On finite Bol Loops of Exponent 2

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November 24, 2008

Abstract

This draft (which still will be split into parts) is the classification of the loop envelopes of the finite Bol Loops of exponent 2, see Theorem 2. It contains the reduction arguments to finite simple groups (reduction to passive groups) in Theorem 5 as well as the classification of the passive groups, see Theorem 1. A main tool for that is the connectivity of certain commuting graphs. This, which will be shown in Section 4, is of interest on its own.

1 Introduction

(Finite) Bol loops of exponent 2 were long considered to be soluble. M. Aschbacher studied the minimal non-soluble finite simple Bol loops of exponent 2, the so called N-loops [Asch], see Definition 6.19. Using the classification of finite simple groups, he could restrict the structure of the related groups considerably, see Theorem 4. In particular, he showed $G/O_2(G) \cong \mathrm{PGL}_2(q), q = 9$ or q is a Fermat prime. The smallest N-loop was found by B. Baumeister and A. Stein and independently by G. Nagy in 2007. Furthermore, G. Nagy produced an infinite family of simple Bol loops of exponent 2.

Notice, that in all the known N-loops $q = 5$.

The notation, which at many places follows [Asch], will be introduced in the next section.

Definition 1.1 *A finite nonabelian simple group S is called* **passive***, if whenever (G, H, K) is a loop folder of a Bol loop of exponent 2 with $F^*(G/O_2(G)) \cong S$, then $G = O_2(G)H$.*

We show the following theorem, which then implies that every nonabelian simple group is either passive or isomorphic to $\mathrm{PSL}_2(q)$, $q = 9$ or $q \geq 5$ a Fermat prime.

Theorem 1 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2 such that $F^*(G/O_2(G))$ is quasisimple. Then either*

- *$G = O_2(G)H$ or*
- *there is an integer q, $q \geq 5$, a Fermat prime or $q = 9$ such that*
  - *(a) $\overline{G} = G/O_2(G) \cong \mathrm{PGL}_2(q)$ or $\overline{G} \cong \mathrm{P\Gamma L}_2(q)$ (only if $q = 9$)*
  - *(b) $|\overline{G} : \overline{H}| = q + 1$*

(c) $\overline{K}$ *consists of the identity* 1 *and all the involutions in* $\mathrm{PGL}_2(q)$ *which are not in* $\mathrm{PSL}_2(q)$.

Using this theorem we are then able to show the following.

Theorem 2 *Let* (G, H, K) *be a loop envelope of a Bol loop of exponent* 2 *Then the following holds.*

(a) $\overline{G} := G/O_2(G) \cong D_1 \times D_2 \times \cdots \times D_k$ *for some non-negative integer* k

(b) $D_i \cong PGL_2(q_i)$ *for* $q_i \geq 5$ *a Fermat prime or* $q_i = 9$

(c) $D_i \cap HO_2(G)/O_2(G) \cong q_i : (q_i - 1)$ *is a Borel subgroup in* D_i *of index* $q_i + 1$

(d) $F^*(G) = O_2(G)$

(e) $\overline{K}$ *is the set of involutions in* $\overline{G} \setminus \overline{G}'$

Roughly speaking, the enveloping group of a Bol loops of exponent 2 looks as if it is a direct product of the enveloping groups of some N-loops. The question of the existence of N-loops with $q > 5$ is still open. Our results do not depend on an answer to this question.

As a consequence to this theorem we get the following result on the general structure of a finite Bol loop of exponent 2.

Corollary 1.2 *Let* (G, H, K) *is a loop folder of a Bol loop* X *of exponent* 2. *Then the following holds.*

(a) $(O_2(G), O_2(G) \cap H, O_2(G) \cap K)$ *and* $(O_2(G)H, H, O_2(G) \cap K)$ *are loop folders of the same soluble Bol subloop of* X.

(b) *(*Sylow's Theorem for $p = 2$*) There exists a Sylow-2-subgroup* P *of* G, *such that* $(P, P \cap H, P \cap K)$ *is a loop folder to a subloop of* X *of size* $|G : H|_2 = |K|_2$. *Every soluble subloop of* X *is contained in an* H*-conjugate of such a subloop.*

(c) Lagrange's Theorem *holds on* X.

2 Notation

Definition 2.1 *A* **(right) Bol loop** $(X, \cdot)$ **of exponent** 2 *is a loop such that*

- *for all* $x, y, z \in X$ *the* **(right) Bol** *identity holds:*

$$x \cdot ((y \cdot z) \cdot y) = ((x \cdot y) \cdot z) \cdot y,$$

  and

- *the loop is of* **exponent** 2*: for all* $x \in X, x \cdot x = 1$.

Remark 2.2 *Bol loops can be translated into the language of group theory, as has been observed by R. Baer [Baer]:*

Given a Bol loop of exponent 2*, we define for* $x \in X$ $\rho(x) : X \to X, y \mapsto y \cdot x$*,*
$G :=$ RMult$(X) := \langle \rho(x) : x \in X \rangle \leq \Sigma(X)$*, the enveloping group of* X*,*
$H := \mathrm{Stab}_G(1)$*,*
$K := \{\rho(x) : x \in X\} \subseteq G$ *and*
$\kappa : K \to X : \rho(x) \mapsto x$*. Then* (G, H, K) *is the* loop envelope of the loop X *and it satisfies the following properties:*

(1) K *is a set of representatives for the set of right cosets for all conjugates of* H*.*

(2) H *is core free.*

(3) $G = \langle K \rangle$*.*

(4) for all $x, y \in K$*:* $x^{-1} \in K$ *and* $xyx \in K$*.*

(5) K *is a union of* G*-conjugacy classes of involutions.*

Definition 2.3 *A triple* (G, H, K) *with* G *a group,* $H \leq G$ *and* $K \subseteq G$ *is called*

- *a* **loop folder***, if it satisfies (1),*
- *a* **faithful** *loop folder, if it satisfies (1) and (2),*
- *a* **loop envelope***, if it satisfies (1) and (3),*
- *a* **loop folder of a Bol loop** *, if it satisfies (1) and (4) and*
- *a* **loop folder of a Bol loop of exponent** 2*, if it satisfies (1), (4) and (5).*

Baer also observed that given a loop folder we can construct a loop, see Remark 1.1 of [Asch]. We generalize this observation as follows: Given a loop folder and a bijection κ from K into some set X we obtain a loop on X by defining $\kappa(k_1) \cdot \kappa(k_2) = \kappa(k_{12})$ with k_{12} the unique element in $K \cap Hk_1k_2$. Denote the inverse map to κ by R, that is $\kappa(R(x)) = x$ for all $x \in X$ and $R(\kappa(k)) = k$ for all $k \in K$. We call X the **loop of the loop folder**. So, by our definition, the loop of a loop folder is only unique up to a bijection.

The distinction between K and elements of X is useful: the symmetric group on X with its subgroups RMult(X), Aut(X), LMult(X) etc. acts naturally on X. The group G projects into this action, if we identify the elements of X with the cosets of H and define an action x^g for $x \in X$, $g \in G$ by the equation $HR(x^g) = HR(x)g$. Notice, that this homomorphism from G into $\Sigma(X)$ covers RMult(X).

On the other hand G acts naturally by conjugation on K, but these two actions are different: The action on X is transitive, while the action on K is not in general. So there are natural actions of G on X and K, but κ does not provide an permutation isomorphism between them.

The folder (G, H, K) comes from a Bol loop if and only if (G, H, K) satisfies (4) and from a Bol loop of exponent 2 if and only if it satisfies (4) and (5).

Subsets K of G with property (4) are called **twisted subgroups** in the literature.

Notice, that conditions (1) and (5) imply the condition $KK \cap H = 1$, which is a useful condition in its own right.

Subloops, homomorphisms, normal subloops, factor loops and simple loops are defined as usual in universal algebra: A **Subloop** is a nonempty subset which is closed under loop multiplication. **Homomorphisms** are maps which commute with loop multiplication. The map defines an equivalence relation on the loop, such that the product of equivalence classes is again an equivalence class. **Normal subloops** are preimages of 1 under a homomorphism and therefore subloops. A normal subloop defines a partition of the loop into blocks (cosets), such that the set of products of elements from two blocks is again a block. Such a construction gives factor loops as homomorphic images with the block containing 1 as the kernel. **Simple loops** have only the full loop and the 1-loop as normal subloops.

Loop folders, which do not satisfy (2) and (3) occure naturally, if one considers embeddings of subloops in larger loops. For more elementary facts and proofs see [Asch].

Finally we recall the definition of a soluble loop given in [Asch]. A loop X is **soluble** if there exists a series $1 = X_0 \leq \cdots \leq X_n = X$ of subloops with X_i normal in X_{i+1} and X_{i+1}/X_i an abelian group.

3 Useful Facts

3.1 Facts from Number Theory

The following lemmata are consequences of Zsygmondy's theorem.

Lemma 3.1 *Let p be a prime.*
If $n \in \mathbb{N}$ with $\Phi_n(p)$ a power of 2, then $n = 1$ and p is 2 or a Fermat prime or $n = 2$ and p is a Mersenne prime.
If $n \in \mathbb{N}$ with $\Phi_n(p)$ a power of 3, then $p = 2$ and $n \in \{1, 2, 6\}$.
If $n \in \mathbb{N}$ with $\Phi_n(p)$ a power of 3 times a power of 5, then $p = 2$ and $n \in \{1, 2, 4, 6\}$.

($\Phi_n(x) \in \mathbb{Z}[x]$ is the n-th cyclotomic polynomial.)

