# Constraint Satisfaction Problems, Twisted Subgroups and Transversals

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We cite from [9]: "Near subgroups of finite groups were introduced by Feder and Vardi [11] as a tool to study the computational complexity of constraint satisfaction problems." Their main motivation was the question whether every problem in the class CSP is either polynomial or NP-complete. Feder and Vardi showed that the class of problems whose constraints are defined by subgroups and their cosets in a given abelian group is polynomially solvable [11]. There it is also proven that every additional constraint type that is not a coset of a subgroup makes the problem NP-complete.

The subgroup-and-their-cosets problem remains polynomial even if the group is not abelian. For non abelian groups additional constraints that are not cosets of subgroups resist attempts at showing NP-completeness with the exception: If for a contraint type the group has an abelian section where a coset of the constraint containing the identity 1 does not define a subgroup, then the problem is NP-complete by the result on abelian groups [10]. A *near subgroup* of a group is a set containing 1 whose every coset containing 1, when restricted to an abelian section, defines a subgroup. Thus sets that are not cosets of near subgroups make the problem for a given group NP-complete, see [10].

Bulatov [7,8] has shown that Mal'tsev constraints have a polynomial time algorithm. In between Aschbacher [2] addressed some questions raised in [11] and showed that near subgroups possess much structure. More recently, Feder [10] showed that near subgroups do indeed characterize the polynomial time solvable cases of group theoretic constraint satisfaction problems, using new structural results for near subgroups obtained by Aschbacher [2,3]. More precisely Feder proved that subgroups, near subgroups, and their cosets are Mal'tsev constraints [10].

The aim of this paper is to exhibit some of the influence of computer science on group theory. In particular, I would like to present the development in group theory which was initiated by Feder and Vardi [11] and its surprising consequences (see also [10, p.3]).

Every near subgroup of a group is a twisted subgroup and for odd order groups the two concepts coincide. A subset $K$ of a group $G$ is a *twisted subgroup* of $G$ if $K$ contains the identity 1 of $G$ and if for all $x, y \in K$ it follows that $xyx$ is in $K$, see [2] or [9].

First of all notice that there is a one-to-one correspondence between twisted subgroups and *Bol loops*, see for instance [6,4]. Bol loops are groups satisfying only a weak axiom of associativity. They are also of importance in physics where they are called *gyrogroups* [17]. If in a twisted subgroup $K$ all the elements different from the identity are involutions, i.e. applied twice are the identity, then the related *Bol loop* is even a *Bruck loop* (in physics called *gyrocommutative gyrogroup* [17]).

Recall that a *transversal* $\mathcal{T}$ to a subgroup $U$ in $G$ is a set of representatives of the set of right cosets of $U$ in $G$, that is $\mathcal{T}$ has precisly one element from every right coset of $U$ in $G$. If the twisted subgroup consists beside the identity of involutions, then $G$ has a subgroup $U$ such that $\mathcal{T} = K$ is a transversal to $U$ in $G$, which satifies the following two conditions.

(1) $1 \in \mathcal{T}$

(2) $\mathcal{T}$ is closed under conjugation by $G$ (that is $\mathcal{T}^G = \{g^{-1}tg \mid g \in G, t \in \mathcal{T}\} = \mathcal{T}$),

see [6] or [4].

Bruck loops have been studied by Glauberman [12, 14]. He generalized the Feit-Thompson Theorem stating that every finite group of odd order is soluble to finite Bruck loops of odd order. In order to do so he proved his famous $Z^\star$ Theorem [13] which became a basic tool in the classification of the finite simple groups.

It was a long standing open question whether, as for finite groups, Bruck loops where $K$ consists beside the identity only of involutions are soluble. Aschbacher answered some questions of Feder and Vardi on near subgroups in [2]. This led him to work on this long standing open question on twisted subgroups just mentionned. He could reduce the problem to a problem on linear groups and their modules. Using these results of Aschbacher Stein and the author answered the question negatively by giving a counter example [6], see also [16]. This surprising answer - almost everybody expected that the Bruck loops would be soluble - revealed that Bruck loops as well as Bol loops of even order are not at all understood up to now.

