A geometry for groups of $J_3$-type.

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7.3.2006

Abstract
The proof of the existence and of the uniqueness of groups of $J_3$-type by G. Higman and J. McKay is based on the fact that a group of $J_3$-type is a faithful completion of an amalgam of $J_3$-type, see [HiMc]. In this paper here, we provide a reference for that fact. The proofs in this paper are elementary and we do not use any character theory.

1 Introduction.
A finite simple group $G$ is said to be of $J_3$-type provided that all involutions of $G$ are conjugate and the centralizer of an involution is a split extension of an extraspecial group of order 32 by $\text{Alt}(5)$. Janko presented the initial evidence of a group of $J_3$-type [Ja], G. Higman and J. McKay showed the existence and the uniqueness of groups of $J_3$-type [HiMc]. Their proof is computer-based and uses, moreover, the fact that a group of $J_3$-type is a faithful completion of an amalgam of $J_3$-type.

An amalgam of rank $n$ is a family $A = (\alpha_{J,K} : P_J \to P_K \mid \emptyset \neq K \subset J \subseteq I)$, where $I = \{1, \ldots, n\}$, of group homomorphisms such that for all $L \subset K \subset J \subseteq I$

$$\alpha_{J,K} \alpha_{K,L} = \alpha_{J,L}.$$ 

To shorten notation we will simply write $A = (P_J \mid \emptyset \neq J \subseteq I)$. A completion $\beta : A \to G$ for $A$ is a family $\beta = (\beta_J : P_J \to G)$ of group homomorphisms such that $G = \langle P_J^{\beta_J} \mid J \subseteq I \rangle$ and for all $K \subset J \subset I$ it holds $\alpha_{J,K} \beta_K = \beta_J$. A completion is said to be faithful if each $\beta_J$ is an injection and a faithful completion $\gamma : A \to G(A)$ is

*2000 MR Subject Classification 20D08, 20D05, 51E24
universal if for every completion $\beta : A \to G$ there is a group homomorphism $\varphi$ of $G(A)$ onto $G$ such that $\gamma J \varphi = \beta J$ for all $J \subseteq I$. These definitions are taken from [Asch3]. In the following we omit brackets in $G_{(i,j)}$ by writing $G_{ij}$.

An amalgam of $J_3$-type is an amalgam $A = \{G_1, G_2, G_3, G_{12}, G_{13}, G_{23}, G_{123}\}$ of rank 3 satisfying the following conditions, where $B := G_{123}$.

(i) $G_1 \cong L_2(16) : 2$, $G_2 \cong 2^4 : GL_2(4)$, $G_3 \cong 3 : PGL_2(9)$;
(ii) $G_{12} \cong 2^4 : (3 \times D_{10})$, $G_{23} \cong GL_2(4) \cong 3 \times \text{Alt}(5)$, $G_{13} \cong \text{Sym}(3) \times D_{10}$;
(iii) $B \cong 3 \times D_{10}$,

It was shown by J. G. Thompson that a group of $J_3$-type has a subgroup isomorphic to $L_2(16) : 2$, but this result was never published.

Meanwhile, there are also existence proofs of a group of $J_3$-type which are not computer dependant [Wei, Asch2, Ba1] and there is a computer-free uniqueness proof due to D. Frohardt [Fro].

In this paper we provide a reference for the fact that a group of $J_3$-type is a faithful completion of an amalgam of $J_3$-type. We show

**Theorem 1** Let $G$ be a group of $J_3$-type. Then $G$ is a completion of an amalgam of $J_3$-type.

The hope is that we can use Theorem 1 to give a more simple uniqueness proof for groups of $J_3$-type because of the following facts. A completion of an amalgam of $J_3$-type acts flag-transitively on a Buekenhout geometry, namely on a dual extended quadrangle DEQ (see [Ba2]) which is a geometry consisting of points, lines and quads such that

(res(p)) For a point $p$ the lines and the quads which are incident with $p$ form a complete graph whose vertices are the lines and whose edges are the quads;
(res(l)) Any point on a line $l$ is incident to any quad which is incident with $l$;
(res(q)) For a quad $q$ the points and the lines which are incident to $q$ form a generalized quadrangle.

