Factorizations of groups and permutation groups

Barbara Baumeister
Martin-Luther-Universität Halle

July 2004

Contents

Chapter 1

Introduction.

In this thesis we investigate groups and in particular finite groups. Thus, in the thesis all the groups are supposed to be finite. A group is said to be *factorizable* if $G = AB$ for some proper subgroups A and B of G. The expression $G = AB$ is called a factorization of G. A permutation group (G, Ω) is a group G which acts faithfully on a set Ω.

There is a surprisingly large number of mathematical questions related to factorizations of groups. First of all they play a constitutive role in group theory. We will discuss this further in the brief history at the end of this chapter. Of course, they also appear in geometric context. A very special, but beautiful example is the following: O. H. Kegel and H. Lüneburg could show that the condition of Desargues and the little Reidemeister condition are equivalent in finite projective planes. Their proof that in finite projective planes the little Reidemeister condition implies the condition of Desargues is based on the following theorem: A group G which is the product $G = AB$ of two subgroups A and B that are both isomorphic to A_5 is either isomorphic to the direct product $A \times B$ or to the alternating group A_6 of degree 6, see [55].

Factorizations are also of importance in the theory of permutation groups. A group G is the product $G = AB$ of two of its subgroups A, B if and only if A acts transitively on the set Ω of cosets of B in G. If moreover, A and B intersect trivially, then the factorization $G = AB$ is called *exact* and A acts *regularly* on Ω. If B is maximal, then G is a primitive permutation group on Ω.

Note that the classification of the primitive permutation groups which admit a regular subgroup will imply a complete list of the primitive graphs which are Cayley graphs (for an introduction to Cayley graphs see for instance [33])! There are far more areas where factorizations of groups do emerge. For instance, the knowledge of the cyclic regular subgroups of the primitive permutation groups is necessary in the classification of polynomials by their monodromy groups, see for instance [39]. If the group G has an exact factorization $G = AB$, then one can construct a semisimple Hopf algebra from these data, see for instance [86] or [29].

The four strongly related problems studied in this thesis can be read as results on factorizations of groups or as results on permutation groups. The first problem is to classify the transitive subgroups of primitive permutation groups (G, Ω) which are conjugate in $\mathrm{Aut}(G)$ to a point stabilizer i.e. to classify those groups G which possess a core-free maximal subgroup A such that $G = AA^\alpha$ for some $\alpha \in \mathrm{Aut}(G)$. The second problem is to find all the irreducible subgroups of $GL_d(q)$ which possess a subgroup of prime power index p^a such that either p is a prime dividing q and $a \geq d$ or p does not divide q and $q^d \leq p^a$. The second problem arose out of a question whose answer we needed to solve the first problem. The third part is a characterization of the finite soluble groups, as well as a proof of an old conjecture of O.H. Kegel [54,

p. 210]. The fourth problem is to provide a list of all the primitive permutation groups (G, Ω) which possess a regular subgroup.

Let us discuss our results in some detail. Recall that all the groups are supposed to be finite. In Chapter 5 we classify all the groups G which posses a core-free maximal subgroup A such that $G = AA^{\alpha}$ for some $\alpha \in \mathrm{Aut}(G)$. We call such a factorization of a group a *primitive factorization.* Let G_i be a group and π_i a permutation representation of G_i on the set Ω_i, for $i = 1, 2$. Then π_1 and π_2 are said to be *isomorphic*, if there exists an isomorphism $\phi : G_1 \to G_2$ and a bijection $\theta : \Omega_1 \to \Omega_2$ such that

$$g^{\pi_1}\theta = \theta g^{\phi\pi_2} \text{ for all } g \in G_1.$$

Two permutation groups are *permutation equivalent* if their permutation representations are isomorphic. We prove the following.

Theorem 1 *Let G be a primitive permutation group on a set Ω of size m. Suppose that $G = G_\omega G_\omega^\alpha$ for G_ω a point stabilizer and for $\alpha \in \mathrm{Aut}(G)$. Then one of the following holds. Conversely each of the listed groups gives rise to such an example.*

(a) *G is affine:*
$G = E_{2^3} L_3(2) \wr X$, *where X is a transitive subgroup of S_n and $m = 2^{3n}$.*

(b) *G is almost simple:*

(b.a) $E(G) \cong P\Omega_8^+(q)$ *and G is an extension of $E(G)$ by field automorphisms.* $E(G)_\omega \cong P\Omega_7(q)$, $E(G)_\omega \cap E(G)_\omega^\alpha \cong G_2(q)$ *and* $m = q^4(q^4 - 1)$*;*

(b.b) $E(G) \cong Sp_4(q)$, *q even, and G is an extension of $E(G)$ by field automorphisms.* $E(G)_\omega \cong L_2(q^2)2$, $E(G)_\omega \cap E(G)_\omega^\alpha \cong \mathrm{Frob}(q^2 + 1 : 4)$ *and* $m = q^2(q^2 - 1)/2$*;*

(b.c) $G \cong Sp_4(2) \cong S_6$, $G_\omega \cong S_5$, $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5 : 4)$ *and* $m = 6$*;*

(b.d) $G \cong Sp_4(2)' \cong A_6$, $G_\omega \cong L_2(4)$, $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5 : 2)$ *and* $m = 6$*;*

(b.e) $G \cong M_{12}$, $G_\omega \cong M_{11}$, $G_\omega \cap G_\omega^\alpha \cong L_2(11)$ *and* $m = 12$.

(c) *G is permutation equivalent to a blow–up K of index n of an almost simple group A:*
$K \leq A \wr S_n$, *where A is a permutation group listed in (b) and* $m = |A : A_\omega|^n$*. Moreover* $E(G)_\omega = E(A)_\omega^n$ *and there exists* $a \in \mathrm{Aut}(A)$ *such that* $A = A_\omega A_\omega^a$ *and such that* $E(G)_\omega \cap E(G)_\omega^\alpha \cong (E(A)_\omega \cap E(A)_\omega^a)^n$.

In the following corollary we formulate the statement of Theorem 1 in terms of permutation representations.

Proposition 1 *Let G be a group which has two isomorphic primitive permutation representations π_1 and π_2 on the sets Ω_1 and Ω_2, respectively. Suppose that a point stabilizer of π_1 acts transitively on Ω_2. Then G and Ω_1, Ω_2 are as listed in Theorem 1 and each of the listed groups gives rise to such an example.*

Notice also that we obtain from Theorem 1 the following geometric application.

Proposition 2 *Let (X, Γ) be a flag–transitive $m \times m$ grid such that the stabilizer of a parallel class of lines, G, acts faithfully and primitively on each of the two parallel classes of lines. Then G and m are as in Theorem 1.*

In order to prove Theorem 1 we have to solve interesting group theoretic questions. One question we generalize to the following: Find all the groups G which admit a subgroup K of index p^a, p a prime, under the assumption that G has an irreducible and faithful $GF(q)$-module in characteristic p whose dimension over $GF(q)$ is at most a. Our result is as follows.