Proof. If $n > 2$ and $(p, n) \neq (2, 6)$ by Zsygmondy's theorem there exists a prime r dividing $\Phi_n(p)$, which does not divide $\Phi_m(p)$ for $m < n$. Since 3 divides $(p-1)p(p+1) = \Phi_1(p)p\Phi_2(p)$ we have $r > 3$. So in the first two cases the question reduces to those primes p, for which $p - 1$ (in case $n = 1$) or $p + 1$ (in case $n = 2$) is a 2-power or a 3-power. For the third case observe, that $n \mid r - 1$, so $n \in \{1, 2, 4\}$ in this case and we have to determine those primes p, for which one of $p - 1, p + 1$ or $p^2 + 1$ is a 3-power times a 5-power. Since in particular $\Phi_n(p)$ is odd, $p = 2$. The statement is immediate. □

Lemma 3.2 *Let q be a prime power.*

(i) If $q - 1$ is a 2-power, then $q = 2$, $q = 9$ or q is a Fermat prime.

(ii) If $q+1$ is a 2-power, then q is a Mersenne prime.

(iii) If q^2-1 is a 2-power, then $q=3$.

(iv) If q^2-1 is a 2-power times a 3-power, then $q \in \{2,3,5,7,17\}$.

(v) If q^2-1 is a 3-power times a 5-power, then $q \in \{2,4\}$.

Proof. Let $q=p^e$. Remember the formulas

$$(p^e)^n - 1 = \prod_{d|en} \Phi_d(p)$$

and

$$(p^e)^n + 1 = \prod_{\substack{d|2en \\ d\nmid en}} \Phi_d(p).$$

For $n=1$ we get $e \leq 2$ in (i) and (ii) by 3.1.
For $n=2$ we get (iii) again by 3.1.
Since 3 divides exactly one of $q-1$, q, $q+1$, we get $q=2$ or q a Mersenne or Fermat prime by (i) and (ii).
For Mersenne primes $p=2^r-1$ we have $p-1=2(2^{r-1}-1)$, which is a 2-power times a 3-power for $r \leq 2$ only by the formula mentioned and 3.1.
For Fermat primes $p=2^m+1$ we can again use the formula on $p+1=2(2^{m-1}+1)$ and 3.1. Finally (v) is a consequence of the above product formula together with 3.1. □

Definition 3.3 *Let q be a power of a prime p and $r \neq p$ another prime. Denote with*

$$d_q(r) := \min\{i \in \mathbb{N} : r \mid q^i - 1\}.$$

So $d_q(r)$ is the order of q modulo r.

Lemma 3.4 *Let q be a power of the prime p and $r \neq p$ another prime. Then $d_q(r)|r-1$ by Lagrange.*

3.2 Facts from group theory

Lemma 3.5 *Let G be a group and $a \in G$ some involution. If a inverts in $G/O_2(G)$ some element of odd prime order p, then a inverts in G some element of order p.*

Proof. This is 8.1 (1) of [Asch], a consequence of the Baer-Suzuki-theorem. □

Lemma 3.6 *Assume p is an odd prime, a is an involution in G, X is an a-invariant subgroup of G and $\overline{X} = X/O_2(X) = \overline{Y} \times \overline{Y}^a$ for some $Y \leq X$ with $p \in \pi(Y)$. Then a inverts an element of order p in X.*

Proof. This is 8.2 of [Asch]. By 3.5 w.l.o.g. $G = \langle a, X\rangle$ and $O_2(G)=1$. Let $y \in Y$ be of order p. Then yy^{-a} is inverted by a. □

3.3 Properties of alternating and sporadic groups

These lemmata seem quite trivial, but have powerful implications on the non-existence of certain loops.

Lemma 3.7 *Let $G \cong \mathrm{Alt}_n$ and $x \in G$ of odd prime order p.*

(1) $O_p(C_G(x))$ contains p-cycles.

(2) If x is a p-cycle, then:

(a) If $p + p < n$, then the commuting graph on x^G is connected.

(b) $F^*(C_G(x)) \cong \langle x \rangle \times A_{n-p}$, unless $n - p = 4$.

(c) If p is not a Fermat prime, then $|N_G(\langle x \rangle) : C_G(x)|$ is divisible by some odd prime r dividing $p - 1$.

(d) If $p + 3 \leq n$, then $C_G(x)$ contains a 3-cycle.

Proof. The centralizer of an element of order p acts on the fixed points and permutes the cycles of lenght p. This gives (1),(2b) and (2d). For (2c) we observe, that in Σ_n all powers of x are conjugate, as they have the same cycle structure. Remains (2a): For a p-cycle x let $M(x) \subseteq \{1, ..., n\}$ be the orbit of length p. Now, if for p-cycles x, y: $|M(x) \cap M(y)| = p - 1$, then x, y are connected in the commuting graph: Since $|M(x) \cup M(y)| = p + 1 \leq n - p$, some p-cycle z exists with $M(x) \cap M(z) = \emptyset = M(y) \cap M(z)$, so $[x, z] = 1 = [y, z]$. But now, given any two p-cycles x, y, we can find p-cycles z_i with: $z_0 := x$, $z_k = y$ and $|M(z_i) \cap M(z_{i+1})| = p - 1$ for $0 \leq i < k$. Therefore the commuting graph on x^G is connected. □

Lemma 3.8 *Let G be a sporadic simple group and $x \in G$ an element of prime order $p > 2$. Then $|N_G(\langle x \rangle) : C_G(x)|$ is not a 2-power, unless p is a Fermat prime. Thus in the non-Fermat case there exists an odd prime $s \mid p - 1$ with $s \mid |N_G(\langle x \rangle) : C_G(x)|$.*

Proof. This lemma can easily verified using the character tables in [ATLAS]. The index $|N_G(\langle x \rangle) : C_G(x)|$ determines the number n_x of conjugacy classes of elements of order p in $\langle x \rangle$. Recall, that $n_x = \frac{p-1}{|N_G(\langle x \rangle):C_G(x)|}$ and can be read off from the character tables, as the corresponding conjugacy classes have the same size and are powers of each other.) □

4 Commuting graphs

The purpose of commuting graphs is to concentrate informations about certain simple groups in useful properties, which can be applied for instance in problems about Loops.
Originally we studied commuting graphs in simple groups, to divide a long proof into short parts. But we also get a better understanding of our original problem: We see in this section, how simple groups look like from the inside.

Later we get results, how groups to loops have to look like.
The combination of these results gives then our final result, that these structures rarely fit together.

We use the following sources about maximal subgroups of groups of Lie type: [KL] for classical groups , [LSS] and [CLSS] for exceptional groups of Lie type. Furthermore the papers [Coo], [K3D4] and [Malle] were useful.

Definition 4.1 *Let G be a finite group and $X \subseteq G$ a normal subset, so for all $x \in X, g \in G : x^g \in X$.*
The undirected graph $\Gamma_{X,G} = \Gamma_X$ is the graph on X with edges (x, y) iff $[x, y] = 1 \in G$.
For $x \in X$ let $\mathcal{C}_x \subseteq X$ be the connected component of Γ_X containing x.
Furthermore let H_x be the stabilizer of $\mathcal{C}_x$ in G.
For some integer n let $\pi(n)$ be the set of prime divisiors of n,
for a group G let $\pi(G) := \pi(|G|)$.
For G a group and ρ a set of integers let $\mathcal{E}_\rho(G) := \{x \in G | o(x) \in \rho\}$.
The graph $\Gamma_{\mathcal{O}}$ is defined as above on the set $\mathcal{O} := \mathcal{E}_{\pi(G)-\{2\}}(G)$, the set of all elements of odd prime order. Similarly we define for $\rho \subseteq \pi(G)$ the graph Γ_ρ on $\mathcal{E}_\rho(G)$.

The following lemma contains trivial observations on commuting graphs, which we later use freely without reference.

Lemma 4.2 *(1) G acts as a group of automorphisms on Γ_X.*

(2) Let $g \in G$. Then x^g and x are connected or equal, iff $g \in H_x$.

(3) $\mathcal{C}_x \subseteq H_x$.

A special case is, if a connected component of $\Gamma_{\mathcal{O}}$ contains a G-conjugacy class:

Lemma 4.3 *Let $X = \mathcal{E}_{\pi_0}(G)$ for a subset $\pi_0 \subseteq \pi(G)$. Suppose there exists some $x \in G$, such that $x^G \subseteq \mathcal{C}_x$, where $\mathcal{C}_x$ is the connected component of x in Γ_X.*
If $y \in \mathcal{C}_x$ with $o(y) = r$, then $\mathcal{E}_{\{r\}}(G) \subseteq \mathcal{C}_x$.

Proof. Let $z \in X$ be of order r. We show, that x and z are connected in Γ_X.
Let $R \in \mathrm{Syl}_r(G)$ with $z \in R$ and $g \in G$ with $y^g \in R$.
Then y^g and z are connected via $Z(R) \neq 1$, as $\mathcal{E}_r(G) \subseteq X$.
Therefore $(y, z^{g^{-1}})$,$(x, z^{g^{-1}})$ and (x^g, z) are connected.
As $x^G \subseteq \mathcal{C}_x$, (x, x^g) are connected, so (x, z) are connected. □

Corollary 4.4 *Let $\emptyset \neq X \subseteq \mathcal{O}$ a subset, such that Γ_X is connected and for all $g \in G, x \in X$: $x^g \in X$.*
Then a subset $\rho \subseteq \pi(G) - \{2\}$ with $\{o(x) : x \in X\} \subseteq \rho$ exists, such that $\mathcal{E}_\rho(G)$ is the connected component in $\Gamma_{\mathcal{O}}$ containing X.

Definition 4.5 *We call a connected component of $\Gamma_{\mathcal{O}}$* **big***, if it contains a conjugacy class of G and* **small** *otherwise.*

Lemma 4.6 *Let G be a group, p a prime and $P \in \mathrm{Syl}_p(G)$. Suppose there exists a set $\mathcal{U}$ of subgroups of P, with $G = \langle O^{p'}(N_G(U)) : U \in \mathcal{U} \rangle$.*
Then the commuting graph on $\mathcal{E}_p(G)$ is connected.

