ be an involution of K:

- σ is an additive automorphism of K.
- $(st)^\sigma = t^\sigma s^\sigma$.
- $\sigma^2 = \mathrm{identity}$.

An involutory set is a triple (K, K_0, σ), where K_0 be an additive subgroup of K containing 1 such that

- $K_\sigma = \{t + t^\sigma \mid t \in K\} \subset K_0 \subset K^\sigma = \{t \in K \mid t^\sigma = t\}$.
- $tKt^\sigma \subset K_0$ for all $t \in K$.

Let $x_i \colon K_0 \to U_i$ for $i = 1$ and 3 and $x_i \colon K \to U_i$ for $i = 2$ and 4 and

$$[x_1(t), x_4(u)] = x_2(tu)x_3(utu^\sigma) \text{ and } [x_2(u), x_4(v)] = x_3(u^\sigma v + v^\sigma u).$$

Involutory sets

An involutory set is a triple (K, K_0, σ), where K_0 be an additive subgroup of K containing 1 such that

- $K_\sigma = \{t + t^\sigma \mid t \in K\} \subset K_0 \subset K^\sigma = \{t \in K \mid t^\sigma = t\}$.
- $tKt^\sigma \subset K_0$ for all $t \in K$.

- If $\mathrm{char}(K) \neq 2$, then $t = (t/2) + (t/2)^\sigma$ for $t \in K^\sigma$, so $K_\sigma = K^\sigma$.
- If $\mathrm{char}(K) = 2$, let $(u + K_\sigma)t = tut^\sigma + K_\sigma$ for all $u \in K^\sigma$ and all $t \in K$. This makes K^σ/K_σ into a right vector space over K!!

Involutory sets

Let (K, K_0, σ) be an involutory set.

- If K is commutative, then $F := K_0$ is a subfield and K/F is a separable quadratic extension.
- Either $K = \langle K_0 \rangle$ (as a subring) or
  - K is commutative.
  - K is a quaternion division algebra and σ is the standard involution of K.

Pseudo-quadratic forms

Let (K, K_0, σ) be an involutory set, let L be a right vector space over K and let f be a skew-hermitian form on L:

- $f(u + v, w) = f(u, w) + f(v, w)$
- $f(ut, w) = tf(u, w)$ and $f(u, wt) = f(u, w)t^\sigma$
- $f(u, w)^\sigma = -f(u, w)$

A map $q\colon L \to K$ is a pseudo-quadratic form if for some skew-hermitian form f:

- $q(u + w) \equiv q(u) + q(w) + f(u, w) \pmod{K_0}$
- $q(ut) \equiv tq(u)t^\sigma \pmod{K_0}$

q is anisotropic if

- $q(u) \equiv 0 \pmod{K_0}$ iff $u = 0$.

Anisotropic pseudo-quadratic forms

Example

- Let (K, K_0, σ) be an involutory set.
- Let $\gamma \in K \backslash K_0$.
- Let $q\colon K \to K$ be given by $q(t) = t\gamma t^\sigma$.
- Let $f(s, t) = s(\gamma - \gamma^\sigma)t^\sigma$ for all s, t.
- Let $L = K$.

Then f is a skew-hermitian form on L and

$$\begin{aligned} q(s+t) &= s\gamma s^\sigma + t\gamma t^\sigma + s\gamma t^\sigma + t\gamma s^\sigma \\ &= q(s) + q(t) + f(s,t) + s\gamma t^\sigma + t\gamma s^\sigma \\ &= q(s) + q(t) + f(s,t) + (t\gamma s^\sigma)^\sigma + (t\gamma s^\sigma) \end{aligned}$$

and $(t\gamma s^\sigma)^\sigma + (t\gamma s^\sigma) \in \{a + a^\sigma \mid a \in K\} \subset K_0$.

Anisotropic pseudo-quadratic forms

- $f(u,u) = q(u) - q(u)^\sigma$ for all $u \in L$.
- If K is finite, then $\dim_K L \leq 1$.