See [Buc] or [Pa] for an introduction to diagram geometries.

Let $A$ be an amalgam of $J_3$-type and let $G$ be a faithful completion of $A$. Then the coset geometry $\Gamma = \Gamma(G, (G_1, G_2, G_3))$, a rank three geometry consisting of points, lines and quads, which are the cosets of $G_i$ for $i = 1, 2, 3$ in $G$, respectively, such that two elements of the geometry are incident if and only if the respective cosets intersect non-trivially, is a DEQ and $G$ acts flag-transitively on $\Gamma$. In [Ba1] it was shown that there is up to isomorphism only one amalgam of $J_3$-type. This shows that there is at most one universal completion of an amalgam of $J_3$-type. By Lemma 2.2 of [Ba1] the latter group is finite. Moreover, in the same paper a DEQ, $\Gamma$ has been constructed which admits a group of $J_3$-type as flag-transitive group of automorphisms.

The two latter facts and Theorem 1 imply the following.
Corollary 1.1 Let $G$ be a group of $J_3$-type. Then $G$ acts flag-transitively on a DEQ which is a quotient of the universal cover of $\hat{\Gamma}$. In particular, $G$ is a quotient of the universal completion of $A$.

To show that there is only one group of $J_3$-type up to isomorphism it remains to determine the universal cover of the geometry $\hat{\Gamma}$ and to study their quotients. Until now, this has been done only with the aid of a computer, see for instance [Ba2].

Theorem 2 [Ba2] The universal cover of $\hat{\Gamma}$ is a triple cover of $\hat{\Gamma}$.

The previous theorem implies that the completion of an amalgam of $J_3$-type is either a group of $J_3$-type or a triple cover of a groups of $J_3$-type and that there is exactly one group of $J_3$-type up to isomorphism.

The proof of Theorem 1 is almost self-contained. We only quote some standard group theory and the result of Bender which states that a group whose involution centralizers are dihedral groups of order 8 is of order either $8 \cdot 3 \cdot 7$ or $8 \cdot 9 \cdot 5$, see [Be]. His proof is very short and elementary. We cite his result to construct the third parabolic subgroup $G_3$. The first parabolic subgroup $G_1$ is constructed using the amalgam method while we choose $G_2$ and $G_3$ as normalizers of an elementary abelian subgroup and a cyclic subgroup of order 16 and 3, respectively.

Contrary to Janko [Ja] we do not use any character theory.

Acknowledgements. I would like to thank A.A. Ivanov as well as G. Stroth for fruitful discussions, C. Parker who pointed out a gap in a previous draft of the paper and last not least U. Meierfrankenfeld for carefully reading the paper.

2 Proof of Theorem 1.

Let $G$ be a group of $J_3$-type. Then all the involutions in $G$ are conjugate and the centralizer of an involution is a split extension of an extraspecial group of order 32 by $\text{Alt}(5)$.

Notation. For $g \in G$ let $C_g = C_G(g)$ and $N_g = N_G(g)$. For $i \in G$ an involution, set $Q_i = O_2(C_i)$ and let $T_i$ be a complement to $Q_i$ in $C_i$.

So $|Q_i| = 32$ and $T_i \cong \text{Alt}(5)$, for every involution $i$ in $G$.

Lemma 2.1 Assume that $i \in G$ is an involution.

(i) $C_{C_i}(Q_i) \leq Q_i$ and $Q_i \cong D_8 * Q_8$.

(ii) $Q_i / \langle i \rangle$ is the even part of the permutation module for $T_i \cong \text{Alt}(5)$.

(iii) $Q_i / \langle i \rangle$ is the $O^-_2(2)$-module for $T_i \cong O^-_2(2)$ and $T_i$ is transitive on the singular subspaces of $Q_i / \langle i \rangle$. 
(iv) $Q_i/(i)$ is a projective module for $T_i$.

(v) Let $s$ be an element of order 3 in $T_i$. Then $C_s \cap Q_i \cong D_8$.