Proof. Let $x \in Z(P)$, $o(x) = p$ and $\mathcal{C}_x$ the connected component of the commuting graph on $\mathcal{E}_p(G)$ containing x. Then $P \leq H_x$, so for all $U \in \mathcal{U} : U \leq H_x$. Let $Q \in \mathrm{Syl}_p(N_G(U))$ for $U \in \mathcal{U}$. Then $Z(Q) \cap U \neq 1$, so $Q \leq H_x$. Therefore $O^{p'}(N_G(U)) \leq H_x$. Now by assumption $G \leq H_x$. As the graph has $|G : H_x|$ connected components, it is connected. □

Corollary 4.7 *Let G be a simple group of Lie type in characteristic $p > 2$, q a p-power, but G not of type $A_1(q)$, ${}^2A_2(q)$ or ${}^2G_2(q)$.*
Then $\Gamma_{\mathcal{O}}$ has a big connected component containing all elements of order p.

Proof. This follows from the Steinberg relations for G and 4.6. □

4.1 Connected conjugacy classes

We determine some conjugacy classes x^G in some groups of Lie type in characteristic 2, such that Γ_{x^G} is connected. In this section q is a 2-power.

Lemma 4.8 *Let $G \cong \mathrm{PSL}_3(q)$ for $q > 4$, q even. Then G has a connected conjugacy class of elements of order r for $r > 3$ some prime divisor of $q - 1$.*

Proof. Notice, that such an r exists. Then there exist elements $a, b \in \mathrm{GF}(q)$ with $1 \neq a$, $a^r = 1$ and $b^2 = \frac{1}{a}$.
Let x_1 the image of $\mathrm{Diag}(a, b, b)$ in G and x_2 the image of $\mathrm{Diag}(b, b, a)$ in G. Then $[x_1, x_2] = 1$, x_1, x_2 are conjugate in G and $\langle x_1, x_2 \rangle \cong \mathbb{Z}_r \times \mathbb{Z}_r$.
Moreover H_{x_1}, the stabilizer of the connected component of x_1 in $\Gamma_{x_1^G}$, contains: $C_G(x_1) \cong \mathbb{Z}_{q_1} \times \mathrm{PSL}_2(q)$, $C_G(x_2) \cong \mathbb{Z}_{q_1} \times \mathrm{PSL}_2(q)$ and $N_G(\langle x_1, x_2 \rangle) \cong (\mathbb{Z}_{q_1} \times \mathbb{Z}_{q-1}) : \Sigma_3$ with $q_1 := \frac{q-1}{(q-1,3)}$. From the list of maximal subgroups therefore $H_{x_1} = G$ and $\Gamma_{x_1^G}$ is connected. □

Lemma 4.9 *Let $G \cong \mathrm{PSL}_4(q)$ for $q > 4$, q even. Then G has a connected conjugacy class of elements of order r for $r > 3$ some prime divisor of $q - 1$.*

Proof. Notice, that such an r exists. Then there exist elements $a, b \in \mathrm{GF}(q)$ with $1 \neq a$, $a^r = 1$ and $b^3 = \frac{1}{a}$.
Let x_1 the image of $\mathrm{Diag}(a, b, b, b)$ in G, x_2 the image of $\mathrm{Diag}(b, a, b, b)$ in G and x_3 the image of $\mathrm{Diag}(b, b, a, b)$ in G.
Then $[x_1, x_2] = 1 = [x_1, x_3] = [x_2, x_3]$, the x_1, x_2, x_3 are conjugate in G and $\langle x_1, x_2, x_3 \rangle \cong \mathbb{Z}_r \times \mathbb{Z}_r \times \mathbb{Z}_r$.

Moreover H_{x_1}, the stabilizer of the connected component of x_1 in $\Gamma_{x_1^G}$, contains:
$C_G(x_1) \cong \mathbb{Z}_{q-1}.\mathrm{PSL}_3(q).\mathbb{Z}_d$, $C_G(x_2) \cong \mathbb{Z}_{q-1}.\mathrm{PSL}_3(q).\mathbb{Z}_d$, $C_G(x_3) \cong \mathbb{Z}_{q-1}.\mathrm{PSL}_3(q).\mathbb{Z}_d$ and $N_G(\langle x_1, x_2, x_3 \rangle) \cong (\mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}) : \Sigma_4$ with $d = (q-1, 3)$.
From the list of maximal subgroups therefore $H_{x_1} = G$ and $\Gamma_{x_1^G}$ is connected. □

Lemma 4.10 *Let $G \cong \mathrm{PSL}_n(q)$ for $n \geq 5$.*
Then G has a connected conjugacy class of elements of order r for r any prime divisor of $q^2 - 1$.

Proof. There exists a maximal subgroup M of type $L_2(q) \oplus L_{n-2}(q)$. Let M_1, M_2 be the components of M with $M_1 \cong \mathrm{SL}_2(q)$ and $M_2 \cong \mathrm{SL}_{n-2}(q)$.
As $\mathrm{SL}_n(q)$ acts transitively on 2-subspaces of its natural module, there exists some $g \in G$, such that $M_1^g \subseteq M_2$. Let r be a prime divisor of $q^2 - 1$ and $x \in M_1$ be some element of order r. We claim, that the conjugacy class x^G is connected:
Let $y := x^G \in M_2$. Then H_x contains $C_G(x)$, so M_2 and $C_G(y)$, so M_1, so $M \leq H_x$. Furthermore $g \in H_x$, but $g \notin M$, so $H_x = G$ and Γ_{x^G} is connected.
□

Lemma 4.11 *Let $G \cong \mathrm{PSU}_3(q)$ for $q > 2$, q even. Then G has a connected conjugacy class of elements of order r for r some prime divisor of $q + 1$.*

Proof. For $q > 8$ and $q = 4$ an $r > 3$ exists with $r \mid q + 1$. Then there exist elements $a, b \in \mathrm{GF}(q^2)$ with $1 \neq a$, $a^r = 1$ and $b^2 = \frac{1}{a}$.
For $q = 8$ there exist elements $a, b \in \mathrm{GF}(64)$ with $1 \neq a$, $a^9 = 1 \neq a^3$ and $b^2 = \frac{1}{a}$. Set $r = 3$ in this case.
Let x_1 the image of $\mathrm{Diag}(a, b, b)$ in G and x_2 the image of $\mathrm{Diag}(b, b, a)$ in G. Then $[x_1, x_2] = 1$, x_1, x_2 are conjugate in G and $\langle x_1, x_2 \rangle \cong \mathbb{Z}_r \times \mathbb{Z}_r$.
Moreover H_{x_1}, the stabilizer of the connected component of x_1 in $\Gamma_{x_1^G}$, contains: $C_G(x_1) \cong \mathbb{Z}_{q_1} \times \mathrm{PSL}_2(q)$, $C_G(x_2) \cong \mathbb{Z}_{q_1} \times \mathrm{PSL}_2(q)$ and $N_G(\langle x_1, x_2 \rangle) \cong (\mathbb{Z}_{q_1} \times \mathbb{Z}_{q+1}) : \Sigma_3$ with $q_1 := \frac{q+1}{(q+1,3)}$. From the list of maximal subgroups therefore $H_{x_1} = G$ and $\Gamma_{x_1^G}$ is connected. □

Lemma 4.12 *Let $G \cong \mathrm{PSU}_4(q)$ for $q > 4$, q even. Then G has a connected conjugacy class of elements of order r for r some prime divisor of $q + 1$.*

Proof. For $q > 8$ an $r > 3$ exists with $r \mid q + 1$. Then there exist elements $a, b \in \mathrm{GF}(q^2)$ with $1 \neq a$, $a^r = 1$ and $b^3 = \frac{1}{a}$.
For $q = 8$ there exist an element $b \in \mathrm{GF}(64)$ with $1 \neq b$, $b^3 = 1$. Set $r = 3$ and $a = 1 \in \mathrm{GF}(64)$ in this case.
Let x_1 the image of $\mathrm{Diag}(a, b, b, b)$ in G, x_2 the image of $\mathrm{Diag}(b, a, b, b)$ in G and x_3 the image of $\mathrm{Diag}(b, b, a, b)$ in G.
Then $[x_1, x_2] = 1 = [x_1, x_3] = [x_2, x_3]$, the x_1, x_2, x_3 are conjugate in G and $\langle x_1, x_2, x_3 \rangle \cong \mathbb{Z}_r \times \mathbb{Z}_r \times \mathbb{Z}_r$.

Moreover H_{x_1}, the stabilizer of the connected component of x_1 in $\Gamma_{x_1^G}$, contains:
$C_G(x_1) \cong \mathbb{Z}_{q+1}.\mathrm{PSU}_3(q).\mathbb{Z}_d$, $C_G(x_2) \cong \mathbb{Z}_{q+1}.\mathrm{PSU}_3(q).\mathbb{Z}_d$, $C_G(x_3) \cong \mathbb{Z}_{q+1}.\mathrm{PSU}_3(q).\mathbb{Z}_d$ and $N_G(\langle x_1, x_2, x_3 \rangle) \cong (\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) : \Sigma_4$ with $d = (q + 1, 3)$.
From the list of maximal subgroups therefore $H_{x_1} = G$ and $\Gamma_{x_1^G}$ is connected. □

Lemma 4.13 *Let $G \cong \mathrm{PSU}_n(q)$ for $n \geq 5$.*
Then G has a connected conjugacy class of elements of order r for r any prime divisor of $q^2 - 1$.