Theorem 2 *Let G be a group which contains a subgroup K of index p^a for a prime p and let V be an irreducible and faithful G-module over $GF(q)$ in characteristic p whose dimension over $GF(q)$ is d. Then one of the following holds.*

(a) $a < d$;

(b) $a \geq d$ *and all the following items hold.*

(b.a) $p = 2$ *and* $d = 3r$, *for some* $r \in \mathbb{N}$;

(b.b) $a \leq 4r - 1$;

(b.c) $E(G) = T_1 \times \ldots \times T_r$, *where* $T_i \cong L_3(2)$, *for* $i = 1, \ldots, r$, *and* $E(G)$ *is a minimal normal subgroup of* G;

(b.e) $E(G) \not\leq K$ *and, for* $i = 1, \ldots, r$, *either* $T_i \leq K$ *or* $K \cap T_i \cong \mathrm{Frob}(7:3)$;

(b.f) $V = [V, T_1] \oplus \ldots \oplus [V, T_r]$, *where* $[V, T_i]$ *is a* 3*-dimensional* $GF(q)$*-module for* T_i;

(b.g) $G/F(G) \cong T_1 \wr X$ *with* X *a transitive subgroup of* S_r *and* $F(G)$ *induces scalar multiplication on* $[V, T_i]$, *for* $i = 1, \ldots, r$.

Remarks. (1) Notice that the statement of Theorem 2 remains true if V is a projective irreducible and faithful G-module in characteristic p.
(2) In fact it can happen that $a \geq d$ and that at the same time $T_i \leq K$, for some $i \in \{1, \ldots, r\}$. Consider the special situation of Theorem 2(b). Let $r = 2$ and let $X \leq S_8$ be isomorphic to $L_2(7)$ and let $G = T_1 \wr X$. Now let K_2 be a subgroup of T_2 isomorphic to the Frobenius group $\mathrm{Frob}(7:3)$ and let S_1 be the stabilizer of T_1 in S. Then, as well, $S_1 \cong \mathrm{Frob}(7:3)$ and $\langle K_2^{S_1} \rangle \cong \mathrm{Frob}(7:3)^7$. Set $K = T_1 \times \langle K_2, S_1 \rangle$. Then the index of K in G is 2^{24} and the dimension d of the module V is 24, so indeed $T_1 \leq K$ and $a \geq d$.

An application of Theorem 2 to permutation groups is as follows. Let (G, Ω) be a primitive permutation group of affine type, that is $G = O_p(G)G_\omega$ where G_ω is the stabilisator of $\omega \in \Omega$ and where $O_p(G)$ is an irreducible and faithful $GF(p)G_\omega$-module. Theorem 2 implies the following.

Corollary 1 *Let (G, Ω) be a primitive permutation group of affine type. Suppose that M is a maximal subgroup of G which acts transitively on Ω and which is conjugate to a point stabilizer G_w in* $\mathrm{Aut}(G)$. *Then the following hold.*

(a) $O_p(G)$ *is elementary abelian of order* 2^{3r} *for some positive integer* r;

(b) $G \cong (2^3 : L_3(2)) \wr X$ *for some transitive subgroup* X *of* S_r *and* $M \cong L_3(2) \wr X$.

As every transitive maximal subgroup of G which does not contain $O_p(G)$ is conjugate to a point stabilizer G_w in $\mathrm{Aut}(G)$, see [84, Chapter 2, Corollary to 8.7], Corollary 1 implies the following statement:

Corollary 2 *Let (G, Ω) be a primitive permutation group of affine type. Suppose that M is a maximal subgroup of G which acts transitively on Ω. Then either $O_p(G) \leq M$ or G and M are as in Corollary 1.*

This corollary generalizes a result of C. E. Praeger, see [74, (Proposition 5.2)], in which the subgroup M is assumed to be primitive.

Corollary 2 also follows from a result of Aschbacher and Scott [4, Theorem 3] together with Proposition 4.3.1 of this thesis. For the convenience of the reader we give two proofs of Corollary 2 in Chapter 4: one based on Theorem 2 and the other one on [4, Theorem 3] and Proposition 4.3.1.

Finally, notice that Corollary 2 implies a classification of the maximal factorizations of the primitive permutation groups of affine type:

Corollary 3 *Let (G, Ω) be a primitive permutation group of affine type. Suppose that $G = A_1A_2$ for two maximal subgroups of G. Set $Q = O_p(G)$. Then one of the following holds.*

(a) $Q \leq A_1, A_2$ *and* $G/Q = (A_1/Q)(A_2/Q)$;

(b) $Q \leq A_i$, $G = Q : A_j$ *and* A_i/Q *is isomorphic to a maximal subgroup of* A_j *for* $\{i, j\} = \{1, 2\}$;

(c) G *is as described in Corollary 1 and* $A_1 \cong A_2 \cong L_3(2) \wr X$.

Consider again the assumptions of Theorem 2. It is natural to ask what happens if we assume that q is the power of a prime t which is different from p still assuming that $a \geq d$. In fact there are many examples satisfying these assumptions:

For instance, if p^a divides $q - 1$, then there is an example with $a \geq d$: Let d be a natural number which is at most a, let $G = GL_d(q)$ and let K be a subgroup of G of index p^a (let U be the subgroup of $GF(q)^{\#}$ of order $(q - 1)/p^a$ and let K be the set of elements of G whose determinant is in U). For instance, $q = 3^5$, $p = 11$ and $a = d = 2$ satisfy these conditions.

Since the latter class of examples are not that interesting, it is better to study those groups G which have a subgroup of prime power index p^a and which act irreducibly on a module of size $q^d \leq p^a$ where q is a power of the prime t. It is also reasonable to assume that p does not divide the order of the Fitting subgroup $F(G)$ of G. If p divides $|F(G)|$, then again many examples can be constructed. For instance, let p be a prime, q a power of a prime $t \neq p$, and d a natural number such that $|P| > q^d$ for $P \in Syl_p(GL_d(q))$. This happens for example, if $p = 2$, $q = 3$ and $d = 4$. Therefore, we assume that p does not divide the order of $F(G)$. There is a family of examples which satisfies all our assumptions:

A family of Examples. Let m be a natural number such that $p = 2^m - 1$ is a Mersenne prime, let P be a Sylow p-subgroup in the symmetric group S_p, and set $G = SL_m(2) \wr P$. Let $V = \oplus_{i=1}^{p} V_i$ be the direct sum of natural $SL_m(2)$-modules, so V is of dimension $d = mp$, and let G act naturally on V. Set $t = 2$. Then G contains a subgroup K of index p^{p+1} and $p^a = p^{p+1} \geq 2^{mp} = q^d$ (see Lemma 4.2.1).