**Proof.** Assume $C_{C_i}(Q_i) \not\subseteq Q_i$. Then, as $C_{C_i}(Q_i)$ is normal in $C_i$, we have $C_{C_i}(Q_i)Q_i = C_i$. Therefore, there is a complement $T$ to $C_{C_i}(Q_i) \cap Q_i = (i)$ in $C_{C_i}(Q_i)$ which is isomorphic to $\text{Alt}(5)$. Let $j$ be an involution in $T$. Then $C_i$ and $C_j$ intersect in a Sylow 2-subgroup $S$. This is not possible, since $i$ is a commutator in $S$, but $j$ is not, which contradicts the fact that all involutions are conjugate in $G$. Hence $C_{C_i}(Q_i) \leq Q_i$.

As $Q_i$ is an extraspecial group, there is a non-degenerate quadratic form on $Q_i$ which is left invariant by $T_i$. The fact that $T_i \cong O_{-}(2)$ implies that $Q_i$ is of $-\text{-type}$, that is $Q_i \cong D_8 \ast Q_8$. This also shows the first part of (iii) and application of the Lemma of Witt yields the second part of (iii).

It follows from (iii) that $T_i$ has two orbits of size 5 and 10 on the set of involutions in $Q_i/(i)$. Hence, $Q_i/(i)$ is not a $GF(4)$-module for $T_i$. It is an easy exercise that there is exactly one module of order $2^4$ which is not a $GF(4)$-module for $T_i$ such that $T_i$ has two orbits of size 5 and 10 on the set of involutions of the module. As the even part of the permutation module for $T_i \cong \text{Alt}(5)$ satisfies these conditions, (ii) holds.

According to Theorem 2.8.7 of [GLS] $Q_i/(i)$ is a projective module for $T_i$ as stated in (iv).

Let $s$ be an element of order 3 in $T_i$. Then $s$ centralizes a subgroup $U$ of order 4 in the even part of the permutation module $Q_i/(i)$. The preimages of two elements of $U$ are of order 4 in $Q_i$, which implies $C_s \cap Q_i \cong D_8$, statement (v). \hfill $\square$

**Lemma 2.2** $G$ acts transitively on the set

$$P := \{(j,W) \mid j \in G \text{ an involution, } j \in W, W \text{ elementary abelian of order } 2^4\}.$$ 

**Proof.** Let $U \subseteq Q_i$, with $i$ an involution, be an elementary abelian subgroup of maximal rank. By Lemma 2.1 $U$ is of order 4, the involution $i$ is in $U$ and $N_{T_i}(U) \cong \text{Alt}(4)$, as $U/Z(Q_i)$ is a singular point in $Q_i/Z(Q_i)$. Set $V = UO_2(N_{T_i}(U))$.

We claim that $V$ is elementary abelian. As every element of order 3 of $N_{T_i}(U)$ acts trivially in $U$ also $O_2(N_{T_i}(U))$ acts trivially on $U$. Thus $V$ is elementary abelian of order 16.

Now let $W$ be some elementary abelian subgroup of order 16 in $C_i$. Then $Q_i \cap W \cong 2^2$ and $(Q_i \cap W)/(i)$ is a singular point.

We claim that all the complements to $Q_i \cap W$ in $W$ are conjugate under $C = C_{C_i}(Q_i \cap W)$. We have $C_{Q_i}(Q_i \cap W) \cong Q_8 \times 2$ and $U \cong (Q_8 \times 2) : \text{Alt}(4)$. We count the elementary abelian subgroups of $Z \setminus Q_i$ of order 4 where $Z = O_2(C)$. Let $f$ be an involution in $Z \cap T_i$. Then we see in the permutation module $Q_i/(i)$ for $T_i$ that $f$ inverts two subgroups $\langle c_1 \rangle, \langle c_2 \rangle$ of order 4 in $C \cap Q_i$ and that $c_1c_2 \in Q_i \cap W$. Hence there are two different elementary abelian subgroups of order 8 in $C \cap Q_i : \langle f \rangle$ and therefore there are precisely four complements to $C \cap Q_i$ in $Z$. It is $|Z \cap T_i| = 4$ and $N_Z(Z \cap T_i)$ is of order $2^4$, which implies, as $|Z| = 2^9$, that $|(Z \cap T_i)^2| = 4$. Thus all
the complements to $C \cap Q_i$ in $Z$ are conjugate. This yields that all the complements to $Q_i \cap W$ in $W$ are conjugate in $C_i$ as asserted.