Proof. There exists a maximal subgroup M of type $U_2(q) \perp U_{n-2}(q)$. Let M_1, M_2 be the components of M with $M_1 \cong \mathrm{SL}_2(q)$ and $M_2 \cong \mathrm{SU}_{n-2}(q)$.
As $\mathrm{SU}_n(q)$ acts transitively on nondegenerated 2-subspaces of its natural module, there exists some $g \in G$, such that $M_1^g \subseteq M_2$.
Let r be a prime divisor of q^2-1 and $x \in M_1$ be some element of order r. We claim, that the conjugacy class x^G is connected:
Let $y := x^G \in M_2$. Then H_x contains $C_G(x)$, so M_2 and $C_G(y)$, so M_1, so $M \leq H_x$. Furthermore $g \in H_x$, but $g \notin M$, so $H_x = G$ and Γ_{x^G} is connected.
□

Lemma 4.14 *Let $G \cong \mathrm{Sp}_4(q)$ for $q > 2$.*
Then G has no connected conjugacy class of elements of odd order, though the graph on $\mathcal{E}_{\pi(q^2-1)}(G)$ is connected.

Proof. We show first, that the commuting graph on $\mathcal{E}_{\pi(q^2-1)}(G)$ is connected.
There exist two classes of maximal subgroups M_1,M_2 of type $(\mathrm{PSL}_2(q)\times\mathrm{PSL}_2(q)).2$, which are interchanged by a graph automorphism.
We can choose M_1 to be of type $(\mathrm{Sp}_2(q) \perp \mathrm{Sp}_2(q)) : 2$, the normalizer of a 2-space decomposition and M_2 to be of type $O_4^+(q)$.
Notice, that these two subgroups contain Sylow-subgroups for all primes dividing q^2-1.
Let r be a prime dividing q^2-1 and $x \in M_1$ some element of order r, such that the centralizer of x contains a $\mathrm{PSL}_2(q)$-component. Then H_x contains M_1, as the commuting graph of elements of odd prime order in M_1 is connected. By conjugation of Sylow-groups we may assume, that $x \in M_2$ too. The commuting graph of elements of odd order in M_2 is connected too, so $M_2 \leq H_x$, therefore $G = \langle M_1, M_2\rangle \leq H_x$.
By Sylow's Theorem G is transitive on the Sylow-r-subgroups. The elements of order r in one Sylow-r-subgroup are in only one connected component of the graph. Therefore G acts transitively on the connected components. As $|G : H_x| = 1$, there is only one component, so the graph is connected.

Further analysis reveals, that for the elements of odd order in $\mathcal{E}_{\pi(q^2-1)}(G)$ we have only the following isomorphism types for a centralizer:
tori of size $(q-\varepsilon)^2$ or subgroups of type $(q-\varepsilon) \times L_2(q)$.
But only the classes with centralizer of type $(q-\varepsilon) \times L_2(q)$ could be connected.
Let $x \in G$ be an element of prime order r with $r \mid q^2-1$.
The component X_1 from $C_G(x)$ has a unique centralizing component X_2 with $x \in X_2$, so x is in a unique group $X = X_1X_2$ of type $\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)$, which is either in a subgroup conjugate to M_1 or to M_2.
By Burnside's Lemma, as Sylow-r-subgroups are abelian, all G-conjugates of x in a Sylow-r-subgroup R are already conjugate in $N := N_G(R)$. From the list of maximal subgroups we conclude $N_G(R) \leq N_G(X)$ and $|N_G(R) : C_G(R)| = 8$.
As $|C_N(x)| = 2|C_G(R)|$, there are exactly 4 conjugates of x in R:
R has two subgroups of order r, which are intersections with the components X_1, X_2 of X. Each of these subgroups contains two conjugates of x. In particular for all $y \in x^G \cap R$ we have: $C_G(y) \leq X$, so the commuting graph on x^G is not connected.
□

Lemma 4.15 *Let $G \cong \mathrm{Sp}_{2n}(q)$ for $n \geq 3$.*
Then G has a connected conjugacy class of elements of order r for r any prime divisor of $q^2 - 1$.

Proof. There exists a maximal subgroup M of type $\mathrm{Sp}_2(q) \perp \mathrm{Sp}_{2n-2}(q)$. Let M_1, M_2 be the components of M with $M_1 \cong \mathrm{SL}_2(q)$ and $M_2 \cong \mathrm{Sp}_{2n-2}(q)$.
As $\mathrm{Sp}_{2n}(q)$ is transitive on nondegenerate 2-spaces, there exists some $g \in G$, such that $M_1^g \subseteq M_2$.
Let r be a prime divisor of $q^2 - 1$ and $x \in M_1$ be some element of order r. We claim, that the conjugacy class x^G is connected:
Let $y := x^G \in M_2$. Then H_x contains $C_G(x)$, so M_2 and $C_G(y)$, so M_1, so $M \leq H_x$. Furthermore $g \in H_x$, but $g \notin M$, so $H_x = G$ and Γ_{x^G} is connected. $\square$

Lemma 4.16 *Let $G \cong \Omega_{2n}^{\varepsilon}(q)$ for $n \geq 3$, $\varepsilon \in \{+, -\}$.*
Then G has a connected conjugacy class of elements of order r for r any prime divisor of $q^2 - 1$.

Proof. There exist maximal subgroups M^+ of type $O_2^+(q) \perp O_{2n-2}^{\varepsilon}(q)$ and M^- of type $O_2^-(q) \perp O_{2n-2}^{-\varepsilon}(q)$.
Let r be an odd prime divisior of $q - 1$, so $q > 2$.
Then M^+ contains a cyclic normal subgroup M_1^+ of size $q - 1$. Furthermore there exists a $g \in G$, such that $M_2^+ := E(M^+) \cong \Omega_{2n-2}^{\varepsilon}(q)$ contains $(M_1^+)^g$. Let $x \in M_1^+$ be an element of order r. Then the conjugacy class of x^G is connected:
Let $y := x^G \in M_2^+$. Then H_x contains $C_G(x)$, so M_2^+ and $C_G(y)$, so M_1^+, so $M^+ \leq H_x$. Furthermore $g \in H_x$, but $g \notin M^+$, so $H_x = G$ and Γ_{x^G} is connected.
Let r be an odd prime divisior of $q + 1$.
Then M^- contains a cyclic normal subgroup M_1^- of size $q + 1$. Furthermore there exists a $g \in G$, such that $M_2^- := E(M^-) \cong \Omega_{2n-2}^{-\varepsilon}(q)$ contains $(M_1^-)^g$. Let $x \in M_1^-$ be an element of order r. Then the conjugacy class of x^G is connected:
Let $y := x^G \in M_2^-$. Then H_x contains $C_G(x)$, so M_2^- and $C_G(y)$, so M_1^-, so $M^- \leq H_x$. Furthermore $g \in H_x$, but $g \notin M^-$, so $H_x = G$ and Γ_{x^G} is connected. $\square$

Lemma 4.17 *Let $G \cong G_2(q)$ for $q > 2$, q even.*
Then G has a connected conjugacy class of elements of order r for $r \neq 3$ any prime divisor of $q^2 - 1$.

Proof. Let $q > 4$. We use the list of maximal subgroups in [Coo]. Let $\varepsilon \in \{+, -\}$ with r a divisor of $q - \varepsilon$. There exist two classes of subgroups of type $(q - \varepsilon) \times \mathrm{PSL}_2(q)$ in a maximal subgroup of type $\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)$. Let C_1, C_2 be representatives of the two classes and $x_1 \in Z(C_1)$, $x_2 \in Z(C_2)$ with $o(x_1) = r = o(x_2)$.
Notice, that there is only one class of maximal subgroups M isomorphic to $A_2^{\varepsilon}(q).2 \cong \mathrm{SL}_3^{\varepsilon}(q).2$ for each ε. We can choose $i \in \{1, 2\}$, such that M does not contain a conjugate of C_i, as M contains a unique class of such subgroups.
Now H_{x_i} contains C_i, but also a subgroup N of shape $(q - \varepsilon)^2 : D_{12} \leq M$. So $H_{x_i} \geq \langle C_i, N \rangle \geq G$ and the class x_i^G is connected. For $q = 4$ we use [ATLAS]. Let $x \in G$ be of order 5. There exists a subgroup $\mathrm{PSU}_3(4)$ and a subgroup

$\mathrm{Alt}_5 \times \mathrm{Alt}_5$. Both subgroups are contained in H_x, as they both contain Sylow-5-subgroups and big connected components containing elements of order 5, see 4.11. Therefore $G = H_x$. $\square$

Lemma 4.18 *Let $G \cong {}^3D_4(q)$ for $q > 4$, q even. Then G has a connected conjugacy class of elements of order r for $r \neq 3$ a prime divisor of $q^2 - 1$.*

Proof. We use the list of semisimple centralizers and maximal subgroups in [K3D4]. Let $\varepsilon \in \{+, -\}$ with r a divisor of $q - \varepsilon$. There exists a subgroup M_1 of type $\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q^3)$. Let $x \in M_1$ with $C_G(x) \cong (q - \varepsilon) \times L_2(q^3)$. Then $C_G(x) \leq H_x$. But there exists a subgroup M_2 of type $(q^2{+}\varepsilon q{+}1).A_2^{\varepsilon}(q).f_{\varepsilon}.2$ with $f_{\varepsilon} = (3, q{-}\varepsilon)$, which contains a torus normalizer N of shape $(\mathbb{Z}_{q^3-\varepsilon}{\times}\mathbb{Z}_{q-\varepsilon}).D_{12}$. Now H_x contains such a torus normalizer, thus $H_x \geq \langle C_G(x), N\rangle \geq G$, so x^G is connected. $\square$

Lemma 4.19 *Let $G \cong {}^2F_4(q)$ for $q > 2$.
Then G has a connected conjugacy class of elements of order r for r any prime divisor of $q^2 + 1$.*

