c-Sections and solvability of finite groups

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Abstract

c-Sections of maximal subgroups in a finite group and their relation to solvability were extensively researched in recent years (see [SW], [W] and [LS]). In this paper we prove (Theorem 1.1) that a finite group G is solvable if and only if every maximal subgroup M of G satisfies $|Sec(M)| < |G : M|^{\beta}$, where $\beta = log(175560)/log(2624832) \simeq 0.817$. We show that β can not be replaced by a larger constant. If G is a finite group in which every maximal subgroup M satisfies $|Sec(M)| < |G : M|$, then each composition factor of G is either cyclic or isomorphic to the $O'Nan$ sporadic simple group (Theorem 1.5).

1 Introduction

Let M be a maximal subgroup of a group G and K/L be a chief factor of G such that $L \leq M$ while $K \nleq M$. According to [SW] we call the group $M \cap K/L$ a *c-section of* M. Shirong and Wang proved that for a given maximal subgroup M of G all the c-sections of M are isomorphic [SW, 1.1]. We denote the abstract group isomorphic to a c-section (and so to all c-sections) of M by $Sec(M)$.

In [W] it was proved (although not using this terminology) that a group is solvable if and only if the c-sections of all its maximal subgroups are trivial. Further solvability conditions were proved in [SW]. In particular, a group is solvable if and only if the c-sections of all its maximal subgroups are 2-closed ([SW], Theorem 2.1), and if and only if the c-sections of all its maximal subgroups are nilpotent ([SW], Theorem 2.2). The case when all the c-sections are supersolvable was discussed in [LS].

In this paper we study further the notion of c-sections and its connection to solvability. In particular, for a maximal subgroup M we consider the relation between the order of the c-section $|Sec(M)|$ and the index $|G : M|$. By the above, if G is solvable then obviously $|Sec(M)| < |G : M|$ for each maximal subgroup M of G. It turns out that the opposite direction is not true, but the counterexamples can occure only for groups in which each non-cyclic composition factor is isomorphic to the $O'Nan$ sporadic simple group (Corollary 1.1 below). However, by a certain refinement, we get an equivalent condition for solvability in Theorem 1.1 below.

In the sequel, a subgoup B of G is called *large* if $|B| \geq |G|^{1/2}$. We shall need the special number $\beta := log(175560)/log(2624832) \simeq 0,817$. Using the classification of finite simple groups, we prove the following solvability criterion.

Theorem 1.1 *Let G be a group. Then G is solvable if and only if $|Sec(M)| < |G : M|^\beta$ for all maximal subgroups M of G.*

We show in Proposition 3.2 that the non-solvable group $G = Aut(O'Nan)$ is a group satisfying $|Sec(M)| \leq |G : M|^\beta$ for all maximal subgroups M of G. Thus β can not be replaced by a larger constant in Theorem 1.1.

In order to formulate conveniently our next result we define the following two conditions on a group G.

Condition 1.1 $|Sec(M)| < |G : M|$ *for every maximal subgroup* M *of* G.

Let G be a group and $B \leq K \leq G$. Then we say that *the action of* G *on* B^G *is controlled by* K, if every G-conjugate of B which is inside K is a K-conjugate of B. In this paper we deal only with the case when K is normal in G.

Condition 1.2 *Let* K/L *be a chief factor of* G. *For each proper large subgroup* B/L *of* K/L, *the action of* G/L *on* $(B/L)^{G/L}$ *is not controlled by* K/L.

Theorem 1.2 *(a classification-free result) Condition 1.1 holds if and only of Condition 1.2 holds for all the non-abelian chief factors of* G.

For proving Theorem 1.1 we shall need the following two propositions, which are of independent interest.

Proposition 1.3 *Let* G *be a simple non-abelian group such that* $G \ncong O'Nan$. *Then* G *has a proper large subgroup* H *such that the action of* $Aut(G)$ *on* $H^{Aut(G)}$ *is controlled by* G.

Proposition 1.4 *Every simple non-abelian group* G *has a proper subgroup* H *such that* $|H| \geq [G : H]^\beta$ *and the action of* $Aut(G)$ *on* $H^{Aut(G)}$ *is controlled by* G.