Therefore we may assume $W = (Q_i \cap W)O_2(N_{T_i}(Q_i \cap W))$. Thus, since $T_i$ acts transitively on the singular points in $Q_i/\langle i \rangle$, the centralizer $C_i$ acts transitively on the elementary abelian subgroups of $C_i$ of maximal rank. As in $G$ there is only one class of involutions, $G$ acts transitively on $P$, as claimed. □

In the following let $V$ be a maximal elementary abelian subgroup of order $2^4$.

**Lemma 2.3** $N_G(V) \cong 2^4 : GL_2(4)$.

**Proof.** By Lemma 2.2 $N_G(V)$ acts transitively on $V^\#$. Let $i$ be an involution in $V$, then $N_{C_i}(V) \cong 2^4 : \text{Alt}(4)$. It can be observed in $C_i$ that $C_i G(V) = V$, so we obtain that $N_{C_i}(V)$ induces on $V$ a group of order 12 which is in fact the stabilizer of an element of $V^\#$ in $N_G(V)$. Thus $N_G(V)$ induces on $V$ a group of order $12 \cdot 15$ which is transitive on $V^\#$. This yields, as $N_G(V)$ is a subgroup of $2^4 : SL_4(2)$, that $N_G(V)/V \cong GL_2(4)$, see [Hu, II (8.27)]. Let $S$ be a Sylow 3-subgroup of $O_{2,3}(N_G(V))$. Then, as $S$ acts fixed point freely on $V$, the Frattini argument implies that the normalizer of $S$ in $N_G(V)$ is a complement to $V$ in $N_G(V)$. Thus $N_G(V)$ splits over $V$, which proves assertion. □

Set

$$G_2 := N_G(V).$$

Let $L_1$ be a subgroup of $N_G(V)$ isomorphic to $2^4 : (3 \times D_{10})$ and let $L_{12}$ be a subgroup of $L_1$ isomorphic to $3 \times D_{10}$. Let $\langle s \rangle = O_3(L_{12})$. Next, we construct $G_3$.

**Lemma 2.4** $N_s \cong (3 \times \text{Alt}(6)) : 2 \cong 3 : PGL_2(9)$.

**Proof.** The element $s$ is centralized by an involution $i$. By 2.1 (vi) we have $C_s \cap C_i \cong D_8 \times 3$. As all the involutions of $G$ are conjugate and as $\langle s \rangle$ is a Sylow-3-subgroup of $C_i$, all the involutions in $C_s$ are conjugate and the centralizer of every involution in $\overline{C}_s = C_s/\langle s \rangle$ is a dihedral group of order 8. By the result of Bender [Be] we have $|\overline{C}_s| = 3 \cdot 8 \cdot 7$ or $8 \cdot 9 \cdot 5$. As 5 divides $|\overline{C}_s|$, the latter holds. Let $R := C_{G_2}(s) \cong 3 \times \text{Alt}(5)$, then $\overline{R}$ is a subgroup of index 6 and it follows that $\overline{C}_s \cong \text{Alt}(6)$. As there is an involution in $C_i$ which inverts $s$, it follows $N_s/\langle s \rangle \cong PGL_2(9)$ or $\text{Sym}(6)$. Assume the latter. Then every Sylow 2-subgroup $U$ of $N_s$ is isomorphic to $D_8 \times 2$. Let $C_i = Q_i : T_i$. As $Q_i/\langle i \rangle$ is the even part of the permutation module for $T_i \cong \text{Alt}(5)$, see 2.1, we see easily that $U \not\cong D_8 \times 2$. Thus $N_s$ is an extension of $\langle s \rangle$ by a group isomorphic to $PGL_2(9)$.