The remaining Moufang polygons

- The remaining Moufang n-gons satisfy $n = 4$, 6 or 8.
- They are parametrized by more exotic algebraic structures: Jordan algebras, Tits endomorphisms, subvector spaces of purely inseparable extensions, etc.

Quaternions

Let E/K be a separable quadratic extension with norm N, so $N(a) = a \cdot a^\sigma$. Let α be in $K \backslash N(E)$ and let

$$Q = \{a + eb \mid a, b \in E\},$$

where

$$a \cdot eb = e(a^\sigma b), \quad eb \cdot a = e(ab), \quad ea \cdot eb = \alpha a^\sigma b.$$

Then Q is a division algebra with center K. Its reduced norm N is given by

$$N(a + eb) = N(a) - \alpha N(b)$$

and its standard involution σ is given by

$$(a + eb)^\sigma = a^\sigma - eb.$$

Octonions

Let Q be a quaternion division algebra with center K.

Let β be in $K \backslash N(Q)$ and let

$$A = \{a + eb \mid a, b \in Q\},$$

where

$$a \cdot eb = e(a^\sigma b), \quad eb \cdot a = e(ab), \quad ea \cdot eb = \beta a^\sigma b.$$

Then A is a (non-associative) division algebra with center K.

Its reduced norm N is given by

$$N(a + eb) = N(a) - \beta N(b)$$

and its standard involution σ is given by

$$(a + eb)^\sigma = a^\sigma - eb.$$

Moufang sets

Let X be a set. For each $x \in X$, let

- U_x be a subgroup of $\mathrm{Sym}(X)$ fixing x such that
- U_x acts *sharply transitively* on $X \backslash \{x\}$.

Let
$$G = \langle U_x \mid x \in X \rangle.$$

Then $(X, (U_x)_{x \in X})$ is a Moufang set if for each $x \in X$:

- U_x is a normal subgroup of the stabilizer G_x.

Moufang sets

Examples

- The group of special fractional linear maps
$$x \mapsto \frac{ax+b}{cx+d}$$
acting on the projective line $K \cup \{\infty\}$.
- The set of neighbors of a fixed vertex of a Moufang polygon.

Spherical Buildings

Coxeter groups

A square symmetric matrix $(m_{ij})_{i,j \in S}$ is a Coxeter matrix if
$$m_{ii} = 1 \text{ and } m_{ij} \in \{2, 3, 4, 5, \ldots, \infty\}.$$

Let $(m_{ij})_{i,j \in S}$ be a Coxeter matrix. Then
$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$
is the corresponding Coxeter group and the pair (W, S) is the corresponding Coxeter system.

The graph with vertex set S and edges all pairs $\{i, j\}$ such that $m_{ij} \geq 3$ labeled by the quantity m_{ij} is called the corresponding Coxeter diagram.

Coxeter groups

Example

The Coxeter group corresponding to the Coxeter diagram having just two vertices and one edge labeled by $n \in \{3, 4, 5, \ldots, \infty\}$ is the dihedral group D_{2n}.

Irreducible and spherical Coxeter matrices

Definition

A Coxeter matrix is irreducible if the Coxeter diagram is connected.

Definition

A Coxeter matrix is spherical if the Coxeter group W is finite.

The spherical Coxeter matrices were classified by Coxeter in the 1930's.

Chamber systems

Let S be a set of colors. An S-colored chamber system is a connected graph whose *edges* each have a color from the set S such that for each vertex x the following hold:

- For each $s \in S$, there exists a vertex y such that $\{x, y\}$ is an edge of color s.
- If y, z are two vertices such that $\{x, y\}$ and $\{x, z\}$ are both edges of color s, then $\{y, z\}$ is also an edge of color s.

Chamber systems

Definitions

A chamber system is *thick* if for each vertex x and each color $s \in S$, there exists at least two s-colored edges containing x.

A chamber system is *thin* if for each vertex x and each color $s \in S$, there exists exactly one s-colored edges containing x.

We sometimes call the vertices of a chamber system chambers.

Examples of chamber systems

Let $(m_{ij})_{x,y \in S}$ be a Coxeter diagram with vertex set S.