Proof. We use the list of maximal and maximal local subgroups in [Malle].
Notice, that $5 \mid q^2 + 1$ in this case.
We can factorize $q^2 + 1 = (q - \sqrt{2q} + 1)(q + \sqrt{2q} + 1)$. Let $\varepsilon \in \{+, -\}$, such that r is a divisor of $q + \varepsilon\sqrt{2q} + 1$ and let $x \in G$ be an element of order r with $C_G(x) \cong \mathbb{Z}_{q+\varepsilon\sqrt{2q}+1} \times {}^2B_2(q)$. Such an element exists in a maximal subgroup M_1 of type $({}^2B_2(q) \times {}^2B_2(q)).2$. Notice, that the outer involution interchanges the components, as ${}^2B_2(q)$ has no outer automorphism of order 2. This gives $M_1 \leq H_x$.
But there exists a subgroup N of type $(\mathbb{Z}_{q+\varepsilon\sqrt{2q}+1} \times \mathbb{Z}_{q+\varepsilon\sqrt{2q}+1}).[96]$, which is maximal for $q > 8$ or $r > 5$, while contained in ${}^2F_4(2)$ for $q = 8$ and $r = 5$.
Then $N \leq H_x$, so from the list of maximal subgroups $H_x = G$ and x^G is connected. $\square$

Lemma 4.20 *Let $G \cong F_4(q)$ for q even. Then G has a connected conjugacy class of order r for r any prime divisor of $q^2 - 1$.*

Proof. By [LSS], G has two classes of maximal subgroups M_1, M_2 isomorphic to $\mathrm{Sp}_8(q) \cong C_4(q)$.
By 4.15, each M_i has a connected conjugacy class for a prime $r \mid q^2 - 1$.
We may choose $x \in M_1$ of order r with $C_G(x) = C_{M_1}(x) \cong (q - \varepsilon) \times \mathrm{Sp}_6(q)$ for for some $\varepsilon \in \{+, -\}$. (The fact, that $C_G(x) = C_{M_1}(x)$ comes from the list of maximal subgroups, which contain a centralizer, see the main theorem of [CLSS].)
Then x is contained in a torus T of type $(q - \varepsilon)^4$, with $W(F_4)$, the full Weyl group, acting on it. As this torus normalizer is not contained in $\mathrm{Sp}_8(q)$ (but in $\Omega_8^+(q).\Sigma_3$), we have $H_x = G$:
H_x contains M_1 as seen in 4.15 and $N_G(T)$, but $\langle M_1, N_G(T)\rangle = G$, as M_1 is a maximal subgroup not containing $N_G(T)$. Therefore the commuting graph on x^G is connected. $\square$

Lemma 4.21 *Let $G \cong E_6(q), {}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ for q even. Then G has a connected conjugacy class of order r for r any prime divisor of q^2-1.*

Proof. By [LSS] there are maximal subgroups M with components $M_1 \cong \mathrm{PSL}_2(q)$ and $M_2 \cong \mathrm{PSL}_6(q), \mathrm{PSU}_6(q), \Omega^+_{12}(q)$ resp. $E_7(q)$, such that a $g \in G$ exists with $M_1^g \subseteq M_2$. The existence of g and these subgroups can also be seen from the Steinberg relations.
Let r be a prime divisor of q^2-1 and $x \in M_1$ be some element of order r. We claim, that the conjugacy class x^G is connected:
Let $y := x^G \in M_2$. Then H_x contains $C_G(x)$, so M_2 and $C_G(y)$, so M_1, so $M \leq H_x$. Furthermore $g \in H_x$, but $g \notin M$, so $H_x = G$ and Γ_{x^G} is connected. □

4.2 Connected components in $\Gamma_{\mathcal{O}}$

We unify results in even and odd characteristic. Notice, that we consider 2 NOT as a Fermat prime.

Lemma 4.22 *Let $G \cong \mathrm{PSL}_3(q)$. Then one of the following holds:*

(i) $\frac{q-1}{(q-1,3)}$ *is not a 2-power. Then $\Gamma_{\mathcal{O}}$ has a unique big connected component, containing all elements of order r with r some odd prime divisor of $(q-1)q(q+1)$.*

(ii) $\frac{q-1}{(q-1,3)}$ *is a 2-power and q is odd. Then $\Gamma_{\mathcal{O}}$ has a unique big connected component, which contains only elements of order p.*

(iii) *$q \in \{2,4\}$ and $\Gamma_{\mathcal{O}}$ has no big connected componet.*

Proof. If q is even, $q > 4$, by 4.8 there is a connected conjugacy class y^G, $o(y) = r$ for $r \neq 3$ some prime divisor of $q-1$. As $q-1$ is not a 3-power, such a y exists. By 4.4 and construction of y, $\mathcal{C}_y$ contains $\mathcal{E}_{\pi(q^2-1)}(G)$.
If q is odd, by 4.7, $\mathcal{E}_p(G)$ is connected. Centralizers of semisimple elements are either tori or of type $\frac{q-1}{(q-1,3)} \cdot L_2(q).2$. If $\frac{q-1}{(q-1,3)}$ is a 2-power, centralizers of semisimple elements contain a characteristic abelian subgroup, which contains all elements of odd order of this centralizer. Therefore the connected component $\mathcal{C}_x$ of a semisimple element x contains only the elements of odd prime order of $C_G(x)$, thus (ii) holds. If $\frac{q-1}{(q-1,3)}$ is not a 2-power, we may find some element $x \in G$, $o(x) = r$ for some odd prime $r \neq p$ such that $C_G(x)$ contains a component isomorphic to $\mathrm{SL}_2(q)$. We may find x also in the normalizer of a torus T of size $\frac{(q-1)^2}{(q-1,3)}$, which contains a Σ_3 acting on top of the torus. Therefore H_x contains $C_G(x)$ and $N_G(T)$. As $G = \langle C_G(x), N_G(T)\rangle$, the connected component $\mathcal{C}_x$ is big and we get (i).
The case (iii) follows from the centralizer size in [ATLAS]. Notice, that for all q a torus of size $\frac{q^2+q+1}{(q-1,3)}$ is self centralizing, so gives small connected components. □

Lemma 4.23 *Let $G \cong \mathrm{PSL}_4(q)$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) q is a Mersenne prime and $d_q(r) = 4$.

(ii) q is a Fermat prime or $q = 2$ and $d_q(r) = 3$.

Proof. Suppose $q > 4$ is even. By 4.9, there is a connected conjugacy class y^G, $o(y) = r$ for $r \neq 3$ some prime divisor of $q-1$. As $q-1$ is not a 3-power, such a y exists. Let $\rho \subseteq \pi(G)$ from 4.4. By construction of y, $\pi(\mathrm{PSL}_3(q)) - \{2\} \subseteq \rho$. There exists a subgroup $\mathbb{Z}_{\frac{q^4-1}{q-1}}$ from the $\mathrm{GL}_2(q^2)$. As it contains elements of order s for s some prime divisor of $q+1$, $\Gamma_{\mathcal{O}}$ is connected.

In case $q = 4$, we have to consider the primes $3 = q-1, 5 = q+1$, $7 = \frac{q^3-1}{9}$ and $17 = q^2+1$. There are abelian subgroups of sizes $3 \cdot 5$, $5 \cdot 17$, $3 \cdot 7$, so the stabilizer of a connected component contains Sylow-subgroups for all odd primes. As no such proper subgroup exist, the graph $\Gamma_{\mathcal{O}}$ is connected.

In case $q = 2$, we use the isomorphism $\mathrm{SL}_4(2) \cong \mathrm{Alt}_8$ and 3.7.

There exists a subgroup M_1 of type $L_2(q) \oplus L_2(q)$. If q is odd, by 4.7, a big connected component containing all elements of odd prime order s with s a divisor of $|\mathrm{PSL}_{n-2}(q)|$ exists.
In case of $d_q(r) = 3$, let M_2 be a subgroup of type $L_1(q) \oplus L_3(q)$. The structure of M_2 is described by Proposition 4.1.4 of [KL]. In particular $Z(F^*(M_2))$ contains elements of odd order, if $\frac{q-1}{(q-1,4)}$ is not a 2-power. This is exactly the case, if q is not a Fermat prime. Then $Z(F^*(M_2))$ contains elements of order s for s some odd prime divisor of $q-1$. As the torus of type q^3-1 is contained in M_2, we get x contained in the big connected component, if $d_q(r) = 3$ and q not Fermat. If q is a Fermat prime, we have (ii).
If $d_q(r) = 4$, let M_3 be a maximal subgroup of type $L_2(q^2)$. The structure of M_3 is described by Proposition 4.3.6 of [KL]. In particular $Z(F^*(M_3))$ has size $\frac{(q-1,2)(q^2-1)}{(q-1)(q-1,4)}$, so contains elements of odd prime order, if q is not a Mersenne prime. As $F^*(M_3)$ contains a torus of type q^4-1, either x is contained in the big connected component or (ii) holds. □

Lemma 4.24 *Let $G \cong \mathrm{PSL}_n(q)$ for $n \geq 5$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) $q = 3$, $n = 5$, $r = 5$.

(ii) $n-1$ is a prime, $\frac{q-1}{(q-1,n)}$ is a 2-power and $d_q(r) = n-1$

(iii) n is a prime and $d_q(r) = n$.

Proof. Let q odd. There exists a subgroup M_1 of type $L_2(q) \oplus L_{n-2}(q)$. By 4.7, a big connected component containing all elements of prime order s with s a divisor of $|\mathrm{PSL}_{n-2}(q)|$ exists.

If q is even, by 4.10, there is a connected conjugacy class y^G, $o(y) = r$ for r some prime divisor of $q^2 - 1$. Let $\rho \subseteq \pi(G)$ from 4.4.
As there exists a subgroup of type $\mathrm{GL}_2(q) \oplus \mathrm{GL}_{n-2}(q)$, $r \notin \pi(\mathrm{SL}_{n-2}(q))$.
So $d_q(r) \geq n - 1$ in both cases.