If we replace (2) above by

(2') $\mathcal{T}$ is closed under conjugation by $U$,

then we obtain a gyrodecomposition of $G$, which were also studied by Feder [9].

In the following we show that the concept of a twisted subgroup can be used to produce a complement to a subgroup in a group. We study the groups satisfying (1) and (2'). If $(G, U)$ satisfies the condition (2'), then we say that $G$ *admits a $U$-invariant transversal in $G$.*

**Notation** In the following the reader is directed to [1] for notation and terminology. Nevertheless we like to recall some definitions.

Let $G$ be a finite group and let $|G| = p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorisation of the order of $G$. Then every subgroup of $G$ of order $p_i^{a_i}$ is a *Sylow $p$-subgroup of $G$*. If the group $G$ is the direct product of its Sylow $p$-subgroups, then $G$ is called *nilpotent*. The group $G$ is *soluble*, if there are normal subgroups $G_1 = G, G_2, \ldots, G_n = 1$ of $G$ (i.e. subgroups $G_i$ such that $g^{-1}G_ig = G_i$ for every $g$ in $G$) such that $G_i/G_{i+1}$ is abelian for $1 \leq i \leq n-1$.

As usual we denote by $\pi(G)$ the set of primes which divide the order of $G$. If $\pi = \pi(G)$, then we say that $G$ is a $\pi$-group. For $\pi \subseteq \pi(G)$, we denote by $O^\pi(G)$ the smallest normal subgroup of $G$ such that $G/O^\pi(G)$ is a $\pi$-group. If $\pi = \{p\}$ consists of a single prime then we omit the brackets.

Gil Kaplan could characterize the groups satisfying (1) and (2') under the further assumption that $U$ is a Sylow $p$-subgroup of $G$. He showed the following.

**Theorem 1** *[15] Let $G$ be a group, $p$ be a prime and $U$ a Sylow $p$-subgroup of $G$. Assume that $U$ has a transversal $\mathcal{T}$ in $G$ which is normalized by $U$. Then $U$ has a normal $p$-complement (that is a normal subgroup $N$ such that $G = NU$ and $N \cap U = 1$).*

Recall that $U$ is a *Hall subgroup* of a group $G$ if $\gcd(|U|, |G|/|U|) = 1$. In particular, every Sylow $p$-subgroup of $G$ is a Hall subgroup of $G$. If $\pi = \pi(U)$, then we say that $U$ is a $\pi$-Hall subgroup.

In [5] we generalized the result of Kaplan to:

**Theorem 2** *Let $U$ be a nilpotent Hall subgroup of $G$ which admits a $U$-invariant transversal in $G$. Then $U$ has a normal complement in $G$.*

It is a natural question to ask what is happening if $U$ is not nilpotent, but soluble? The following example shows that a soluble Hall subgroup $U$ which has a by $U$ normalized transversal does not have a normal complement in general:

**Example 1** Let $G = S_5$, the group of permutations of the set $\{1, \ldots, 5\}$, and let $U$ be the stabilizer of 5 in $G$. Set

$$\mathcal{T} = \{id, (15), (25), (35), (45)\},$$

where $(ij)$ is the transposition interchanging $i$ and $j$. Then $U \cong S_4$ is a soluble Hall subgroup of $G$ and $\mathcal{T}$ an $U$-invariant transversal to $U$ in $G$. Clearly, $U$ does not have a normal complement in $G$.

Assume that $U$ is a $\pi$-Hall subgroup of $G$ which admits a normal complement $N$. Then $N$ is a $U$-invariant transversal to $U$ in $G$. Moreover, $U \cong G/N$ is a $\pi$-group and $N$ is not divisible by any prime in $\pi$. This shows that $N$ is contained in $O^{\pi}(G)$.

We prove that this necessary condition is already sufficient.