We show that β can not be replaced by a larger constant in Proposition 1.4 (see Remark 2.6). The following result follows from Theorem 1.2 and Proposition 1.3.

Theorem 1.5 *If Condition 1.1 holds for a group* G *then every composition factor of* G *is either cyclic or isomorphic to* $O'Nan$.

The proofs of Propositions 1.3 and 1.4 are given in Section 2. The proofs of Theorems 1.1, 1.2 and 1.5 are given in Section 3.

2 Proofs of Propositions 1.3 and 1.4

We prove Proposition 1.3 separately for the sporadic simple groups, the simple groups of Lie type and the alternating groups; see Proposition 2.1, Corollary 2.4 and Proposition 2.5, respectively.

Proposition 2.1 *Let G be a sporadic simple group which is not isomorphic to O'Nan. Then G has a proper large subgroup H such that the action of $Aut(G)$ on $H^{Aut(G)}$ is controlled by G.*

Proof. It was proved in [L] that each simple non-abelian group G has a large maximal subgroup. When $Out(G) = 1$ this large subgroup H certainly satisfies our control condition. In Table 1 we give for each sporadic group G with $Out(G) > 1$, except $O'Nan$, a corresponding large maximal subgroup H such that the action of $Aut(G)$ on $H^{Aut(G)}$ is controlled by G. The information is based on [At]. This information completes the proof. □

Table 1: Proper large subgroups with control

G	H	$\|H\|$	$\|G:H\|$
M_{12}	$L_2(11)$	660	144
M_{22}	$L_3(4)$	20160	22
Suz	$G_2(4)$	251596800	1782
HS	M_{22}	443520	100
M^CL	$U_4(3)$	3265920	275
He	$S_4(4):2$	1958400	2058
HN	A_{12}	239500800	1140000
J_2	$U_3(3)$	6048	100
J_3	$L_2(16):2$	8160	6156
Fi_{22}	$2\cdot U_6(2)$	18393661440	3510
Fi_{24}'	Fi_{23}	4089470473293004800	306936

Recall that a Borel subgroup of a group of Lie type G in characteristic p is the normalizer of a Sylow p-subgroup of G.

Proposition 2.2 *Let G be a simple group of Lie type ${}^{\sigma}\mathcal{L}_l(q)$ of rank l defined over the field with q elements. If $q > 2$, then every Borel subgroup B of G is a large subgroup of G such that the action of $Aut(G)$ on $B^{Aut(G)}$ is controlled by G.*

Proof. Since B is a Sylow normalizer and all the Sylow p-subgroups of G are conjugate in G, the control condition is satisfied. It is left to show that B is large in G. We deal separately with the cases when G is twisted or not.

Case 1. G is a non-twisted group of Lie type.
Then according to [Ca, 9.4.10]

$$|G| = \frac{1}{d}q^N(q^{d_1}-1)\cdots(q^{d_l}-1),$$

$$|B| = \frac{1}{d}q^N(q-1)^l$$

and

$$|G:B| = (q^{d_1}-1)\cdots(q^{d_l}-1)/(q-1)^l,$$

where d is as in 9.4.10 of [Ca], $N = |\Phi^+|$ is the number of positive roots of the root system related to G and $d_1 + \cdots + d_l = N + l$ [Ca, 9.3.4].

By assumption $q \geq 3$. Assume $l = 1$. Then even $q \geq 4$, $N = 1$ and $d_1 = N + l = 2$. Hence

$$|G:B| = (q^2-1)/(q-1) = q+1$$

and

$$|B| = q(q-1)/(q-1,2).$$

As $q(q-1) \geq 3q$ and $3q > 2(q+1)$, the assertion follows.

Now let $l \geq 2$. If $l = 2$ and $q = 3$, then either $d = 1$ and $G \cong L_3(3)$ or $G_2(3)'$, or $d = 2$ and $G \cong PSp_4(3)$. In the first case $|B| = 2^2 \cdot 3^3$ or $2^2 \cdot 3^6$ and $|G:B| = 2^2 \cdot 13$ or $2^4 \cdot 7 \cdot 13$, respectively. Thus B is a large subgroup of G. If $G \cong PSp_4(3)$ then $|B| = 2 \cdot 3^4 = 162$ and $|G:B| = 2^5 \cdot 5 = 160$ and the assertion holds again.