Let
$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$
be the corresponding Coxeter group.

Let Σ be the S-colored graph with vertex set W whose s_i-colored edges are all pairs of the form
$$\{x, xs_i\}$$
for some $x \in W$.

Σ is a *thin* chamber system.

Examples of chamber systems

Let Γ be a connected bipartite graph.

We call the two sets in the bipartition of Γ's vertex set B and W and let $S = \{B, W\}$.

Let Δ_Γ be the graph whose vertices are the edges of Γ, where two edges of Γ are joined by an edge of color $s \in S$ in Δ_Γ precisely when the two edges of Γ intersect in a vertex of Γ contained in s.

Γ is a circuit if and only if Δ_Γ is a circuit (of the same length).

Examples of chamber systems

In particular, a generalized n-gon can be thought of as a chamber system with $S = \{B, W\}$.

Residues and panels in chamber systems

- Let Δ be an S-colored chamber system.
- Let J be a subset of S.
- Let Δ_J be the graph obtained from Δ by discarding all the edges whose color is not contained in J.

Definition

A J-residue of Δ is a connected component of Δ_J.

- Each vertex of Δ lies in a unique J-residue.
- The set J is the type of a J-residue and the cardinality of J is the rank of a J-residue.
- The cardinality of S is the rank of Δ.

Residues and panels in chamber systems

- A residue of rank one is called a panel.
- Panels are complete graphs.

Buildings

- Let $M = (m_{ij})_{i,j\in S}$ be a Coxeter diagram with vertex set S.
- Let Σ be the corresponding S-colored thin chamber system.
- Let Δ be an arbitrary S-colored chamber system.

Definition

An apartment (of type M) in Δ is a subgraph isomorphic to Σ.

Buildings

Let $M = (m_{ij})_{i,j\in S}$ be our Coxeter diagram with vertex set S.

Definition

A building of type M is an S-colored chamber system Δ such that the following hold:

- For each vertex x and each panel P, there exists a unique vertex in P nearest to x.
- Every two vertices are contained in an apartment.
- For every two apartments Σ_1 and Σ_2, there is an isomorphism from Σ_1 to Σ_2 fixing every vertex contained in $\Sigma_1 \cap \Sigma_2$.

Buildings and generalized polygons

Let $M = (m_{ij})_{i,j\in S}$ be an irreducible Coxeter matrix with $|S| = 2$ and let n be the unique label m_{ij}.

Let Δ be a building of type M.

Let Γ be the corresponding bipartite graph with $S = \{B, W\}$.

- If $n < \infty$, then Γ is a generalized n-gon.
- If $n = \infty$, then Γ is a tree.

Buildings of rank two

In fact:

- If $n < \infty$, then a building of type M is the same thing as a generalized n-gon.
- If $n = \infty$, then a building of type M is the same thing as a tree with no vertices of valency 1.

Other examples of buildings

Example

A building of rank one is just a complete graph whose apartments are its 2-element subsets.

Example

- Let $M = (m_{ij})_{i,j \in S}$ be a Coxeter diagram with vertex set S.
- Let Σ be the corresponding thin S-colored chamber system.

Then Σ itself is the unique thin building of type M.

From now on, all buildings are assumed to be thick.

Irreducible buildings

Let Δ be a building of type M.

Definition

- Δ is called irreducible if the Coxeter diagram corresponding to M is connected.

Spherical buildings

Definition

- A building Δ is called spherical if its apartments are finite.

A basic property of buildings

Let Δ be a building of type M.

Let $J \subset S$, let M_J be the matrix $(m_{ij})_{i,j \in J}$ and let R be a J-residue of Δ.

Then R is a building of type M_J.

Roots in buildings

Suppose: Δ is a building and Σ is an apartment of Δ.

If e is an edge and x a vertex of Σ, then x is nearer to one vertex in e then it is to the other. The nearer vertex in e is called $\mathrm{proj}_e(x)$.

Two edges e and e' of Σ are *parallel* if the map proj_e is a bijection from e' to e. This is an equivalence relation.