**Theorem 3** *Let $U$ be a soluble Hall subgroup of $G$. Then $G$ has a normal complement to $U$ if and only if $U$ admits a transversal $\mathcal{T} \subseteq O^{\pi}(G)$ with $\mathcal{T}^U = \mathcal{T}$.*

Example 1 is not a counterexample to this theorem, as in the example $U$ is a $\{2,3\}$-Hall subgroup of $G$, $O^{\{2,3\}}(G) = A_5$, the subgroup of even permutations of $S_5$, and there is no $U$-invariant transversal to $U$ in $G$ which is contained in $A_5$.

Clearly, the immediat question arises: what can be said if $U$ is an arbitrary Hall subgroup which has a $U$-invariant transversal, but none of the normalized transversals is contained in $O^\pi(G)$? In a forthcoming paper we further investigate these groups.

The condition that $\gcd(|U|, |\mathcal{T}|) = 1$ is really needed - else we get counter examples, see [5].

In the next section we provide some general facts which are necessary to prove Theorem 3. The last section includes the proof of Theorem 3.

# 1 General properties

The first lemma is an easy exercise in group theory.

**Lemma 1.1** *Let $G$ be a group and $\pi \subseteq \pi(G)$. Then*

*(a)* $O^\pi(G) \leq \cap_{p\in\pi} O^p(G)$

*(b)* $O^\pi(O^p(G)) = O^\pi(G)$ *for all* $p \in \pi$.

**Proof.** The first part follows directly from the definition and the second from the facts that on the one hand if $O^p(G)/A$ is a $\pi$-group, then also $G/A$ and on the other hand that $O^\pi(G)$ is a subgroup of $O^p(G)$ and $O^p(G)/O^\pi(G)$ a $\pi$-group. □

Now let us focus on groups which have a subgroup $U$ admitting a $U$-invariant transversal.

**Lemma 1.2** *Let $U$ be a subgroup of $G$ which has a transversal $\mathcal{T}$ such that $\mathcal{T}^U = \mathcal{T}$. Then the following holds.*

*(a) There is precisely one element $u_0$ in $U \cap \mathcal{T}$.*

*(b) Let $u_0$ be as in (a). Then $u_0$ commutes with $U$, i.e. $u_0 u = u u_0$ for all $u_0$ in $U$.*

(c) *$U$ controls fusion of its $p$-elements, i.e. if $u$ is an element of $U$ whose order is a power of $p$, then $u^G \cap U = u^U$.*

**Proof.** As $\mathcal{T}$ is a transversal for $U$ in $G$, there is precisely one element $u_0$ in $U \cap \mathcal{T}$, which is (a). Now $\mathcal{T}^U = \mathcal{T}$ implies (b).

Let $u$ be a $p$-element of $U$ such that $u^t \in U$ for some $t \in \mathcal{T}$. Then $t^{-1}utu^{-1} \in U$. As $utu^{-1}$ is in $\mathcal{T}$ and as $\mathcal{T}$ is a transversal for $U$ in $G$, it follows that $utu^{-1} = t$ and therefore $ut = tu$. The fact that $G = UT$ implies now $u^G \cap U = u^U$, which is the assertion (c). □

# 2 Proof of Theorem 3

In the following we need to distinguish between the prime divisors of $|G|$ which divide the order of $U$ and which don't. Set

$$\pi := \pi(U).$$

Moreover, recall the definition of the *commutator subgroup*

$$G' := \langle g^{-1}h^{-1}gh \mid g, h \in G \rangle$$

of $G$.

**Lemma 2.1** *Let $U$ be a subgroup of $G$ such that $G$ admits a $U$-invariant transversal in $G$. If $O^p(U)U' < U$ for a prime $p$ in $\pi$, then $O^p(G)G' < G$.*

**Proof.** As $U$ controls fusion of its $p$-elements by Lemma 1.2(c), it follows that

$$(O^p(G)G') \cap U = O^p(U)U' \text{ for all } p \in \pi,$$

see [1, 37.5]. If $O^p(G)G' = G$, then we get the contradiction

$$U = G \cap U = (O^p(G)G') \cap U = O^p(U)U' < U.$$

□

**Proposition 2.2** *Let $U$ be a subgroup of $G$ such that $G$ admits a $U$-invariant transversal in $G$. If $O^p(G)G' = G$ for all prime $p$ in $\pi$, then $U$ is a perfect group, i.e. $U = U'$.*