From now on we assume $l \geq 3$ if $q = 3$ and $l \geq 2$ otherwise. We aim to show

$$(q^{d_1}-1)\cdots(q^{d_l}-1) < \frac{1}{d}q^N(q-1)^{2l}.$$

We have

$$(q^{d_1}-1)\cdots(q^{d_l}-1) < q^{\sum_{i=1}^{l} d_i} = q^{N+l}$$

and claim that

$$q^l < (q-1)^{2l-1},$$

which then yields the assertion.

First let $q = 3$. Then $l \geq 3$, $(\frac{4}{3})^l > 2$ and so $2^{2l-1} > 3^l$ as required. Now suppose $q \geq 4$. Then $(q-1)^{2l} > (q^2-2q)^l = q^l(q-2)^l$. Thus it remains to show that $(q-2)^l \geq q-1$. This holds, as $(q-2)^l \geq (q-2)^2 = q^2-4q+4$ and $q^2 \geq 5(q-1)$.

Case 2. G is a twisted group of Lie type.
We choose the notation as it is given in [Ca]. So G is isomorphic to one of the following groups:

$${}^2A_l(q^2), {}^2B_2(q^2), {}^2D_l(q^2), {}^3D_4(q^3), {}^2E_6(q^2), {}^2F_4(q^2), {}^2G_2(q^2),$$

where $q^2 = 2^{2m+1}$ (resp. $q^2 = 3^{2m+1}$) if $\mathcal{L}$ is of type B_2 or F_2 (resp. of type G_2).

Let B be a Borel subgroup of T. Then by [Ca, 14.1.2]

$$|B| = \frac{1}{d}q^N(q-\eta_1)(q-\eta_2)\cdots(q-\eta_l),$$

where N is the number of positive roots in the root system related to $\mathcal{L}_l(q)$, d will be indicated in each case and $\eta_1,\ldots,\eta_l$ are the eigenvalues of the isometry τ of the vector space spanned by the roots which is related to the symmetry of the diagram for $\mathcal{L}_l(q)$. By [Ca, 14.1.3] and [Ca, 14.3.1] we have

$$|G| = \frac{1}{d}q^N(q-\eta_1)(q-\eta_2)\cdots(q-\eta_l)\sum_{w\in W^1} q^{l(w)} = \frac{1}{d}q^N(q^{d_1}-\in_1)(q^{d_2}-\in_2)\ldots(q^{d_l}-\in_l)$$

where W^1 is the Weyl group of G and d_i as well as $\in_i$ are as in [Ca, Section 14.2]. Now we discuss all the possibilities.

Let $G \cong {}^2A_l(q^2)$ be a unitary group. We distinguish between the cases l even and l odd.

l even. Then $d = (q+1, l+1), N = l(l-1)/2, \eta_1 = \ldots = \eta_{l/2} = 1$ and $\eta_{l/2+1} = \ldots = \eta_l = -1$. So,

$$|B| = \frac{1}{d} q^{l(l-1)/2}(q-1)^{l/2}(q+1)^{l/2} \text{ and } |G:B| = \prod_{i=1}^{l}(q^{i+1} - (-1)^{i+1})/(q-1)^{l/2}(q+1)^{l/2}.$$

Notice that $(q^m - 1)(q^{m+1}+1) < q^{m+m+1}$. Thus $|G:B| < q^{2+3+\cdots+(l+1)}/(q-1)^{l/2}(q+1)^{l/2} = q^{l(l-1)/2+l}/(q-1)^{l/2}(q+1)^{l/2}$. So it is enough to show that $q^l \leq \frac{1}{d}(q-1)^l(q+1)^l$, or $q \leq \frac{1}{d^{1/l}}(q-1)(q+1)$. Since the "worst" case is $d = q+1$, it suffices to show $q \leq (q-1)(q+1)^{1-1/l}$. Since even $q \leq (q-1)(q+1)^{1/2}$ holds for every $q > 2$, we are done.

l odd, $l \geq 3$. Then $d = (q+1, l+1), N = l(l-1)/2, \eta_1 = \ldots = \eta_{(l+1)/2} = 1$ and $\eta_{(l+1)/2+1} = \ldots = \eta_l = -1$. So,

$$|B| = \frac{1}{d} q^{l(l-1)/2}(q-1)^{(l+1)/2}(q+1)^{(l-1)/2} \text{ and}$$

$$|G:B| = \prod_{i=1}^{l}(q^{i+1} - (-1)^{i+1})/(q-1)^{(l+1)/2}(q+1)^{(l-1)/2}.$$