We show the following:

Theorem 3 *Let G be a group which contains a subgroup K of index p^a for a prime p that does not divide $|F(G)|$ and let V be an irreducible and faithful G-module over the field $GF(q)$ of dimension d such that $p^a \geq q^d$. Then there are natural numbers $r, m \in \mathbb{N}$ such that the following holds.*

(a) $q = 2$ *and* $d = rm$;

(b) $E(G) = T_1 \times \ldots \times T_r$, *where* $T_i \cong SL_m(2)$ *with* $2^m - 1$ *a Mersenne prime, for* $i = 1, \ldots, r$, *and* $E(G)$ *is a minimal normal subgroup of* G;

(c) $V = [V, T_1] \oplus \ldots \oplus [V, T_r]$, *where* $[V, T_i]$ *is a natural* $SL_m(2)$*-module for* T_i*, for* $i = 1, \ldots, r$*;*

(d) $G \cong T_1 \wr X$ *with* X *a transitive subgroup of* S_r*;*

(e) Either

  (e.a) $p = q$ *or*

  (e.b) $p = 2^m - 1$*, a Mersenne prime.*

Chapter 6 is devoted to a characterization of the soluble groups. The questions we address there are again related to factorizations of groups and to permutation groups. Recall that for a group G and a subgroup M of G we say that a subgroup A of G is a *supplement* to M in G, if $G = MA$. We prove the following theorem.

Theorem 4 *Let* G *be a group which satisfies the condition that every maximal subgroup of* G *admits an abelian supplement. Then* G *is soluble.*

This result proves a conjecture of Kegel [54, p. 210]. Theorem 4 may also be formulated in terms of permutation groups as follows.

Corollary 4 *Let* G *be a group such that for every primitive action* (G, Ω) *of* G*, there exists an abelian subgroup of* G *which acts transitively on* Ω*. Then* G *is soluble.*

Upon replacing "action" by "permutation action", the statement of the theorem is no longer true: If there is to be a transitive abelian subgroup only for every *faithful* primitive action (G, Ω) of G, then G is not necessarily soluble. For example let G be an extension of an elementary abelian group N by a non soluble group. Let (G, Ω) be a faithful and primitive action of G. Then the stabilizer G_ω of an element $\omega \in \Omega$ is a maximal subgroup of G and N is not contained in G_ω. Therefore, $G = G_\omega N$ which implies that N acts transitively on Ω. Thus for every faithful primitive action of G there is an abelian transitive subgroup in G, but G is not soluble.

Unfortunately, Theorem 4 does not characterize the soluble groups as the following example shows.

Example. Let $G = SL_2(3) \cong Q_8 : \mathbf{Z}_3$. Let $Z = Z(O_2(G))$ and let T be a subgroup of order 3 of G. Then $M = ZT$ is maximal in G, but does not have an abelian supplement.

The soluble groups can be characterized among the finite groups by a condition which is a little more general than this one in Theorem 4.

Theorem 5 *A group* G *is soluble if and only if every maximal subgroup* M *of* G *admits a supplement whose commutator subgroup is contained in* M*.*

Which groups G are such that every maximal subgroup of G admits a nilpotent or a soluble supplement? The answers are given in the following two theorems.

Theorem 6 *Let* G *be a group such that every maximal subgroup* M *of* G *admits a nilpotent supplement. Then* $G/O_\infty(G)$ *is isomorphic to* 1 *or* $L_2(7)$*.*

Theorem 7 *Let* G *be a group such that every maximal subgroup* M *of* G *admits a soluble supplement. Then the composition factors of* G *are either* p*-groups,* p *a prime, or isomorphic to* $L_2(q)$*, with* $q \in \{5, 7, 11\}$*. There is no chief factor which is a direct product* $T_1 \times \cdots \times T_n$ *with* $n > 1$ *and* $T_i \cong L_2(7)$ *or* $T_i \cong L_2(11)$*.*

In order to prove these theorems we also classify the almost simple groups G satisfying the property that every maximal subgroup admits a supplement which does not contain $soc(G)$, see Proposition 6.2.4.

Assume that T is a simple non abelian group which satisfies the assumption of Theorem 5, i.e. every maximal subgroup M of T admits a supplement C with $C' \leq M$. Assume further that $C \cap M \neq 1$. Then, as $C \cap M$ is normal in C, it follows that $(C \cap M)^G = (C \cap M)^M \leq M$. Therefore T has a non-trivial normal subgroup, which is impossible. This shows that C is an abelian complement to M in T and in particular, C acts regularly on the set of cosets of M in G.

In Chapter 7 we address the problem of classifying the primitive permutation groups which admit a regular subgroup. If this classification will be finished it will have many applications.

Recall the subdivision of the primitive permutation groups into different types, see 2.3. Every primitive permutation group (G, Ω) of affine type, of diagonal type or of twisted product action type possesses a regular subgroup, see 7.2. Thus the goal is to determine those (G, Ω) possessing a regular subgroup with G almost simple or (G, Ω) of product action type. In this thesis we focus on G being almost simple. We give a complete list of those (G, Ω) with $soc(G)$ an alternating or a sporadic or an exceptional group of Lie type. Further we present all the examples which are known to us for G a classical group. Our particular interest is in the 8-dimensional orthogonal groups of Witt index 4. This family of classical groups seems to be the most difficult one to analyse, as the group $P\Omega_8^+(q)$ has lots of maximal subgroups, see [57], and admits lots of factorizations, see [67, Theorem A]. It also seems that most examples live in small dimension – and in small characteristic, namely in characteristic 2 and 4. We determine all the pairs (G, Ω) such that G has a regular subgroup for $soc(G) \cong \mathrm{P}\Omega_8^+(q)$. In the end of Section 7.3 of Chapter 7 we outline how to finish the classification of the case T a classical group. Our results are as follows.

The alternating and symmetric groups.

See the definition of $M(p^e+1)$ in Chapter 2.

Theorem 8 *Let (G,Ω) be a primitive permutation group with $soc(G)=T\cong \mathrm{A}_n$, $n\geq 5$ and suppose that there is a subgroup X of G which acts regularly on Ω. Then one of the following holds, where ω is an element in Ω and $\Delta=\{1,\ldots,n\}$.*

(a) $G=\mathrm{A}_n$.