Suppose $r \mid q^n - 1$.
If n is a prime, we have (iii), so suppose $n = a \cdot b$ with $a \neq 1 \neq b$ and b a prime.
If n is not a 2-power, we choose b odd.
There exists a subgroup M_2 of type $L_{n/b}(q^b)$ in class $\mathcal{C}_3$.
By Proposition 4.3.6 of [KL], this subgroup is local with a cyclic normal subgroup of size $\frac{(q-1,n/a)(q^b-1)}{(q-1)(q-1,n)}$.
By Zsygmondy, some odd prime $t \mid q^b - 1$ exists with $d_q(t) = b$, unless $b = 2$ and q is a Mersenne prime. If $Z(F^*(M_2))$ contains elements of odd prime order, then as $F^*(M_2)$ contains a section isomorphic to $\mathrm{PSL}_2(q)$, and a torus of type $q^n - 1$, x is contained in the big connected component.
If n is a 2-power, then $n \geq 8$ and there exists a subgroup $M_3 \leq M_2$ of type $L_{n/4}(q^4)$.
Now $Z(F^*(M_3))$ has elements of odd order, as there exists a Zsygmondy-prime t with $d_q(t) = 4$. As $F^*(M_3)$ contains a torus of type $q^n - 1$ and $PSL_2(q)$-section, again x is in the big connected component.

Suppose now $r \mid q^{n-1} - 1$. There exists a subgroup M_1 of type $L_1(q) \oplus L_{n-1}(q)$. By Proposition 4.1.4 of [KL], $Z(F^*(M_4))$ contains elements of odd order s with $s \mid q - 1$, if $\frac{q-1}{(q-1,n)}$ is not a 2-power. In that case $F^*(M_4)$ contains a torus of type $q^{n-1} - 1$, so x is contained in the big connected component.
If $Z(F^*(M_4))$ contains no elements of odd prime order, $F^*(M_4)$ contains a component of type $L_{n-1}(q)$. The connected components of the commuting graph for $F^*(M_4)$ can be determined by induction. We have to distinguish the case $n = 5$, where we use 4.23 and $n > 5$.

If $n = 5$, the exception (ii) in 4.23 is handled by M_1. The exception (i) occurs only, if q is a Mersenne prime. The case $q = 3$ is (i). If $q > 3$, then $q - 1$ is divisible by 3, so $Z(F^*(M_4))$ contains elements of order 3 and x is in the big connected component.

If $n > 5$, exceptions of type (i) and (ii) in $F^*(M_4)$ are handled by the subgroup M_1. Exceptions of type (iii) in $F^*(M_4)$ produce (ii). $\square$

Lemma 4.25 *Let $G \cong \mathrm{PSU}_3(q)$ for $q > 2$. Then one of the following holds:*

- *(i)* $\frac{q+1}{(q+1,3)}$ *is not a 2-power. Then $\Gamma_{\mathcal{O}}$ has a unique big connected component, containing all elements of order r with r some odd prime divisor of $(q-1)q(q+1)$.*
- *(ii)* $\frac{q+1}{(q+1,3)}$ *is a 2-power. Then $\Gamma_{\mathcal{O}}$ has no big connected component.*

Proof. If q is even, by 4.11 there is a connected conjugacy class y^G, $o(y) = r$ for r some prime divisor of $q + 1$. By 4.4 and construction of y, $\mathcal{C}_y$ contains $\mathcal{E}_{\pi(q^2-1)}(G)$, so (i) holds.
So let q odd. The Borel subgroup B is strongly p-embedded, so $\mathcal{E}_p(G)$ is not

connected. Therefore big connected components contain semisimple elements. Centralizers of semisimple elements are either tori or of type $\frac{q+1}{(q+1,3)} \cdot L_2(q).2$. If $\frac{q+1}{(q+1,3)}$ is a 2-power, centralizers of semisimple elements contain a characteristic abelian subgroup, which contains all elements of odd order of this centralizer. Therefore the connected component $\mathcal{C}_x$ of a semisimple element x contains only the elements of odd prime order of $C_G(x)$, thus (ii) holds.
If $\frac{q+1}{(q+1,3)}$ is not a 2-power, we may find some element $x \in G$, $o(x) = r$ for some odd prime $r \neq p$ such that $C_G(x)$ contains a component isomorphic to $\mathrm{SL}_2(q)$. We may find x also in the normalizer of a torus T of size $\frac{(q+1)^2}{(q+1,3)}$, which contains a Σ_3 acting on top of the torus. Therefore H_x contains $C_G(x)$ and $N_G(T)$. As $G = \langle C_G(x), N_G(T)\rangle$, the connected component $\mathcal{C}_x$ is big and we get (i).
Notice, that for all q a torus of size $\frac{q^2-q+1}{(q+1,3)}$ is self centralizing. □

Lemma 4.26 *Let $G \cong \mathrm{PSU}_4(q)$ for $q > 2$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) q is a Fermat prime and $d_q(r) = 4$.

(ii) q is a Mersenne prime and $d_q(r) = 6$.

Proof. Consider first $q > 4$, q even. By 4.12, there is a connected conjugacy class y^G, $o(y) = r$ for r some prime divisor of $q+1$. Let $\rho \subseteq \pi(G)$ from 4.4. By construction of y, $\pi(\mathrm{PSU}_3(q)) - \{2\} \subseteq \rho$. There exists a subgroup $\mathbb{Z}_{\frac{q^4-1}{q+1}}$ in a Levi complement of a parabolic subgroup of type $q^4 : \mathrm{GL}_2(q^2)$. This subgroup contains elements of order s for s some prime divisor of $q-1$, so $\Gamma_{\mathcal{O}}$ is connected.

In case $q = 4$ we have to consider the primes $3 = q-1$, $5 = q+1$, $17 = q^2+1$ and $13 = \frac{q^3+1}{q+1}$. There are abelian subgroup of sizes $3 \cdot 5$, $5 \cdot 13$, $17 \cdot 3$, so the stabilizer of a connected component is of 2-power index. As no such proper subgroup exists, the graph $\Gamma_{\mathcal{O}}$ is connected.

So q is odd. There exists a subgroup M_1 of type $U_2(q) \perp U_2(q)$. By 4.7, a big connected component containing all elements of odd prime order s with s a divisor of $|\mathrm{PSL}_{n-2}(q)|$ exists.
So remain the cases $d_q(r) \in \{4, 6\}$.
In case of $d_q(r) = 6$, let M_2 be a subgroup of type $U_1(q) \perp U_3(q)$. The structure of M_2 is described by Proposition 4.1.4 of [KL]. In particular $Z(F^*(M_2))$ contains elements of odd order, if $\frac{q+1}{(q+1,4)}$ is not a 2-power. This is exactly the case, if q is not a Mersenne prime. Then $Z(F^*(M_2))$ contains elements of order s for s some odd prime divisor of $q+1$. As the torus of type q^3+1 is contained in M_2, we get x contained in the big connected component, if $d_q(r) = 6$ and q not Mersenne. If q is a Mersenne prime, we have (ii).
If $d_q(r) = 4$, let M_3 be a maximal subgroup of type $\mathrm{GL}_2(q^2)$ in class $\mathcal{C}_2$. The structure of M_3 is described by Proposition 4.2.4 of [KL]. In particular $Z(F^*(M_3))$ has size $\frac{(q-1)(q+1,2)}{(q+1,4)}$, so contains elements of odd prime order, if q is not a Fermat prime. As $F^*(M_3)$ contains a torus of type q^4-1, either x is contained in the big connected component or (ii) holds. □

Lemma 4.27 *Let $G \cong \mathrm{PSU}_n(q)$ for $n \geq 5$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) $q = 3$, $n = 5$, $r = 5$.

(ii) $n-1$ *is a prime,* $\frac{q+1}{(q+1,n)}$ *is a 2-power and* $d_q(r) = 2(n-1)$

(iii) n *is a prime and* $d_q(r) = 2n$.

Proof. If q even, by 4.13, there is a connected conjugacy class y^G, $o(y) = r$ for r some prime divisor of $q^2 - 1$.
If q is odd, by 4.7, a big connected component exists, containing all elements of order p.
There exists a subgroup M_1 of type $U_2(q) \perp U_{n-2}(q)$. Therefore a big connected component exists, which contains all elements of prime order s with s a divisor of $|\mathrm{PSU}_{n-2}(q)|$.
So $r \mid (q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})$.

Suppose n even and $r \mid q^n - 1$.
There exists a torus of type $q^n - 1$ in a subgroup M_2 of type $\mathrm{GL}_{n/2}(q^2).2$ in class $\mathcal{C}_2$. If $n/2$ is even, then $n/2 \geq 4$. Let t be some Zsygmondy prime with $d_q(t) = 4$.
If $n/2$ is odd and $(q, n) \neq (2, 6)$, let t be some Zsgmondy prime with $d_q(t) = n/2$. If $(q, n) = (2, 6)$ let $t = 3$. Now the torus of type $q^n - 1$ contains elements of order t, but $t \mid |\mathrm{SU}_{n/2}(q)|$, so x is in the big connected component.

Suppose n odd, but not a prime and $r \mid q^n + 1$. Let $n = a \cdot b$ with $a \neq 1 \neq b$ and b a prime.
There exists a subgroup M_3 of type $U_{n/b}(q^b)$ in class $\mathcal{C}_3$.
By Proposition 4.3.6 of [KL], this subgroup is local with a cyclic normal subgroup of size $\frac{(q+1,n/a)(q^b+1)}{(q+1)(q+1,n)}$.
By Zsygmondy, some odd prime $t \mid q^b - 1$ exists with $d_q(t) = b$.
So $Z(F^*(M_3))$ contains elements of odd prime order, while $F^*(M_3)$ contains a $\mathrm{PSL}_2(q)$-section and a torus of type $q^n + 1$. Therefore x is contained in the big connected component.
If n is a prime and $r \mid q^n + 1$, we have case (iii) or $r \mid q + 1$ and x is contained in the big connected component.