It remains to show that this extension splits. Let $\sigma$ be an element of order 3 in $R' \cong \text{Alt}(5)$. Then $N_R(\langle \sigma \rangle) = \langle s \rangle \times A$ with $A \leq R'$ and $A \cong \text{Sym}(3)$ and there is an involution which inverts $\sigma$ and centralizes $s$. As there is no involution in $N_s$ which inverts $s$ and centralizes an element of order 3, the subgroups $\langle s \rangle$ and $\langle \sigma \rangle$ are not conjugate in $G$. As $s \cdot \sigma$ centralizes an involution in $O_2(G_2)$, this element is conjugate to $s$. If $N_s$ were a non-split extension, then a Sylow 3-subgroup of $N_s$ would be an
This yields the assertion. \( \square \)

Recall the definition of \( L_1 \) and \( L_{12} \) just before Lemma 2.4. Set
\[
G_3 := N_s.
\]

The next result follows from Lemma 2.4.

**Lemma 2.5** \( N_G(L_{12}) \cong \Sym(3) \times D_{10} \).

Set
\[
L_2 := N_G(L_{12}).
\]

Notice, that \( L_{12} = L_1 \cap L_2 \).

Let \( f \) be an involution and \( w \) an element of order 5 in \( L_{12} \). Then \( f \) inverts \( \langle w \rangle \) and centralizes \( \langle s \rangle \) and \( C_{L_1}(f) \cong 2^2 : 3 \times 2 \cong \Alt(4) \times 2 \) and \( C_{L_2}(f) \cong \Sym(3) \times 2 \). Set
\[
L_3 := \langle C_{L_1}(f), C_{L_2}(f) \rangle.
\]

**Lemma 2.6** \( L_3 \cong \Alt(5) \times 2 \).

**Proof.** We have \( N_{C_f}(\langle s \rangle) \cong 3 : D_{16} \) and all the subgroups isomorphic to \( C_{L_2}(f) \cong \Sym(3) \times 2 \) are conjugate in \( N_{C_f}(\langle s \rangle) \). Therefore, we may choose a complement \( T_f \cong \Alt(5) \) to \( Q_f \) in \( C_f \) such that \( T_f \cap C_{L_2}(f) \cong \Sym(3) \).

It remains to show that \( C_{L_1}(f) \) is contained in a conjugate of \( \langle f \rangle \times T_f \) under the action of the normalizer of \( C_{L_2}(f) \) in \( C_f \).

Assume \( C_{L_1}(f) \cap Q_f > \langle f \rangle \). Then \( C_{L_1}(f) \cap Q_f \) is elementary abelian of order 8, which contradicts the fact that \( Q_f \cong D_8 \ast Q_8 \) is of minus-type, see Lemma 2.1. Therefore, we have \( C_{L_1}(f) \cap Q_f = \langle f \rangle \).

We claim that all the subgroups isomorphic to \( 2 \times \Alt(4) \) which intersect \( Q_f \) precisely in \( \langle f \rangle \) and which contain \( s \) are conjugate in \( C_f \cap C_s \). Let \( X \) be such a subgroup. Let \( U \) be the projection of \( XQ_f / Q_f \) onto \( T_f \) and let \( u \) be an involution in \( U \). Then \( \tilde{C} = C_{Q_f / Q_f}(u) = 2^2 \) with preimage \( K \cong 4 \times 2 \) and \( u \) inverts every element of order 4 of \( K \). Let \( C_K(s) = \langle f, b \rangle \). Then \( b \) is an involution and notice, if \( \langle bu, (qu)^s \rangle \cong 2^2 \) for some \( q \in K \), then \( \langle bu, (bu)^s \rangle \not\cong 2^2 \). This shows that there are precisely two subgroups \( \langle qu, (qu)^s \rangle \) with \( q \in K \) which are elementary abelian of order 4. We have \( C_{Q_f}(s) \cong D_8 \) and \( C_{Q_f}(\langle s, u \rangle) \cong 2^2 \) which implies that the two subgroups are conjugate under \( C_{Q_f}(s) \). This proves the claim.