- *(a.a)* $\Omega=\Delta$ *and* $G_\omega=\mathrm{A}_{n-1}$.
- *(a.b)* G_ω *is sharply k-transitive on Δ and X is the pointwise stabilizer of a k-subset of Δ, for some $k\in\{3,4,5\}$, and one of the following holds.*
  - *(a.b.a)* $n=p^2+1$, *with p a prime congruent to* 3 *modulo* 4, $k=3$ *and* $G_\omega\cong M(p^2+1)$;
  - *(a.b.b)* $n=11$, $k=4$ *and* $G_\omega\cong \mathrm{M}_{11}$;
  - *(a.b.c)* $n=12$, $k=5$ *and* $G_\omega\cong \mathrm{M}_{12}$.
- *(a.c)* G_ω *is k-homogeneous, but not k-transitive on Δ, for some $k\in\{2,3,4\}$, and one of the following holds. In the last two items p is a prime congruent to* 3 *modulo* 4, *but* $p\neq 3,7,11,23$.
  - *(a.c.a)* $n=9, k=4$, $G_\omega\cong \mathrm{P\Gamma L}_2(8)$ *and* $X\cong \mathrm{S}_5$;
  - *(a.c.b)* $n=33, k=4$, $G_\omega\cong \mathrm{P\Gamma L}_2(32)$ *and* $X\cong(\mathrm{A}_{29}\times\mathrm{A}_3):2$;
  - *(a.c.c)* $n=p+1$, $k=3$, $G_\omega\cong \mathrm{L}_2(p)$ *and* $X\cong \mathrm{S}_{p-2}$;
  - *(a.c.d)* $n=p$, $k=2$, $G_\omega\cong \mathrm{Frob}(p:(p-1)/2)$ *and* $X\cong \mathrm{S}_{p-2}$.
- *(a.d)* Ω *is the set of k-subsets of Δ, for some $k\in\{2,3\}$, and one of the following holds. In the last item q a is prime power congruent to* 3 *modulo* 4.
  - *(a.d.a)* $n=8$, $k=3$ *and* $X\cong \mathrm{AGL}_1(8)$;
  - *(a.d.b)* $n=32$, $k=3$ *and* $X\cong \mathrm{A\Gamma L}_1(32)$;
  - *(a.d.c)* $n=q$, $k=2$ *and* $X\cong \mathrm{AGL}_1(q)/\langle -1\rangle\cong \mathrm{Frob}(q:(q-1)/2)$.
- *(a.e)* $n=8$, $G_\omega\cong 2^3:L_3(2)$, $|\Omega|=15$ *and* $X\cong \mathbb{Z}_{15}$.

(b) $G=\mathrm{S}_n$.

- *(b.a)* $G_\omega\cap\mathrm{A}_n$ *is a subgroup of index* 2 *in* G_ω *and is as in (a.a), (a.b.a) or as a group listed in (a.d).*
- *(b.b)* G_ω *is sharply k-transitive on Δ, for some $k\in\{2,3\}$, $X\cong \mathrm{S}_{n-k}$ and one of the following holds. In both cases p is a prime and $p\geq 5$.*
  - *(b.b.a)* $n=p$, $k=2$ *and* $G_\omega\cong \mathrm{Frob}(p:(p-1))$;
  - *(b.b.b)* $n=p+1$, $k=3$ *and* $G_\omega\cong \mathrm{PGL}_2(p)$.
- *(b.c)* $n=6$, $G_\omega\cong \mathrm{PGL}_2(5)$ *is transitive on Δ and X is a subgroup of G of order* 6;

Conversely, each of the listed items satisfies the assumptions of the theorem.

Remark. (a) If (G,Ω) is as in (b.b.a), then $(G_\omega\cap\mathrm{A}_n,\Omega)$ is as in item (a.c.d) of the theorem.

The sporadic groups.

Theorem 9 *Let (G,Ω) be a primitive permutation group with $\mathrm{soc}(G)=T$ a sporadic simple group. Suppose that there is a subgroup X of G which acts regularly on Ω. Let A be the stabilizer in G of an element in Ω. Then (G,A,X) are as follows.*

(a) $T=G\cong \mathrm{M}_{11}$ *and one of the following holds.*

- *(a.a)* $A\cong \mathrm{M}_{10}$ *and* Ω *is the set of points of the Steiner system* $\mathcal{S}=S(4,5,11)$ *related to* T*; and* $X\le T$ *with* $X\cong \mathbb{Z}_{11}$*;*
- *(a.b)* $A\cong \mathrm{M}_9.2\cong 3^2:SD_{16}$ *and* Ω *is the set of duads of the Steiner system* $\mathcal{S}$*; and* $X\le T$ *with* $X\cong \mathrm{Frob}(11:5)$.

(b) $T\cong \mathrm{M}_{12}$ *and one of the following holds.*

- *(b.a)* $A\cap T\cong \mathrm{M}_{11}$ *and* Ω *is the set of points* $\mathcal{P}$ *of the Steiner system* $S(5,6,12)$ *related to* T*; and* $X\le T$ *with* $X\cong 2^2\times 3, \mathrm{A}_4$ *or* $2\times \mathrm{S}_3$*;*
- *(b.b)* $A\cap T\cong \mathrm{L}_2(11)$*; and* $X\le T$ *with* $X\cong 3^2:SD_{16}$.

(c) $T\cong \mathrm{M}_{22}$*,* $G\cong \mathrm{Aut}(\mathrm{M}_{22})$*, and* Ω *is the set of points of the Steiner system* $S(3,6,22)$ *related to* T*; and* $X\not\le T$ *with* $X\cong \mathrm{Frob}(11:2)$.

(d) $T=G\cong \mathrm{M}_{23}$ *and one of the following holds.*

- *(d.a)* $A\cong \mathrm{M}_{22}$ *and* Ω *is the set of points of the Steiner system* $\mathcal{S}=S(4,7,23)$ *related to* T*; and* $X\le T$ *with* $X\cong \mathbb{Z}_{23}$*;*
- *(d.b)* $A\cong \mathrm{M}_{21}:2$ *and* Ω *is the set of duads of* S*; and* $X\le T$ *with* $X\cong \mathrm{Frob}(23:11)$*;*
- *(d.c)* $A\cong \mathrm{Frob}(23:11)$*; and* $X\le T$ *with* $X\cong \mathrm{M}_{21}:2$ *or* $2^4:\mathrm{A}_7$*;*
- *(d.d)* $A\cong 2^4:\mathrm{A}_7$ *and* Ω *is the set of blocks of* $\mathcal{S}$*; and* $X\le T$ *with* $X\cong \mathrm{Frob}(23:11)$.

(e) $T=G\cong \mathrm{M}_{24}$ *and one of the following holds.*

- *(e.a)* $A\cong \mathrm{M}_{23}$ *and* Ω *is the set of points* P *of the Steiner system* $S(5,8,24)$ *related to* T*; and* $X\le T$ *with* $X\cong D_8\times 3, (2^2\times 3):2, \mathrm{S}_4$ *or* $2^2\times \mathrm{S}_3$*;*
- *(e.b)* $A\cong \mathrm{L}_2(23)$*; and* $X\le T$ *with* $X\cong 2^4:\mathrm{A}_7$ *or* $\mathrm{M}_{21}:2$.

(f) $T\cong \mathrm{J}_2$*,* $G\cong \mathrm{Aut}(\mathrm{J}_2)$ *and* Ω *is the set of points of the rank three graph for* T*;* $A\cap T\cong \mathrm{U}_3(3)$*; and* $X\not\le T$ *with* $X\cong 5^2:4$ *where* $X/O_5(X)$ *acts as a central or a non-central element of order* 4 *of* $\mathrm{Aut}(O_5(X))\cong \mathrm{GL}_2(5)$ *on* $O_5(X)$.

(g) $T\cong \mathrm{HS}$ *and* Ω *is the set of points of the Higman-Sims graph related to* T*;* $A\cap T\cong \mathrm{M}_{22}$*; and* $X\not\le T$ *with* $X\cong 5\times 5:4$ *or* $5^2:4$ *and* $O_5(X)$ *is self-centralizing in* X *and* $X/O_5(X)$ *acts either as a central or as a non-central element of* $\mathrm{Aut}(O_5(X))\cong \mathrm{GL}_2(5)$ *on* $O_5(X)$.