**Proof.** Assume that $U$ is not perfect. Then $U' < U$. Let $p$ be a prime which divides the order of $U/U'$. Then $O^p(U)U'$ is a proper subgroup of $U$. Thus, we get a contradiction to Lemma 2.1. □

**Corollary 2.3** *Let $U$ be a Hall subgroup of a perfect group $G$ such that $G$ admits a $U$-invariant transversal in $G$. Then $U$ is a perfect group.*

**Proof.** As $G$ is perfect, we get $O^p(G)G' = G$ for all $p$ in $\pi$, and therefore Proposition 2.2 yields the assertion. □

**Proof of Theorem 3.**
Suppose that $U$ has a normal complement $N$. Then $N$ is a $\pi'$-group and therefore it is is a transversal which is normalized by $U$ and which is contained in $O^\pi(G)$.

Now assume that there is a transversal $\mathcal{T} \subseteq O^\pi(G)$ with $\mathcal{T}^U = \mathcal{T}$. Then in particular, $\mathcal{T}$ is contained in $O^p(G)$ for all $p$ in $\pi$, see Lemma 1.1(a). Let $p$ be a prime dividing the order of $U/U'$. Then $O^p(G) < G$ by Lemma 2.1. Clearly, $O^p(G) = \mathcal{T}(U \cap O^p(G))$. As moreover $\mathcal{T} \subseteq O^\pi(G) = O^\pi(O^p(G))$ by Lemma 1.1(b) we can use induction and get that there is a normal $\pi$-complement $N$ to $U \cap O^p(G)$ in $O^p(G)$. Thus, $N$ is a characteristic subgroup of $O^p(G)$ and therefore it is normal in $G$. Further,

$$G = O^p(G)U = N(U \cap O^p(G))U = NU$$

and $N \cap U \leq N \cap O^p(G) \cap U = 1$. □

# References


[1] M. Aschbacher, Finite Group Theory, Cambridge University Press, 1986.

[2] M. Aschbacher, Near subgroups of finite groups, *J. Group Theory* **1** (1998), 113-129.

[3] M. Aschbacher, Manuscript, 2001.

[4] M. Aschbacher, On Bol loops if exponent 2, *J. Algebra* **288** (2005), 99-136.

[5] B. Baumeister,Transversals with further properties, preprint 2007.

[6] B. Baumeister, A. Stein, Self-invariant 1-Factorizations of Complete Graphs and Finite Bol Loops of Exponent 2, submitted (16 pages).

[7] A.A. Bulatov, Tractable constraint satisfaction problems on a three-element set, *Electronic Colloquium on Computational Complexity*, Report No. TR02-032 (2002).

[8] A.A. Bulatov, Mal'tsev constraints are tractable, *Electronic Colloquium on Computational Complexity*, Report No. TR02-034 (2002).

[9] T. Feder, Strong near subgroups and left gyrogroups, *J. Algebra* **259** (2003), 177-190.

[10] T. Feder, Constraint satisfaction on finite groups with near subgroups, *Electronic Colloquium on Computational Complexity*, Report No. 5 (2005).

[11] T. Feder, M. Vardi, The computational structure of monotone monadic SNP and constrainst satisfaction: a study through datalog and group theory, *SIAM J. Comput.* **28** (1998), 57-104.

[12] G.Glauberman, On Loops of odd order I, *J.Algebra* **1** (1964), 374-396.

[13] G.Glauberman, Central elements in core-free groups, *J. Algebra* **4** (1966), 403-420.

[14] G.Glauberman, On Loops of odd order II, *J.Algebra* **8** (1968), 393-414.

[15] G. Kaplan, Personal Communication, 2006.

[16] G.Nagy, A class of simple proper Bol loops, *Preprint*

[17] A.A.Ungar, Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovectors Spaces Kluwer Academic Publishers, Doldrechts-Boston-London 2001