Hence, \( C_{L_1}(f) \) is conjugate to a subgroup of \( \langle f \rangle \times T_f \) under the action of the normalizer of \( C_{L_2}(f) \) in \( C_f \). So, we may assume that \( T_f \) is chosen such that \( C_{L_1}(f) \leq \langle f \rangle \times T_f \). This yields the assertion. \( \square \)
Set 
\[ L = \langle L_1, L_2 \rangle. \]

Then \( L_3 \leq L \). Recall that

\[ L_1 \cong 2^4 : (3 \times D_{10}), L_2 \cong \text{Sym}(3) \times D_{10} \text{ and } L_3 \cong 2 \times \text{Alt}(5). \]

To prove that \( L \cong L_2(16) : 2 \), we show the following.

**Lemma 2.7** Let \( H \) be a group and \( H_1, H_2, H_3 \) subgroups of \( H \) such that

(i) \( H = \langle H_1, H_2 \rangle \);

(ii) \( H_1 \cong 2^4 : (3 \times D_{10}), C_{H_1}(O_2(H_1)) = O_2(H_1); H_2 \cong \text{Sym}(3) \times D_{10}; H_3 \cong \text{Alt}(5) \times 2 \); and

(iii) \( H_1 \cap H_2 \cong 3 \times D_{10}; H_1 \cap H_3 \cong \text{Alt}(4) \times 2; H_2 \cap H_3 \cong \text{Sym}(3) \times 2. \)

Then \( H \) is a triply transitive permutation group of degree 17; in this action \( H_1 \) is the stabilizer of a point and \( |H| = 2 \cdot 15 \cdot 16 \cdot 17. \)

**Proof.** Let \( \langle s \rangle = O_3(H_2), \langle w \rangle = O_5(H_2) \) and let \( b, i \) be involutions in \( H_2 \cap H_3 \) with

\[ s^b = s^{-1}, w^b = w \text{ and } s^i = s, w^i = w^{-1}. \]

Let \( \Theta \) be a graph whose vertices are the cosets of \( H_1 \) in \( H \) and whose edges are the sets \( \{H_1 x, H_1 b x\} \) with \( x \in H \).

As by (i) \( H = \langle H_1, H_2 \rangle = \langle H_1, b \rangle \) this graph is connected.

We claim that \( \Theta \) is a graph of valency 16. Clearly, \( b \) normalizes \( H_1 \cap H_2 \). If \( b \) would also normalize \( H_1 \), then \( H = H_1 \langle b \rangle \) in contradiction to \( 2 \times \text{Alt}(5) \cong H_3 \leq H \).

Since \( C_{H_1}(O_2(H_1)) = O_2(H_1) \), the intersection \( H_1 \cap H_2 \) is maximal in \( H_1 \) which implies \( H_1 \cap H_1^b = H_1 \cap H_2 \cong 3 \times D_{10} \) is the stabilizer of the two neighbours \( H_1 \) and \( H_1 b \) in \( H \). Thus \( \Theta \) is of valency \( |H_1 : H_1 \cap H_1^b| = 16 \), as claimed.

Therefore, it follows that \( O_2(H_1) \) acts regularly on \( \Theta(H_1) \). Moreover, as \( H_1 \cap H_2 \) is transitive on \( O_2(H_1)^\# \), it follows that \( H_1 \) acts doubly transitively on its neighbours \( \Theta(H_1) \).

Next, we show that \( \Theta \) is a complete graph. Notice, that the facts \( H_3 \cong 2 \times \text{Alt}(5), b \in (H_2 \cap H_3) \setminus H_1 \) and \( H_1 \cap H_3 \cong 2 \times \text{Alt}(4) \) yield that there is an \( h \in H_1 \cap H_3 \) such that \( (bh)^3 \in \langle i \rangle \). Hence

\[ H_1 b h b = H_1 h b h = H_1 b h \]

is a common neighbour of \( H_1 \) and \( H_1 b \). This shows that there is a triangle in \( \Theta \). Now, the fact that \( H_1 \) acts doubly transitively on \( \Theta(H_1) \) implies that every vertex in \( \Theta(H_1) \) is a neighbour of \( H_1 b \), so \( \Theta \) is a complete graph.