In particular, the rank 3 graphs for J_2 and HS, respectively, are Cayley graphs.

The exceptional groups of Lie type.

If G is an exceptional group of Lie type, then there is no example:

Theorem 10 *Let (G, Ω) be a primitive permutation group with $soc(G) = T$ an exceptional group of Lie type. Then there is no regular subgroup in G.*

The classical groups.

Let G be a classical group. In Section 7.3 of Chapter 7 we present all the examples known to us. Then we first prove the following theorem in order to be able to study $\mathrm{P}\Omega_8^+(q)$.

Theorem 11 *Let (G, Ω) be a primitive permutation group with $T = soc(G) \cong \mathrm{P}\Omega_7(q)$, q even or odd. Suppose that $G_\omega \cap T \cong \mathrm{G}_2(q)$, for ω in Ω. Then G has a regular subgroup if and only if $q \in \{2, 4\}$.*

The next theorem, which is a negative statement, we only need for $n = 3$, but prove in full generality.

Theorem 12 *Let (G, Ω) be a primitive permutation group with $T = soc(G) \cong \mathrm{P}\Omega_{2n+1}(q)$, q odd. Let V be the natural module for T and assume that Ω equals the set of totally singular subspaces of dimension i of V, for some i in $\{1, \ldots, n\}$. Then there is no regular subgroup in G.*

Then we are showing the main result of this section.

Theorem 13 *Let (G, Ω) be a primitive permutation group with $T = soc(G) \cong \mathrm{P}\Omega_8^+(q)$. Suppose there is a subgroup X of G which acts regularly on Ω. Then (G, Ω) is one of the examples (f), (g), (i) or (l) listed in Section 7.3 of Chapter 7.*

In order to characterize the examples for the unitary groups in their action on the set of totally isotropic i-subspaces of their natural module we show the following:

Theorem 14 *Let $T \cong \mathrm{U}_n(q)$ and let $T \leq G \leq \mathrm{Aut}(T)$. Suppose there is a subgroup X of G which acts regularly on the set Ω of totally isotropic subspaces of dimension i of the natural T-module V, for some $1 \leq i \leq n/2$ if n is even or $1 \leq i \leq (n-1)/2$ if n is odd. Then $(n, q) = (4, 2)$ or $(3, 8)$ and the pair (T, X) is as in Example (a) or (n) in Section 7.3 of Chapter 7.*

Shortly before submitting the thesis the author learned from C.E. Praeger that she as well as M.W. Liebeck and J. Saxl would also be working on the classification of regular subgroups of primitive permutation groups.

Note that the thesis provides new evidence for the fact that $L_2(7)$ is a special group: Two of our results imply characterizations of the simple group $L_2(7)$. Theorem 6 implies that $L_2(7)$ is the only non-abelian simple group T such that every maximal subgroup of T has a nilpotent supplement (which is in fact a complement). Moreover, it follows from Theorem 2 that $L_2(7)$ is the only non-abelian simple group T which is a transitive subgroup of a primitive permutation group of affine type.

In Chapter 2 we fix some notation and sum up definitions that will be used throughout the present work. In that chapter also included is the necessary background on primitive permutation groups, as well as the classification of the maximal subgroups of the classical groups by M. Aschbacher, and lower bounds on the dimensions of the irreducible modules for the groups of Lie type in defining and in cross characteristic.

Some words on the methods of proofs used in this thesis. First of all, note that all the results of this thesis rely on the classification of the finite simple groups. In quite a few chapters, namely in Chapters 4, 6 and 7 Zsigmondy primes (see Chapter 2 for their definition) play a prominent role. In Chapters 4 and 6 we first solve the problems discussed therein for the almost simple groups and then try (successfully) to reduce the general case to the special case of the almost simple groups. Therefor Lemma 6.3.1 of L. G. Kovács is of great use in the latter chapter. In Chapter 5 we determine all the permutations groups (G, Ω) admitting a primitive factorization. Each section of this chapter is devoted to one of the different types of the primitive permutation groups (see the partition given in Chapter 2). For each of these types it turned out to be necessary to devise a new method of proof. In Chapters 4, 6 and 7 we quote the classification of maximal factorizations of the almost simple groups by Liebeck, Praeger and Saxl a lot. In the Chapter 7 we can not apply general methods but need studying the groups in detail.

A brief history.

We recall some important results on factorizations of groups. A group G has a non-trivial factorization if and only if G acts on a set Ω and possesses a non-trivial transitive subgroup. This connection suggests that there are many many theorems which can be read as a result on factorizations of groups. Thus we are only able to mention some of the theorems on factorizations.

We do not start with the work of Cauchy or Galois, who both clearly worked also on factorizations of groups, as they studied transitive permutation groups, but much later with W. Burnside. It seems to me that there are four principal questions, closely related to each other, which have been and still are asked about factorizations of groups.

The first question. Given a group $G = AB$ which factorizes in two subgroups A and B of G, does G inherite certain properties of A and B? The main emphasis has been and still is put on properties of A and B which force G to be soluble. In this context a first result is the fundamental p^aq^b-theorem by W. Burnside published in 1904, which states if $G = AB$ with A and B p-groups of not necessarily the same prime, then G is soluble, [19].

Thirty years later P. Hall characterized the soluble groups: A group G is soluble if and only if every Sylow p-subgroup of G admits a complement if and only if G possesses a Sylow-system (a Sylow-system is a set of Sylow-subgroups of G containing exactly one Sylow-subgroup for every prime in $\pi(G)$ such that every two elements in the system commute setwise), see [40,41,42].

Then, in 1955, N. Ito proved that if $G = AB$ with A and B abelian, then G is metabelian, i.e. G' is abelian, [48]. This theorem also holds for infinite groups. The Theorem of Burnside as well as the Theorem of Ito have been generalized by H. Wielandt and O.H. Kegel. They showed that if $G = AB$ with A and B nilpotent, then G is soluble, [91, 52]. Notice that the proof of W. Burnside of his theorem is based on character theory. Later D. Goldschmidt gave a group theoretic proof of the p^aq^b-theorem for p and q odd, see [34], and H. Bender extended the proof of Goldschmidt for all primes [14].

Properties of A and B which force G to be not a non-abelian simple group gained much attention, as well. In 1963 J. Szép suggested if $G = AB$ with $Z(A) \neq 1 \neq Z(B)$, then G is not non-abelian simple, [85]. E. Fisman and Z. Arad proved the conjecture using the classification of the finite simple groups [32]. It seems that a classification-free proof is not possible for the time being. Later C. Hering proposed that if G is a group of Lie type the conjecture could be deduced from quite elementary facts of groups of Lie type and certain geometrically significant collineations of buildings [45]. He demonstrated this for the case $G \cong \mathrm{L}_n(q)$. Notice that the p^aq^b-theorem of Burnside is a consequence of the Conjecture of Szép. Maybe more special cases of the conjecture can be proven without using the classification by applying the methods of

Goldschmidt and Bender to the problem.