Suppose now $r \mid q^{n-1} - (-1)^n$. There exists a subgroup M_4 of type $U_1(q) \oplus U_{n-1}(q)$. By Proposition 4.1.4 of [KL], $Z(F^*(M_4))$ contains elements of odd order s with $s \mid q + 1$, if $\frac{q+1}{(q+1,n)}$ is not a 2-power. In that case $F^*(M_4)$ contains a torus of type $q^{n-1} - (-1)^n$, so x is contained in the big connected component.

If $Z(F^*(M_4))$ contains no elements of odd prime order, $F^*(M_4)$ contains a component of type $U_{n-1}(q)$. We use the knowledge about the commuting graph of that component, but have to distinguish the case $n = 5$ with $q > 2$, where we use 4.26, the case $n > 5$ and $(q, n) = (2, 5)$.

If $n = 5$, $q > 2$, the exception (ii) in 4.26 is handled by M_1. The exception (i) occurs only, if q is a Fermat prime. The case $q = 3$ is (i). If $q > 3$, then $q + 1$

is divisible by 3, so $Z(F^*(M_4))$ contains elements of order 3 and x is in the big connected component.

If $n > 5$, exceptions of type (i) and (ii) in $F^*(M_4)$ are handled by the subgroup M_1. Exceptions of type (iii) in $F^*(M_4)$ produce (ii).

If $n = 5$ and $q = 2$, elements of order 5 commute with elements of order 3, so are contained in the big connected component. □

Lemma 4.28 *Let $G \cong \mathrm{PSp}_4(q)$ for $q > 2$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then $r \mid q^2 + 1$.

Proof. If q is even, by 4.14, the subset $\mathcal{E}_{\pi(q^2-1)}(G)$ is connected.
If q is odd, by 4.7 there exists a big connected component, containing all elements of order p. There exists a subgroup of type $\mathrm{Sp}_2(q) \perp \mathrm{Sp}_2(q)$. Therefore, if $r \mid (q-1)q(q+1)$, then x is in the big connected component.
Notice, that self centralizing subgroups of size $\frac{q^2+1}{(q-1,2)}$ exist. □

Lemma 4.29 *Let $G \cong \mathrm{PSp}_{2n}(q)$ for $n \geq 3$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) n is a 2-power and $r \mid q^n + 1$.

(ii) n is a prime, q is a Fermat prime or $q = 2$ and $d_q(r) = n$.

(iii) n is a prime, q is a Mersenne prime and $d_q(r) = 2n$.

Proof. If q is odd, by 4.7 there exists a big connected component, containing all elements of order p.
If q is even, by 4.15 there is a big connected component containing all elements of prime order r for r a divisor of $q^2 - 1$.
There exists a subgroup M_1 of type $\mathrm{Sp}_2(q) \perp \mathrm{Sp}_{2n-2}(q)$. Therefore, if $r \mid |\mathrm{Sp}_{2n-2}(q)|$, then x is in the big connected component.
So $r \mid (q^n-1)(q^n+1)$. If n is even, then $r \mid q^n+1$, else $\mathrm{Sp}_n(q)$ contains elements of order r.
Let $n = a \cdot b$ with a a 2-power and b odd. If $b = 1$, we have (i). There exists a subgroup M_3 of type $\mathrm{Sp}_{2b}(q^a)$. This subgroup contains a subgroup M_4 of type $\mathrm{GL}_b(q^a)$, which contains a torus of type $q^n - 1$, and M_5 of type $\mathrm{GU}_b(q^a)$, which contains a torus of type q^n+1. The structure of M_4 is described by Proposition 4.2.5, while those of M_5 is described by 4.3.7 for q odd and 4.3.18 for q even. In particular $Z(F^*(M_4))$ contains no elements of odd order, iff q is a Fermat prime or $q = 2$ and $a = 1$.
Furthermore $Z(F^*(M_5))$ contains no elements of odd order, iff q is a Mersenne prime and $a = 1$. Both subgroups contain a $\mathrm{PSL}_2(q)$-section. If $a > 1$, then x is in the big connected component. Remains the case of $a = 1$ and b composite, so $b \geq 9$. We use 4.24 and 4.27 for the connected components of $F^*(M_4)$ and

$F^*(M_5)$ and get x is in the big connected component. $\square$

Lemma 4.30 *Let $G \cong \mathrm{P}\Omega_{2n}^+(q)$ for $n \geq 4$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) n is a prime, q a Fermat prime or $q = 2$ and $d_q(r) = n$.

(ii) $n-1$ is a prime, q is a Fermat prime or $q = 2$ and $d_q(r) = n-1$.

(iii) $n-1$ is a prime, q is a Mersenne prime and $d_q(r) = 2n-2$.

(iv) $n-1$ is a 2-power, q is a Mersenne prime and $d_q(r) = 2n-2$.

Proof. Notice, that the statement is also true for $n = 3$ by 4.23.
If q is odd, by 4.7 there exists a big connected component, containing all elements of order p. There exists a subgroup M_1 of type $O_3(q) \perp O_{2n-3}(q)$, so $r \mid (q^n-1)(q^{n-1}-1)(q^{n-1}+1)$.
If q is even, by 4.16, there exists a connected component containing all elements of prime order r for r some divisor of q^2-1. Let M_1 in class $\mathcal{C}_1$ of type $O_2^-(q) \perp O_{2n-2}^-(q)$. By the structure of M_1, elements of order r are in the big connected component, if r is a prime divisor of $|\Omega_{2n-2}^-(q)|$.
So remain primes r, which divide $(q^n-1)(q^{n-1}-1)$.
Let n even.

Suppose $r \mid q^n-1$. If q is odd, then $q^n-1||\Omega_{n+1}(q)|$ and $n+1 \leq 2n-3$. If q is even, then $q^n-1||\Omega_{n+2}^-|$ and $n+2 \leq 2n-2$ This implies, that x is in the big connected component by M_1 in both cases.

Suppose $r \mid q^{n-1}-1$. A torus of type $q^{n-1}-1$ can be found in a subgroup M_2 of type $\mathrm{GL}_n(q).2$ in class $\mathcal{C}_2$. The structure of M_2 is described by Proposition 4.2.7 of [KL]. If q is not a Fermat prime and $q > 2$, then $Z(F^*(M_2))$ contains elements of odd order, so x is in the big connected component.
We use 4.23 and 4.24 for the connected components of M_2, if q is a Fermat prime. Therefore $n-1$ is a prime and we have (ii).

Suppose $r \mid q^{n-1}+1$, so q odd. A torus of type $q^{n-1}+1$ is contained in a subgroup M_3 of type $\mathrm{GU}_n(q)$ in class $\mathcal{C}_3$. The structure of M_3 is described by Proposition 4.3.18. If q is not a Mersenne prime, $Z(F^*(M_3))$ contains elements of odd order and x is in the big connected component. We use 4.26 and 4.27 for the connected component of M_3, if q is a Mersenne prime. Therefore $n-1$ is a prime and we have (iii).

Let n odd.

Suppose $r \mid q^n-1$. A torus of type q^n-1 can be found in a subgroup M_4 of type $\mathrm{GL}_n(q).2$ in class $\mathcal{C}_2$. The structure of M_4 is described by Proposition 4.2.7 of [KL]. If q is not a Fermat prime and $q > 2$, then $Z(F^*(M_4))$ contains elements of odd order, so x is in the big connected component.
We use 4.23 and 4.24 for the connected components of M_4, if q is a Fermat prime or $q = 2$. Therefore n is a prime and we have (i).

Suppose $r \mid q^{n-1}-1$. If q is odd, then $q^{n-1}-1||\Omega_n(q)|$ and $n \leq 2n-3$. If q is even, then $q^{n-1}-1||\Omega_{n+1}^-|$ and $n+1 \leq 2n-2$ This implies, that x is in the big connected component by M_1 in both cases.

Suppose $r \mid q^{n-1}+1$, so q is odd. Let $n-1 = a \cdot b$ with a a 2-power and b odd. Notice, that $a \neq 1$ and $b = 1$ gives (iv). We can find a torus of type $q^{n-1}+1$ in a subgroup M_5 of type $\mathrm{GU}_b(q^a)$. This subgroup is contained in a subgroup M_6 of type $O^-_{2b}(q^a)$, which is contained in a subgroup M_7 of type $O^-_2(q) \perp O^-_{2n-2}(q)$. If q is not a Mersenne prime, then $Z(F^*(M_7))$ contains elements of odd order and x is in the big connected component. The structure of M_5 is described by Proposition 4.3.18 of [KL]. By Zsygmondy, $Z(F^*(M_5))$ contains elements of odd order s with $s \mid q^a+1$. As $q^{2a}+1 \mid |\Omega_{2a+1}(q)|$, $a \leq \frac{n}{3}$ and $n \geq 4$, we have $2a+1 \leq 2n-3$, so x is in the big connected component by M_1. $\square$

Lemma 4.31 *Let $G \cong \mathrm{P}\Omega^-_{2n}(q)$ for $n \geq 4$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) n is a prime, q a Mersenne prime and $d_q(r) = 2n$.

(ii) n is a 2-power and $d_q(r) = 2n$.

(iii) $n-1$ is a prime, $q = 3$ and $d_q(r) \in \{n-1, 2n-2\}$.

(iv) $n-1$ is a 2-power, q is a Fermat prime or $q = 2$ and $d_q(r) = 2n-2$.

Proof. Notice, that the statement is also true for $n = 3$ by 4.26.
If q is odd, by 4.7 there exists a big connected component, containing all elements of order p.
There exists a subgroup M_1 of type $O_3(q) \perp O_{2n-3}(q)$, so $r \mid (q^n+1)(q^{n-1}-1)(q^{n-1}+1)$.
If q is even, by 4.16, there exists a connected component containing all elements of prime order r for r some prime divisor of q^2-1.
Let M_1 in class $\mathcal{C}_1$ be of type $O^-_2(q) \perp O^+_{2n-2}(q)$. By the structure of M_1, elements of order r are in that connected component, if $r \mid |\Omega^+_{2n}(q)|$, so remain primes r, which divide $(q^n+1)(q^{n-1}+1)$.