Similarly to the previous case we obtain $|G:B| < q^{l(l-1)/2+l}/(q-1)^{(l+1)/2}(q+1)^{(l-1)/2}$. Thus it is enough to show $q^l \leq \frac{1}{d}(q-1)^{l+1}(q+1)^{l-1}$. Again we take the worst case $d = q+1$, so it suffices to show $q^l \leq (q-1)^{l+1}(q+1)^{l-2}$, or $q \leq (q-1)^{1+1/l}(q+1)^{1-2/l} = \frac{1}{((q+1)^2/(q-1))^{1/l}}(q-1)(q+1)$. Since $q \leq \frac{1}{((q+1)^2/(q-1))^{1/3}}(q-1)(q+1)$, holds for every $q > 2$, this case is completed as well.

Let $G \cong {}^2B_2(q^2)$ be a Suzuki group. Then $d = 1, N = 4, \eta_1 = 1$ and $\eta_2 = -1$. Thus

$$|B| = q^4(q^2-1),\ |G:B| = q^4 + 1$$

and the assertion holds for every q (including $q = 2$).

Let $G \cong {}^2D_l(q^2)$ be an orthogonal group of minus type. Then $d = (4, q^l+1), N = l(l-1), \eta_1 = \ldots = \eta_{l-1} = 1$ and $\eta_l = -1$. Thus

$$|B| = \frac{1}{d} q^{l(l-1)}(q-1)^{l-1}(q+1) \text{ and } |G:B| = (q^l+1)(\prod_{i=1}^{l-1}(q^{2i}-1))/(q-1)^{l-1}(q+1).$$

Then $|G:B| < q^{l-1}2^{l-1}\prod_{i=1}^{l-1} q^{2i-1} = 2^{l-1}\prod_{i=1}^{l-1} q^{2i} = 2^{l-1}q^{l(l-1)}$. If $q > 2$, then the latter is at most $q^{l(l-1)}(q-1)^{l-1}$. Hence B is a large subgroup in that case.

Let $G \cong {}^3D_4(q^3)$. Then $d = 1, N = 12$ and $\eta_i = \alpha^{i-1}$ with $\alpha \neq 1$ a third root of unity, for $1 \leq i \leq 3$. Hence

$$|B| = q^{12}(q-1)(q-\alpha)(q-\alpha^2) = q^{12}(q^3-1) \text{ and}$$

$$|G:B| = (q^8+q^4+1)(q^3+1)(q^2-1) < 2q^{13} < q^{12}(q^3-1),$$

and the assertion holds for every q (including $q = 2$).

Let $G \cong {}^2E_6(q^2)$. Then $d = (3, q+1), N = 36$, $\eta_1 = \ldots = \eta_4 = 1$, $\eta_5 = \eta_6 = -1$, $|B| = \dfrac{1}{d}q^{36}(q-1)^4(q+1)^2$ and $|G:B| = (q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)/(q-1)^4(q+1)^2$. If $q = 2$, then B is not a large subgroup of G. Let $q \geq 3$. Then

$$|G:B| < 2q^{11}q^8 2q^7 2q^5(q^5+1) = 2^3q^{31}(q^5+1)$$

and $q^5(q-1)^4(q+1) > 2^3(q^5+1)$, which shows the assertion.

Let $G \cong {}^2F_4(q^2)$. Then $d = 1$, $N = 24$, $\eta_1 = \eta_2 = 1$ and $\eta_3 = \eta_4 = -1$. So $|B| = q^{24}(q-1)^2(q+1)^2 = q^{24}(q^2-1)^2$ and $|G:B| = (q^{12}+1)(q^8-1)(q^6+1)(q^2-1)/(q^2-1)^2$. If $q^2 = 2$, then $|B| = 2^{12} < |G:B| = 65 \cdot 15 \cdot 9$.