In 1965 Kegel published a paper [54] in which he considers conditions on a group which are in the spirit of the Theorem of Hall and of his theorem with Wielandt. Kegel showed, if G is a group such that every maximal subgroup of G admits a supplement which is cyclic and of prime-power order, then G is soluble, see [54, Proposition 1]. In the same paper Kegel conjectured Theorem 4 of this thesis stated above, [54, p.210]. It is well known that every maximal subgroup of a soluble group is of prime power index in that group. The theorem of Kegel can be seen as a partial converse of this fact. Note that in the simple group $\mathrm{L}_2(7)$ every maximal subgroup is of prime power index, as well. R. M. Guralnick showed that if G is a group such that every maximal subgroup has prime power index in G, then $G/O_\infty(G) \cong 1$ or $L_2(7)$, see [37, Corollary 3].

Another characterization of soluble groups was obainted by Y. Wang. He showed that a group G is soluble if and only if for every maximal subgroup H of G there exists a normal subgroup $N = N(H)$ of G such that $G = HN$ and $H \cap N \leq Core(H)$ [88, Theorem 3.1]. In their paper [13] J. C. Beidleman and D. J. S. Robinson considered groups G having the property that for every proper subgroup H of G there is a cyclic group $X = X(H)$ which is not in H such that $XH = HX$. They proved that every group G enjoying this property is soluble [13, Theorem A]. See also the related work by Z. Yaoqing [94] and by W. E. Deskins [27]. Notice that the proofs of Corollary 3 of [37] and of Theorem A of [13] are based on the classification of finite simple groups. Also notice that Theorem 4 implies Corollary 3 of [37], Theorem A of [13] as well as Theorem 3.1 of [88].

In this context research on permutable subgroups ($H \leq G$ is called permutable, if $HK = KH$ for all $K \leq G$), see for instance [12] or [28] and the references therein, and on formations and Fitting classes, see for instance [43], should also be mentioned. Finally, I would like to quote a result by G. Xiuyun, K.P. Shum and A. Ballester-Bolinches. They showed that a group G is soluble if there is a maximal subgroup M in G such that every minimal subgroup of M has a complement in G, see [93, (4.1)].

The second question is the extension problem. Given two groups A and B, find all the groups G possessing two subgroups A' and B', which are isomorphic to A and B, respectively, such that $G = A'B'$. There are quite a few papers by L. Rédei and J. Szép on this difficult problem, see for instance [77]. U. Preiser classifies in [76] all the groups G which contain simple groups A and B with $G = AB$ such that every Sylow 2-subgroup of A and of B has rank 2 or is elementary abelian, respectively.

The third question is as follows. Given a group G, a normal subgroup N of G (or a subgroup U of G), does there exist a complement to N (or to U) in G? Partial answers to this question are provided by the Theorems of Gaschütz (see for instance Theorem 10.4 of [5]) and Schur and Zassenhaus (see Theorem 18.1 of [5]). There is also the *Normal p-complement Theorem* by Burnside, stating if a Sylow p-subgroup of G is in the center of its normalizer then every Sylow p-subgroup has a normal complement in G (a normal complement to a Sylow p-subgroup is also called *normal p-complement*), see Theorem 39.1 of [5], as well as the *Normal p-complement Theorem* by G. Frobenius, stating G has a normal p-complement if and only if each p-local subgroup of G has a normal p-complement if and only if $\mathrm{Aut}_G(P)$ is a p-group for each p-subgroup P of G, see Theorem 39.4 of [5], and the *Normal p-complement Theorem* by J. Thompson. It is as follows: Let p be an odd prime and $P \in Syl_p(G)$. If $N_G(J(P))$ and $C_G(\Omega_1(Z(P)))$ have normal p-complements, then G has a normal p-complement, where $J(P)$ is the Thompson-subgroup of P, see Theorem 39.5 of [5]. A paper related to the third question has been written by M. Liebeck, C.E Praeger and J. Saxl in the language of permutation groups. They determine the structure of a group G which acts primitively on a set and contains a transitive subgroup which does not contain a subnormal subgroup of G, see [69].

The last question is as follows. Does there exist a (special) factorization for a given group G? In a fundamental work Liebeck, Praeger and Saxl proved: Let T be a finite simple group and let G be a group such that $T \leq G \leq \mathrm{Aut}(T)$. Suppose that $G = AB$, where A and B are maximal subgroups of G not containing T. Then the triples (G, A, B) are explicitly known and they present explicit lists of the triples (G, A, B), see [67]. Theorem 1 of this thesis can also be seen as an answer to the fourth question. Finally, there is a paper by R.W. Baddeley and C.E. Praeger where they determine all the factorizations $T = AB$ for the simple groups T such that A is maximal in T and such that $|T|, |A|$ and $|B|$ are divisible by the same primes. This result is related to but independent of Theorem 1.

Special attention obtain and obtained the primitive permutation groups which have a regular subgroup (see also the third question). Let G be a group which acts faithfully and primitively on a set Ω. In 1911 W. Burnside proved that if (G, Ω) contains a cyclic regular subgroup of prime power order, then G is doubly transitive or of prime degree [21, p. 343]. Later I. Schur could extend this result to all cyclic groups: if G contains a cyclic regular subgroup, then G is doubly transitive or of prime degree [79]. In order to prove this fact Schur introduced the powerful theory of S-rings. In 1935 H. Wielandt [90] generalized this theorem to abelian groups with a cyclic Sylow subgroup: if G contains an abelian regular subgroup which has a cyclic Sylow subgroup, then G is doubly transitive or of prime degree. In his book [92] Wielandt introduced the notion of a Burnside group. A group X is called a *Burnside-group* (or short a *B-group*) if each primitive permutation group which contains a regular subgroup isomorphic to X is necessarily 2-transitive. Hence the above theorem states in fact that every abelian group which has a cyclic Sylow subgroup is a B-group, see for a survey of B-groups [92, Chapter IV § 25].

All the pairs (G, Ω) such that G contains a cyclic regular subgroup have then been classified by Feit [31] (the insoluble ones), see also [36, Theorem 1.49], and by J. F. Ritt [78, p. 27] (the soluble ones) – see also G. A. Jones [49].

C. Li classified those (G, Ω) such that G contains an abelian regular subgroup. As a corollary of this classification he obtained that an abelian group X which is not a p-group is not a B-group if and only if $X = X_1 \times \cdots \times X_n$ with $|X_i| = |X_j|$ and $n \geq 2$. In the last chapter we continue the classification of the pairs (G, Ω) such that G contains a regular subgroup and discuss some results on B-groups.

Acknowledgements

First of all I would like to thank Prof. Dr. Gernot Stroth for his permanent interest in my work.

I also thank the referees of the papers [8,10,11] who indirectly contributed to this thesis (specially the referee of [11]).