Thus \( \Theta \) consists of 17 vertices and \( |H : H_1| = 17 \) which implies \( |H| = |H_1| \cdot 17 = 2 \cdot 15 \cdot 16 \cdot 17 \) and \( H \) acts triply transitively on the cosets of \( H_1 \) in \( H \). \( \square \)
Corollary 2.8 Let $L$ be a faithful completion of an amalgam

$$B = \{H_1, H_2, H_3, H_{12}, H_{13}, H_{23}\},$$

where the groups $H_1, H_2$ and $H_3, H_{ij} = H_i \cap H_j$ ($1 \leq i < j \leq 3$) are as described in Lemma 2.7. Then $|L| = 2 \cdot 15 \cdot 16 \cdot 17$. In particular, every faithful completion of such an amalgam is already universal.

Notice that $H = L_2(16) : 2$ possesses such an amalgam $B$: Let $H_1$ be a point stabilizer in $H$ in its action of degree 17. Then $H_1 \cong 2^4 : (3 \times D_{10})$. Let $H_2$ be the setwise stabilizer of two points such that $H_1 \cap H_2 \cong 3 \times D_{10}$. Finally, let $f$ be an involution in $H_1 \cap H_2$ and set $H_3 = C_H(f)$. Then $H_1, H_2, H_3, H_{ij} = H_i \cap H_j$ ($1 \leq i < j \leq 3$) form an amalgam as described in Lemma 2.7. By Lemma 2.8 a completion of an amalgam of type $B$ is a triply transitive permutation group of degree 17.

Lemma 2.9 The embeddings of $H_1 \cong 2^4 : (3 \times D_{10})$ and of $H_2 \cong \text{Sym}(3) \times D_{10}$ in $\text{Sym}(17)$ as the stabilizer of a point and of a 2-set containing that point, respectively, are unique up to conjugation in $\text{Sym}(17)$.

Proof. Let $H_1$ be the stabilizer of 1. Then $O_2(H_1)$ acts regularly on $\{2, \ldots, 17\} =: \Omega$. Let $K = \text{Sym}(\Omega)$. Then $N_K(O_2(H_1)) \cong 2^4 : L_4(2)$. We may assume that $O_3(H_1 \cap H_2)$ fixes 2 $\in \Omega$. As $O_3(H_1 \cap H_2)$ acts fixed point freely on $O_2(H_1)^\#$, it follows that $H_1 \cap H_2 = C_{H_1}(O_3(H_1 \cap H_2))$ is a subgroup of the stabilizer of 2 in $H_1$ and therefore $H_1 \cap H_2$ is the stabilizer of 2 in $H_1$. Moreover, the action of $H_1 \cap H_2$ on $\Omega$ is uniquely determined up to conjugation in $N_K(O_2(H_1))$. Let $a$ be an involution in $H_2 \setminus H_1 \cap H_2$ which centralizes $O_3(H_1 \cap H_2)$. Then $a$ interchanges 1 and 2 and it fixes all 5 3-cycles of $O_3(H_1 \cap H_2)$ on the set $\Omega \setminus \{2\}$. We may assume the action of $a$ on one of the 3-cycles which then determines uniquely the action of $a$ on $\Omega$. □

The previous lemma yields that the amalgam $B$ is uniquely determined. This shows the following.

Corollary 2.10 The universal completion of $B$ is isomorphic to $L_2(16) : 2$. In particular, $L$ is isomorphic to $L_2(16) : 2$.

Set $G_1 := L$.

Lemma 2.11 $A = \{G_1, G_2, G_3, G_{12}, G_{13}, G_{23}, B\}$ is an amalgam of type $J_3$.

Proof. By construction $A$ is of type $J_3$. □

Lemma 2.11 completes the proof of Theorem 1.
References