A root of Σ is a connected component of the graph obtained from Σ by removing all the edges in a parallel class.

A root of Δ is a root of one of its apartments. A root can be the a root in many apartments simultaneously.

Moufang buildings

Let Δ be an *irreducible spherical building of rank at least two*.

Let α be a root of Δ.

The root group U_α is the pointwise stabilizer in $\mathrm{Aut}(\Delta)$ of the set of all vertices adjacent to at least two chambers in α.

The root group U_α acts trivially on α.

Δ is Moufang if for every root α, the root group U_α acts transitively on the set of apartments containing α.

A local-to-global principle

Let $M = (m_{ij})_{i,j \in S}$ be an irreducible spherical Coxeter diagram of rank at least three.

Definition

For each vertex x of a building Δ of type M, let $E_2(x)$ be the subgraph spanned by all the irreducible rank 2 residues of Δ containing x.

Theorem

Let Δ and Δ' be two buildings of type M and let $x \in \Delta$ and $x' \in \Delta'$ be vertices. Then an isomorphism from $E_2(x)$ to $E_2(x')$ (if one exists) always extends to an isomorphism from Δ to Δ'.

A local-to-global principle

Let $M = (m_{ij})_{i,j \in S}$ be an irreducible spherical Coxeter diagram of rank at least three.

Corollary

A building of type M is uniquely determined by the irreducible rank 2 residues containing a fixed vertex.

Corollary

Every building of type M is Moufang, as is every irreducible residue of rank at least two of such a building.

⋆⋆⋆ *Thus the irreducible rank 2 residues are Moufang polygons!*

The classification of simply laced spherical buildings

Let Δ be a building of type M.

- Suppose $M = (m_{ij})_{i,j \in S}$ is simply laced.

Simply laced means that $m_{ij} \leq 3$ for all $i, j \in S$.

The classification of simply laced spherical buildings

Suppose $M = (m_{ij})_{i,j \in S}$ is simply laced.

- All irreducible rank 2 residues of Δ are Moufang triangles defined by the same field or skew field K.
- If the Coxeter diagram M has a vertex of degree 3, then K must be commutative.
- For each field or skew field (commutative if M has a vertex of degree 3), there exists a unique building whose residues are Moufang triangles defined over K.

The classification of spherical buildings

Suppose that M is not simply laced and let x be a vertex of Δ.

Then:

- There is a unique edge $J = \{i, j\}$ of S such that the J-residue R of Δ containing x is a generalized quadrangle.
- The building Δ is uniquely determined by this residue and its orientation.

The classification of spherical buildings

Suppose that M is of type B_ℓ for $\ell \geq 3$.

Let K be the field or skew field or octonion division algebra defining the residue of type $A_{\ell-1}$ containing a fixed chamber x.

Then Δ is uniquely determined by

- An anisotropic quadratic space (K, L, q) OR
- An involutory set (K, K_0, σ) OR
- An anisotropic pseudo-quadratic space (K, K_0, σ, L, q) OR
- An honorary involutory set (K, K_0, σ).

This last case can only occur if $\ell = 3$.

The classification of spherical buildings

An honorary involutory set is a triple (K, K_0, σ), where

- K is an octonion division algebra
- K_0 is its center
- σ is its standard involution.

Buildings of type F_4

Buildings of type F_4 are classified by the following families of involutory sets (K, F, σ), both genuine and honorary:

- $\mathrm{char}(K) = 2$, K is a purely inseparable extension of the field F of exponent 1 and $\sigma = \mathrm{id}$.
- $F = K$ and $\sigma = \mathrm{id}$.
- K/F is a separable quadratic extension and σ is the non-trivial element in $\mathrm{Gal}(K/F)$.
- K is a quaternion division algebra, F is its center and σ is its standard involution.
- K is an octonion division algebra, F is its center and σ is its standard involution.

The field of definition

In every case the relevant algebraic structure is defined over a field or a skew field or (in a few cases) an octonion division algebra K. We call K the field of definition of the spherical building Δ. It is an invariant of Δ.