Now let $r := q^2 = 2^{2m+1} > 2$. Then $|G:B| = (r^6+1)(r^3+r^2+r+1)(r^3+1) \leq (r^6+1)2r^3(r^3+1) < r^{12}(r-1)^2 = |B|$ and B is a large subgroup of G.

Let $G \cong {}^2G_2(q^2)$. Then $d = 1$, $N = 6, \eta_1 = 1$ and $\eta_2 = -1$. Then

$$|B| = q^6(q^2-1) \text{ and } |G:B| = (q^6+1)$$

and the assertion holds in all cases. □

We note that Proposition 2.2 is generally false in the case $q = 2$, but a Borel subgroup is a large subgroup of G if $G \cong {}^2B_2(2), {}^3D_4(2)$ as was shown in the proof of Proposition 2.2.

We still need to consider the linear groups defined over $GF(2)$. We have the following more general result.

Proposition 2.3 *Let G be a special linear group of rank $l \geq 2$ defined over the field with q elements. Let V be the natural module for T and (V_1, V_l) be two subspaces of dimension 1 and l, respectively, such that $V_1 \subseteq V_l$. Let P_i be the stabilizer of V_i in T, for $i = 1, l$. If $(l, q) \neq (2, 2)$, then $N := P_1 \cap P_l$ is a large subgroup of G, such that the action of $Aut(G)$ on $N^{Aut(G)}$ is controlled by G.*

Proof. Recall that the field and diagonal automorphisms of G act on the set of maximal parabolic subgroups of type i, for $1 \leq i \leq l$ [Ca] and that the graph automorphisms interchange the sets of maximal parabolics of type 1 and l. As P_l acts transitively on the 1-dimensional subspaces of V_l, the action of $Aut(G)$ on $N^{Aut(G)}$ is controlled by G.

$n := |G : N|$ is the number of flags (W_1, W_l), where W_i an i-dimensional subspace of V and $W_1 \subseteq W_l$. We have $n = (q^{l+1} - 1)(q^l - 1)/(q - 1)^2$. As

$$|G| = \frac{1}{d} q^{l(l+1)/2}(q^{l+1} - 1) \cdots (q^2 - 1),$$

where $d = (q - 1, l + 1)$, we get

$$|N| = \frac{1}{d} q^{l(l+1)/2}(q^{l-1} - 1) \cdots (q^2 - 1)(q - 1)^2.$$

We have to show that $|G : N| \leq |N|$. If $l = 2$ and $q \geq 3$ then

$$|G : N| = (q^3 - 1)(q^2 - 1)/(q - 1)^2 = (q^2 + q + 1)(q + 1) < \frac{1}{q - 1} q^3 (q - 1)^2 \leq |N|.$$

If $l = 3$ then

$$|G : N| = (q^4 - 1)(q^3 - 1)/(q - 1)^2 < \frac{1}{q - 1} q^6 (q^2 - 1)(q - 1)^2 \leq |N|.$$

Finally, if $l \geq 4$ then

$$|G : N| = (q^{l+1} - 1)(q^l - 1)/(q - 1)^2 < q^{2l+1} < q^{l(l+1)/2} < |N|,$$

completing the proof. □

Notice that the assertion in Lemma 2.3 is false for $G \cong L_3(2)$.

Corollary 2.4 *Let G be a simple group of Lie type. Then G has a proper large subgroup H such that the action of $Aut(G)$ on $H^{Aut(G)}$ is controlled by G.*

Proof. If G is not defined over $GF(2)$ or if $T \cong {}^2B_2(2), {}^3D_4(2)$, then the assertion follows by Proposition 2.2 and the remark after it. Therefore we may assume that G is defined over $GF(2)$.

If G is of type A_l, $l > 2$, then the statement is a consequence of Proposition 2.3. If $G \cong B_2(2)' \cong A_6 \cong L_2(9)$, $G \cong A_2(2) \cong L_3(2) \cong L_2(7)$ or $G \cong G_2(2)' \cong U_3(3)$, then we obtain the assertion by Proposition 2.2. If G is as listed in Table 2, then H is a large subgroup of G such that the action of $Aut(G)$ on $H^{Aut(G)}$ is controlled by G (the details are taken from [At]).