I am also grateful to Dr. B. Wilkens, A. Hirn and R. Knop for their proof reading and for finding many misprints.

I am indebted to many other people who have contributed in one or the other way to this thesis.

Last not least I owe thanks to my little son David for being rarely sick and for being a continuous sunshine.

Bibliography

[1] Z. Arad and E. Fisman, On finite factorizable groups, *J. Algebra* **86** (1984), 522–548.

[2] E. Artin, The orders of the linear groups, *Comm. Pure and Appl. Math.* **8** (1955), 355–366.

[3] M. Aschbacher, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76** (1984), 469–514.

[4] M. Aschbacher, L. Scott, Maximal subgroups of finite groups, *J. Algebra* **92** (1985), 44–80.

[5] M. Aschbacher, *Finite Group Theory*, Cambridge University Press, Cambridge, 1986.

[6] R. W. Baddeley, C. E. Praeger, On classifying all full factorisations and multiple-factorisations of the finite almost simple groups, *J. Algebra* **204** (1998), 129–187.

[7] B. Baumeister, Dual extended grids or a characterization of the third group of Janko, PhD-thesis, FU-Berlin, 1996.

[8] B. Baumeister, Factorizations of primitive permutation groups, *J. Algebra* **194** (1997), 631–653.

[9] B. Baumeister, A characterization of $L_2(3^r)$ and $U_3(2^r)$, Preprint.

[10] B. Baumeister, A characterization of the finite soluble groups, *Archiv Math.* **72** (1999), 167–176.

[11] B. Baumeister, On groups with subgroups of prime power index, *J. London Math. Soc.* (2) **62** (2000), 407–422.

[12] J. C. Beidleman, H. Heineken, Finite soluble groups whose subnormal subgroups permute with certain classes of subgroups , *J. Group Th.* **6** (2003), 139–158.

[13] J. C. Beidleman, D. J. S. Robinson, On finite groups satisfying the permutizer condition, *J. Algebra* **191** (1997), 686–703.

[14] H. Bender, A group theoretic proof of Burnside's $p^a q^b$-Theorem, *Math. Z.* **126**, (1972), 327–338.

[15] R. D. Bercov, The doubly transitivity if a class of permutation groups, PhD-thesis, California Institute of Technology Pasadena, California, 1962.

[16] F. Buekenhout, On a theorem of O'Nan and Scott, *Bull. Soc. Math. Belg. – Tijdschr. Belg. Wisk. Gen.* (ser. B) **40** (1988), 1–9.

[17] F. Buekenhout, A. Delandtsheer, J. Doyen, Finite linear spaces with flag transitive groups, *J. Comb. Th.* (A) **49** (1988), 268–293.

[18] W. Burnside, On some properties of groups of odd order, *Proc. London Math. Soc.* **33** (1900), 162–185.

[19] W. Burnside, On groups of order $p^a q^b$, *London Math. Soc.* **1**, (1904), 338–392.

[20] W. Burnside, On certain simply-transitive permutation groups, *Proc. Cambridge Phi. Soc.* **20** (1921), 482–484.

[21] W. Burnside, *Theory of Groups of Finite Order*, 2nd ed., Cambridge Univ. Press, London, 1911.

[22] P. Cameron, W. M. Kantor, 2-transitive and antiflag transitive collineation groups of finite projective spaces, *J. Algebra* **60** (1979), 384–422.

[23] R. W. Carter, *Simple Groups of Lie type*, John Wiley and Sons, London, 1972.

[24] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[25] B. N. Cooperstein, Maximal subgroups of $G_2(2^n)$, *J. Algebra* **70** (1981), 23–36.

[26] M. R. Darafsheh, G. R. Rezaeezadeh, Factorization of groups, involving symmetric and alternating groups, *Int. J. Math. Sci.* **27** No. 3, (2001), 161–167.

[27] W. E. Deskins, A note on the index complex of a maximal subgroup, *Arch. Math.* **54** (1990), 236–240.

[28] M. R. Dixon, S. E. Stonehewer, The join of two disjoint soluble subgroups, both of which are permutable, in *Proc. of the intensive Bimester Dedicated to Reinhold Baer, 2002,* 5–13, Editors F. de Giovanni et all, Aracne, Rom, 2003.

[29] P. Etinghof, S. Gelaki, R. M. Guralnick, J. Saxl, Biperfect Hopf algebras, *J. Algebra* **232** (2000), 331–335.

[30] W. Feit, On symmetric balanced incomplete block designs with doubly transitive autiomorphism groups, *J. Combin. Theory Ser. A* **14** (1973), 221–247.

[31] W. Feit, Some consequences of the classification of finite simple groups. In *The Santa Cruz conference on finite groups*, Proc. Symp. Pure Math. 37 (American Mathematical Society, 1980), pp. 175–181.

[32] E. Fisman, Z. Arad, A proof of Szép's conjecture on nonsimplicity of certain finite groups, *J. Algebra* **. 108**, (1987), 340–354.

[33] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer, New York, Berlin, 2001.

[34] D. Goldschmidt, A group theoretic proof of the $p^a q^b$ theorem for odd primes, *Math. Z.* **113**, (1970), 373–375.

[35] D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1968.

[36] D. Gorenstein, *Finite Simple Groups*, Plenum Press, New York, 1982.

[37] R. M. Guralnick, Subgroups of prime power index in a simple group, *J. Algebra* **81** (1983), 304–311.

[38] R. M. Guralnick, T. Penttila, C. E. Praeger, J. Saxl, Linear groups with orders having certain primitive divisors, *Proc. London Math. Soc.* (3) **(78)** (1999), No. 1, 167–214.

[39] R. M. Guralnick, J. Saxl. Monodromy groups of polynomials. In *Groups of Lie type and their geometries*, London. Math. Soc. Lecture Note Ser. 207 (Cambridge University Press, 1995), pp. 125–150.

[40] P. Hall, A note on soluble groups, *J. London Math. Soc* **3** (1934), 98–105.

[41] P. Hall, A characteristic property of soluble groups, *J. London Math. Soc* **12** (1937), 188–200.

[42] P. Hall, On the Sylow-systems of a soluble group, *Proc. London Math. Soc* **2** (1937), 198–200.

[43] P. Hauck, A. Martínez-Pastor, M. D. Pérez-Ramos, Products of pairwise totally permutable groups, *Proc. Edingbourgh Math. Soc.* **46** (2003), 147–157.

[44] P. Hegedüs, Regular subgroups of the affine group, *J. Algebra* **225** (2000), 740–742.

[45] C. Hering, On Széps conjecture concerning factorizations of finite groups, *Rend. Circ. Mat. Palermo* (2) Suppl. No. **19** (1988), 105–111.

[46] B. Huppert, *Endliche Gruppen I*, Springer–Verlag, Berlin, Heidelberg, 1979.

[47] B. Huppert, N. Blackburn, *Endliche Gruppen III*, Springer–Verlag, Berlin, Heidelberg, 1982.

[48] N. Ito, Über das Produkt von zwei abelschen Gruppen, *Math. Z.* **62**, (1955), 400–401.