The algebraic structure itself is also an invariant, more or less. For example, two anisotropic quadratic spaces yield the same building of type B_ℓ if and only if they are similar.

Conclusion

There is a Moufang spherical building corresponding to every absolutely simple algebraic group of F-rank at least 2. Here F is the center $Z(K)$ of the defining field K or, in the unitary case, $F = Z(K) \cap K^\sigma$.

The only Moufang spherical buildings which do not arise in this way are those that involve:

- an infinite dimensional vector space,
- a skew field of infinite dimension over its center,
- a bilinear (or skew-hermitian form) that is degenerate or
- a purely inseparable field extension in characteristic 2 or 3.

Affine Buildings

Affine Coxeter matrices

The affine Coxeter diagrams are the Coxeter diagrams underlying the extended Dynkin diagrams.

They are all of the form $\tilde{Q}_\ell$, where Q_ℓ is one of the spherical Coxeter diagrams $A_\ell, B_\ell, \ldots, G_\ell$.

Affine buildings

An (irreducible) affine building is a building of type $\tilde{Q}_\ell$ for some affine Coxeter diagram $\tilde{Q}_\ell$. Affine buildings are sometimes called Euclidean buildings.

The apartments of an affine building of type $\tilde{Q}_\ell$ have a canonical representation as a tesselation of Euclidean space of dimension ℓ.

Example

An apartment A of a building X of type $\tilde{A}_2$ looks like a Euclidean space of dimension 2 tesselated by regular hexagons, each subdivided into 6 equilateral triangles. These triangles are the chambers of A.

The building at infinity

Let X be a building of type $\tilde{Q}_\ell$.

Apartments contain sectors. A sector of X is a sector in one of its apartments.

Two sectors are equivalent if their intersection is a sector.

The set of sector classes is the vertex set of a building X^∞ of type Q_ℓ. The building X^∞ is called the building at infinity of X. It is spherical and

$$A \mapsto A^\infty$$

is a bijection from the set of apartments of X to the set of apartments of X^∞.

Bruhat-Tits buildings

Definition

A Bruhat-Tits building is an irreducible affine building whose building at infinity is Moufang.

The root groups of X^∞

Let X be a Bruhat-Tits building, let A be an apartment of X and let a be a "half-space" of A. Then the following hold:

- $\alpha = a^\infty$ is a root of the apartment A^∞.
- Every element g in the root group U_α^* of X^∞ is induced by a unique element $\hat{g} \in \mathrm{Aut}(X)$.
- The fixed point set of $\hat{g}$ in A is a half-space of A contained in or containing the half-space a. This gives rise to a function $\varphi_\alpha\colon U_\alpha^* \to \mathbf{Z}$ such that

  $$\varphi_\alpha(g) = \varphi_\alpha(-g) \quad \text{and} \quad \varphi_\alpha(g_1 + g_2) \geq \min\{\varphi_\alpha(g_1), \varphi_\alpha(g_2)\}.$$

- The map

  $$d_\alpha(g_1, g_2) = 2^{-\varphi_\alpha(g_1 - g_2)}$$

  is a metric on U_α.
- U_α is complete with respect to the metric d_α.

The classification of Bruhat-Tits buildings

Theorem

A Bruhat-Tits building is uniquely determined by its building at infinity.

(A Bruhat-Tits building is not, however, uniquely determined by its residues.)

The classification of Bruhat-Tits buildings

Theorem

Let Δ be a spherical building satisfying the Moufang condition and let K be its field of definition. Then Δ is the building at infinity of a Bruhat-Tits building iff

- *K is complete with respect to a discrete valuation and*
- *for each root α, the root group U_α is complete with respect to a certain metric (which turns out to be the metric d_α).*

Two final comments:

- The second condition follows from the first if Δ is the spherical building associated with an absolutely simple algebraic group or if Δ is simply laced.
- The proof of the Theorem is not finished: There is a family of Moufang quadrangles arising from certain groups of type E_8 for which the result has not yet been proved.

The End