Table 2: Proper large subgroups with control

G	H	$\|H\|$	$\|G : H\|$
$D_4(2)$	$3^4 : 2^3.S_4$	15552	11200
$F_4(2)$	$[2^{20}]A_6.2$	754974720	4385745
${}^2F_4(2)'$	$2.[2^8] : 5 : 4$	10240	1755

If G is one of the remaining groups of Lie type defined over $GF(2)$, then it is easily verified that the large subgroup of G given by Table II of [L] satisfies our control condition. This completes the proof. □

It remains to consider the alternating groups. The case $G \cong A_6 \cong L_2(9)$ has already been handled in Proposition 2.2.

Proposition 2.5 *Let $G \cong A_n$, $n \geq 5$, $n \neq 6$, and let H be a point stabilizer in G (with respect to the action of G on the set $\{1, \cdots, n\}$). Then H is a large subgroup of G such that the action of $Aut(G)$ on $H^{Aut(G)}$ is controlled by G.*

Proof. Clearly H is large in G. The control condition is satisfied since $Aut(A_n) = S_n$ for $n \geq 5$, $n \neq 6$. □

Now Proposition 1.3 is a consequence of Proposition 2.1, Corollary 2.4 and Proposition 2.5.

Proof of Proposition 1.4. By Proposition 1.3 every simple non-abelian group G such that $G \ncong O'Nan$ has a proper subgroup H such that $|H| \geq |G : H|$ and the action of $Aut(G)$ on $H^{Aut(G)}$ is controlled by G. So the assertion certainly holds for every such group. It is left to consider $G = O'Nan$. By [At] G has a (maximal) subgroup $H \cong J_1$, $|H| = 175560$, $|G : H| = 2624832$ which satisfies our control condition. Since $|H| = |G : H|^\beta$, we are done. □

Remark 2.6 The number β can not be replaced by a larger constant in Proposition 1.4. Indeed, let $T := O'Nan$ and let $A < T$ be such that the action of $G := Aut(T)$ on A^G is controlled by T. We show that $|A| < |T : A|^\beta$. By Frattini's argument $Aut(T) = TN_G(A)$ and so $|T : A| \geq |T : T \cap N_G(A)| = |G : N_G(A)|$. The list of maximal subgroups of $G = Aut(T) \cong O'Nan : 2$ is determined in [Wi]. By this list $S := J_1 \times 2$ is the largest maximal subgroup of $Aut(T)$ which is different from T. Thus $|T : A| \geq |G : S| = 2624832$, which implies $|A| < |T : A|^\beta$ as required.

3 Proofs of Theorems 1.1, 1.2 and 1.5

The following lemma is useful.

Lemma 3.1 *Let $N = T^m$, where T is a simple non-abelian group, and let $N \leq G \leq Aut(N) = Aut(T)$ wr S_m. Suppose $B \leq T$ and the action of $Aut(T)$ on $B^{Aut(T)}$ is controlled by T. Let $A := B^m$, $A \leq N$. Then the action of G on A^G is controlled by N.*

Proof. Let $g = (h_1, ..., h_m)s \in G$, where $h_i \in Aut(T), s \in S_m$. Then $A^g = (B \times \cdots \times B)^g = B^{h_{s(1)}} \times \cdots \times B^{h_{s(m)}}$. Since the action of $Aut(T)$ on $B^{Aut(T)}$ is controlled by T, there exist $f_i \in T$ such that $B^{h_{s(i)}} = B^{f_i}$ for all $1 \leq i \leq m$. Let $u = (f_1, ..., f_m) \in N$, then we have

$A^g = B^{f_1} \times \cdots \times B^{f_m} = A^u$, which completes the proof. $\square$

Proof of Theorem 1.1. The *only if* part is known, as mentioned in the introduction. We prove the *if* part. Let G be a minimal counterexample. Since the condition on the c-sections of G is inherited by quotients of G, we have that G/N is solvable for each $1 < N \trianglelefteq G$. Hence G has a unique minimal normal subgroup N, and N is non-abelian, $N = T^m$, where T is a simple non-abelian group. Furthermore $N = T^m \leq G \leq AutT\ wr\ S_m = Aut(N)$. By Proposition 1.4 there exists a proper subgroup H of T such that $|H| \geq |G : H|^\beta$ and the action of $Aut(T)$ on $H^{Aut(T)}$ is controlled by T. Define $A = H^m$, $A < N$. Then it is easily verified that $|A| \geq |N : A|^\beta$, and by Lemma 3.1 the action of G on A^G is controlled by N. By the argument of Frattini we get $G = NN_G(A)$. Notice that $A < N$ forces that A is not normal in G. Let M be a maximal subgroup of G containing $N_G(A)$. Then $N \not\leq M$ and since N is minimal normal we have $Sec(M) \cong M \cap N$. Now $M \cap N \geq A$, implying that $|M \cap N| \geq |N : A|^\beta \geq |N : M \cap N|^\beta$. But since $G = MN$ we have $|N : M \cap N| = |G : M|$, hence $|Sec(M)| \geq |G : M|^\beta$, the desired contradiction. $\square$