[49] G. A. Jones, Cyclic regular subgroups of primitive permutation groups, *Journal of Group Theory* **5** (2002), 403–407.

[50] W. Jones, B. Parshall, On the 1-cohomology of finite groups of Lie Type, *Proceedings on ht econference of finite groups*, (Univ. Utah, Park City, Utah, 1975), 313–328, Academic Press, New York, 1976.

[51] W. M. Kantor, k-Homogeneous Groups, *Math. Z.* **124**, (1972), 261–265.

[52] O. H. Kegel, Produkte nilpotenter Gruppen, *Arch. Math.* **7**, (1961), 90–93.

[53] O. H. Kegel, On the solvability of some factorized linear groups, *Illinios J. Math.* **9**, (1965), 535–547.

[54] O. H. Kegel, On Huppert's characterization of finite supersoluble groups, in *Proc. Internat. Conf. Theory of groups, Canberra, 1965*, 209–215, Gordon and Breach, New York, 1967.

[55] O. H. Kegel, H. Lüneburg, Über die kleine Reidemeisterbedingung.II, *Arch. Math.* **14**, (1972), 7–10.

[56] P. B. Kleidman, *The Low-dimensional Finite Simple Classical Groups and their Subgroups*, Preprint.

[57] P. B. Kleidman, The maximal subgroups of the finite 8–dimensional orthogonal groups $P\Omega_8^+(q)$ and of their automorphism groups, *J. Algebra* **117** (1988), 30–71.

[58] P. B. Kleidman, M. W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, London Math. Soc. Lecture Notes **129**, Cambridge University Press, 1990.

[59] L. G. Kovaćs, Primitive subgroups of wreath products in product action, *Proc. London Math. Soc.* (3) **58** (1989) 306–322.

[60] H. Kurzweil, B. Stellmacher, *Theorie der endlichen Gruppen*, Springer–Verlag, Berlin, Heidelberg, 1998.

[61] V. Landazuri, G. Seitz, On the minimal degree of projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418–443.

[62] C. H. Li, The finite primitive permutation groups containing an abelian regular subgroup, to appear in *Proc. London Math. Soc.*

[63] M. W. Liebeck, On the orders of maximal subgroups of the finite classical groups, *Proc. London Math. Soc.* (3), **50** (1985), 426–446.

[64] S. Linton, personal communication, 2003.

[65] M. W. Liebeck, C. E. Praeger, J. Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups, *J. Algebra* **111** (1987), 365–383.

[66] M. W. Liebeck, C. E. Praeger, J. Saxl, On the O'Nan–Scott Theorem for finite primitive permutation groups, *J. Austral. Math. Soc.* (ser. A) **44** (1988), 389–396.

[67] M. W. Liebeck, C. E. Praeger, J. Saxl, The maximal factorizations of the finite simple groups and their automorphism groups, *Memoirs of the AMS* **(86)**, Number 432, 1990.

[68] M. W. Liebeck, C. E. Praeger, J. Saxl, On factorizations of almost simple groups, *J. Algebra* **185** (2) (1996), 409–419.

[69] M. W. Liebeck, C.E. Praeger, J. Saxl, Transitive subgroups of primitive permutation groups, *J. Algebra* **234** (2000), no. 2, 291–361.

[70] F. Luebeck, *Smallest degrees of representations of exceptional groups of Lie type*, Preprint 2003.

[71] U. Meierfrankenfeld, On the generalized Fitting group of locally finite, finitary groups, to appear in *J. of Group Theory.*

[72] L. L. Scott, Representations in characteristic p, in *Santa Cruz conference on finite groups, Proc. Sympos. Pure Math.*, **37**, Amer. Math. Soc., Providence, R.I., (1980) 318–331.

[73] P. P. Palfy, J. Saxl, Congruence lattices of finite algebras and factorizations of groups, *Comm. in Algebra* **18** (1990), 2783–2790.

[74] C. E. Praeger, The inclusion problem for finite primitive permutation groups, *Proc. London Math. Soc.* (3) **60** (1990), 68–88.

[75] C. E. Praeger, C. Schneider, Factorisations of characteristically simple groups, *J. Algebra* **255** No. 1 (2002), 198–220.

[76] U. Preiser, Factorizations of finite groups, *Math. Z.* **185** (1984), 373–402.

[77] L. Rédei, J. Szép, Die Verallgemeinerung der Theorie des Gruppenproduktes von Zappa–Casadio, *Acta Sci. Math. Szeged* **16** (1955), 165–170.

[78] J. F. Ritt, On algebraic functions which can be expressed in terms of radicals, *Trans. Amer. Math. Soc.* **24** (1922), 21–30, 324.

[79] I. Schur, Zur Theorie der einfach transitiven Permutationsgruppen, *S. B. Preuss. Akad. Wiss. Phys.-Math.Kl.* (1933), 598–623.

[80] G. Seitz, A. Zalesski, On the minimal degrees of projective representations of the finite Chevalley groups. II, *J. Algebra* **158** (1993), 233–243.

[81] D. H. Smith, Primitive and imprimitive graphs, *Quart. J. Math. Oxford* (2) **32** (1971), 551–557.

[82] R. Solomon, On finite simple groups and their classification, *Notices AMS* **42** (1995), 231–239.

[83] R. Steinberg, Representations of algebraic groups, *Nagoya Math. J.* **22** (1963), 33–56.

[84] M. Suzuki, *Group Theory I*, Springer–Verlag, Berlin, Heidelberg, 1982.

[85] J. Szép, Sui gruppi fattorizzabili non semplici, *Rend. Mat. e Appl.* (5) **22**, (1963), 245–252.

[86] M. Takeuchi, Matched pairs of groups and bimash products of Hopf algebras, *Comm. Algebra* **9**, No. 8 (1981), 841–882.

[87] D. E. Taylor, *The Geometry of the Classical Groups*, Helderman–Verlag, Berlin, 1992.

[88] Y. Wang, C-Normality of groups and its properties, *J. Algebra* **180** (1996), 954–965.

[89] J. Wiegold, A. G. Williamson, The factorizations of the alternating and the symmetric groups, *Math. Z.* **175**, (1980), 171–179.

[90] H. Wielandt, Zur Theorie der einfach transitiven Permutationsgruppen, *Math. Z.* **40** (1935), 582–587.

[91] H. Wielandt, Über Produkte nilpotenter Gruppen, *Illinois J. Math.* **2**, (1958), 611–618.

[92] H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964.

[93] G. Xiuyun, K. P. Shum, A. Ballester-Bolinches, On Complemented minimal subgroups in finite groups, *J. Group theory* **6**, (2003), 159–167.

[94] Z. Yaoqing, On the index complex of a maximal subgroup and the supersolvability of a finite group, *Comm. in Algebra* **24(5)**, (1996), 1785–1791.

[95] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.* **3** (1892), 265–284.

[96] M. Aschbacher, G. Seitz, Involutios in Chevalley groups over fields of even order, *Nagoya Math. J.* **63** (1976), 1–91.