As noted in the introduction, the following shows that Theorem 1.1 can not be improved by replacing β by a larger constant.

Proposition 3.2 *Let $G = Aut(O'Nan)$. Then $|Sec(M)| \leq [G : M]^\beta$ for each maximal subgroup M of G.*

Proof. Denote $T = Soc(G)$. Let M be a maximal subgroup of G. If $M = T$ then clearly $Sec(M) = 1$, so we may assume that $M \not\geq T$, hence $M \cap T < T$ and $G = MT$. For $g \in G$ there exist $u \in M, t \in T$ such that $g = ut$ and so $(M \cap T)^g = M^g \cap T = M^t \cap T = (M \cap T)^t$. This shows that the action of G on $M \cap T$ is controlled by T, and thus by Remark 2.6 $|M \cap T| \leq |T : M \cap T|^\beta = |MT : M|^\beta = |G : M|^\beta$. $T/1$ is a chief factor of G such that $T \not\leq M$, $1 \leq M$, and therefore $Sec(M) \cong M \cap T$, so by the above $|Sec(M)| \leq |G : M|^\beta$ as required. $\square$

Proof of Theorem 1.2. Suppose that Condition 1.2 is not satisfied by G with respect to all the non-abelian chief factors of it. So let K/L be a non-abelian chief factor of G, and let

B/L be a large proper subgroup of K/L such the action of G/L on $(B/L)^{G/L}$ is controlled by K/L. We shall show that G/L has a maximal subgroup M/L such that $|Sec(M)| \geq |G : M|$. It is no loss here to assume that $L = 1$. By the argument of Frattini $G = KN_G(B)$. Since B is not normal in G we can choose M, a maximal subgroup of G containing $N_G(B)$. Then $M \not\geq K$ and K is minimal normal, hence $Sec(M) = M \cap K$. But $M \cap K \geq B$ and B is a large subgroup of K. Thus $|M \cap K| \geq |K : M \cap K| = |G : M|$, which implies $|Sec(M)| \geq |G : M|$.

In the other direction, suppose G has a maximal subgroup M with $|Sec(M)| \geq |G : M|$. Let K/L be a chief factor of G satisfying $L \leq M$ and $K \not\leq M$. Then $G = KM$ implying $|G : M| = |K : M \cap K|$ and so $|(M \cap K)/L| \geq |K : M \cap K|$. Thus $(M \cap K)/L$ is a large proper subgroup of K/L. It is left to show that the action of G/L on $((M \cap K)/L)^{G/L}$ is controlled by K/L. It is no loss here to assume $L = 1$. Let $g \in G$, then $g = mk$, where $m \in M$, $k \in K$. Thus $(M \cap K)^g = M^g \cap K = M^{mk} \cap K = M^k \cap K = (M \cap K)^k$. The proof is now completed. □

Proof of Theorem 1.5 Let G be a group satisfying Condition 1.1. By Theorem 1.2, Condition 1.2 must hold for each non-abelian chief factor of G. Suppose that the corollary is not true, so there exists a chief factor $K/L = T^m$, T is a simple non-abelian group and $T \not\cong O'Nan$. By Proposition 1.1 there exists a large proper subgroup B of T such that the action of $Aut(T)$ on $B^{Aut(T)}$ is controlled by T. Let $A = B^m$, $A < K/L$. Then it is easily verified that A is a large proper subgroup of K/L, and by Lemma 3.1 the action of G/L on $A^{G/L}$ is controlled by K/L. Thus Condition 1.2 does not hold for K/L, a contradiction. □

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