Let n even.
Suppose $r \mid q^n+1$. Let $n = a \cdot b$ with a a 2-power and b odd. Notice, $a \neq 1$ and $b = 1$ gives (ii). A torus of type q^n+1 is contained in a subgroup M_2 of type $\mathrm{GU}_b(q^a)$, which is contained in a subgroup M_3 of type $O^-_{2b}(q^a)$. The structure of M_2 is described by Proposition The structure of M_2 is described by Proposition 4.3.18 of [KL]. By Zsygmondy, $Z(F^*(M_2))$ contains elements of odd order s with $s \mid q^a+1$. As $F^*(M_2)$ contains a $\mathrm{PSL}_2(q)$-section, x is in the big connected component.

Suppose $r \mid q^{n-1}-1$, so q is odd. A torus of type $q^{n-1}-1$ can be found in a subgroup M_4 of type $O^-_2(q) \perp O^+_{2n-2}(q)$. If q is not a Mersenne prime, $Z(F^*(M_4))$ containes elements of odd order, so x is in the big connected component. Else we may use 4.30 for the connected components of $F^*(M_4)$. This gives one case of (iii).

Suppose $r \mid q^{n-1}+1$. A torus of type $q^{n-1}+1$ can be found in a subgroup M_5 of type $O^+_2(q) \perp O^-_{2n-2}(q)$. If q is not a Fermat prime and $q > 2$, $Z(F^*(M_5))$ containes elements of odd order, so x is in the big connected component. Else we use induction for the connected components of $F^*(M_5)$. This gives the other case of (iii).

Let n odd. Suppose $r \mid q^n + 1$. A torus of type $q^n + 1$ can be found in a subgroup M_6 of type $\mathrm{GU}_n(q)$. The structure of M_6 is described by Proposition 4.3.18 of [KL]. If q is not a Mersenne prime, then $Z(F^*(M_6))$ contains elements of odd order and x is contained in the big connected component. We use 4.26 and 4.27 for the connected components of $F^*(M_6)$. This gives (i).

Suppose $r \mid q^{n-1} - 1$, so q is odd. As $q^{n-1} - 1 \mid |\Omega_n(q)|$ and $n \leq 2n - 3$, x is in the big connected component by M_1.

Suppose $r \mid q^{n-1} + 1$. A torus of type $q^{n-1} - 1$ can be found in a subgroup M_7 of type $O_2^+(q) \perp O_{2n-2}^-(q)$. If q is not a Fermat prime and $q > 2$, then $Z(F^*(M_7))$ contains elements of odd order, so x is in the big connected component.
Else we get the structure of the connected components of $F^*(M_7)$ by induction. This gives (iv). □

Lemma 4.32 *Let $G \cong \mathrm{P}\Omega_{2n+1}(q)$ for $n \geq 3$, so q is odd. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) n is a prime, q a Mersenne prime and $d_q(r) = 2n$.

(ii) n is a prime, q a Fermat prime and $d_q(r) = n$.

(iii) n is a 2-power and $d_q(r) = 2n$.

Proof. Notice, that the statement is also true for $n = 2$ by 4.28.
By 4.7 there exists a big connected component, containing all elements of order p.
There exists subgroup M_1 of type $O_3(q) \perp O_{2n-2}^+(q)$ and M_2 of type $O_3(q) \perp O_{2n-2}^-(q)$ so $r \mid (q^n - 1)(q^n + 1)$.
Suppose $r \mid q^n - 1$. A torus of type $q^n - 1$ can be found in a subgroup M_3 of type $O_1(q) \perp O_{2n}^+(q)$. We use 4.30 for the structure of the connected components of $F^*(M_3)$. The exception (i) gives (ii), while the other exceptions are handled by M_1.
Suppose $r \mid q^n + 1$. A torus of type $q^n + 1$ can be found in a subgroup M_4 of type $O_1(q) \perp O_{2n}^-(q)$. We use 4.31 for the structure of the connected components of $F^*(M_4)$. The exceptions (i) and (ii) give (i) and (iii), respectively. The other exceptions are handled by M_1.
Compare this with 4.29. □

Lemma 4.33 *Let $G \cong G_2(q)$ for $q \neq 2$. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) $3 \nmid q - 1$ and $d_q(r) = 3$.

(ii) $3 \nmid q + 1$ and $d_q(r) = 6$.

Proof. If q odd, by 4.7 there exists a big connected component, containing all elements of order p.
If q even, by 4.17, there is a connected conjugacy class y^G, $o(y) = r$ for $r \neq 3$ some prime divisor of $q^2 - 1$. By [LSS] there exists a subgroup M_1 of type $\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$. Therefore $d_q(r) \in \{3, 6\}$.

Suppose $d_q(r) = 3$. By [LSS] there exists a subgroup M_2 of type $\mathrm{SL}_3(q)$, which has a nontrivial center, if $3 \mid q - 1$. This gives (i).

Suppose $d_q(r) = 6$. By [LSS] there exists a subgroup M_3 of type $\mathrm{SU}_3(q)$, which has a nontrivial center, if $3 \mid q + 1$. This gives (ii). □

Lemma 4.34 *Let $G \cong {}^3D_4(q)$ for q odd. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then $d_q(r) = 12$.

Proof. If q is odd, by 4.7 there exists a big connected component, containing all elements of order p.
If q is even, by 4.18, there is a connected conjugacy class y^G, $o(y) = r$ for $r \neq 3$ some prime divisor of $q^2 - 1$. By [LSS] there exists a subgroup M_1 of type $\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q^3)$. Therefore $d_q(r) = 12$. □

Lemma 4.35 *Let $G \cong {}^2F_4(q)$ for $q > 2$, q even. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then $d_q(r) = 12$.

Proof. Recall, that $3 \mid q + 1$ and $5 \mid q^2 + 1$, as q is an odd power of 2.
By 4.19, there is a connected conjugacy class y^G, $o(y) = r$ for r some prime divisor of $q^2 + 1$. Let $\rho \subseteq \pi(G)$ from 4.4. By [Malle], subgroups of type $\mathrm{SU}_3(q)$, ${}^2B_2(q) \times {}^2B_2(q)$ and $\mathrm{Sp}_4(q) \geq \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)$ exist. Therefore $\pi(q^2+1) \subseteq \rho$, so $\pi(q-1) \subseteq \rho$, so $\pi(q+1) \subseteq \rho$, so $3 \in \rho$, so $\pi(q^3+1) \subseteq \rho$. As self centralizing subgroups of size $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$ and $q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$ exist with $(q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1)(q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1) = q^4 - q^2 + 1$, the proof is complete. □

Lemma 4.36 *Let $G \cong F_4(q)$ for q odd. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then $d_q(r) \in \{8, 12\}$.

Proof. If q is odd, by 4.7 there exists a big connected component, containing all elements of order p.
If q is even, by 4.20 we have a unique connected component containing all elements of order r for r some divisor of $q^2 - 1$.
By [LSS] there exist subgroup M_1 of type $\Omega_9(q)$ (q odd) or $\mathrm{Sp}_8(q)$ (q even) and M_2 of type ${}^3D_4(q)$. From the group order formula, $r \mid |M_1||M_2|$. By 4.32,4.29 and 4.34, $d_q(r) \in \{8, 12\}$. □

Lemma 4.37 *Let $G \cong E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ for q odd. Then G has a unique big connected component.*
Let $x \in G$ be of prime order r. If x is not contained in this big connected component, then one of the following cases holds:

(i) $G \cong E_6(q)$ *and* $d_q(r) = 9$

(ii) $G \cong {}^2E_6(q)$ *and* $d_q(r) = 18$.

(iii) $d_q(r) = 8$ *and* $(G, r) \in \{(E_6(3), 41), (E_6(7), 1201), ({}^2E_6(2), 17), ({}^2E_6(3), 41), ({}^2E_6(5), 313)\}$.

(iv) $G \cong E_7(q)$, *q a Mersenne prime and* $d_q(r) \in \{14, 18\}$.

(v) $G \cong E_7(q)$, *q a Fermat prime or* $q = 2$ *and* $d_q(r) \in \{7, 9\}$.

(vi) $G \cong E_8(q)$ *and* $d_q(r) \in \{15, 24, 30\}$.

(vii) $G \cong E_8(q)$, $5 \nmid q^2 + 1$ *and* $d_q(r) = 20$.

(viii) $G \cong {}^2E_6(2)$, $r = 13$.

Proof. If q is odd, by 4.7 there exists a big connected component, containing all elements of order p.
If q is even,By 4.21 there exists a connected component containing all elements of order r for r some prime divisor of $q^2 - 1$.
Consider $G \cong E_6(q)$. By [LSS] there exists a subgroup of type $\mathrm{SL}_2(q) \circ \mathrm{SL}_6(q)$. Therefore $d_q(r) \in \{8, 9, 12\}$.

Suppose $d_q(r) = 12$. By [LSS] there exists a subgroup of type $({}^3D_4(q) \circ \frac{q^2+q+1}{(q-1,3)}$, therefore x is in the big connected subgroup.

Suppose $d_q(r) = 8$. By [LSS] there exists a subgroup of type $\Omega_{10}^+(q) \times \frac{(q-1)}{(q-1,3)}$. We use 4.30 for the connected components of this group, if $\frac{q-1}{(q-1,3)}$ is a 2-power. We get x centralized by a subgroup of size $(q+1)\frac{q-1}{(q-1,3)}$. This is a 2-power, iff $q = 3$ or $q = 7$ by 3.2.

The case $d_q(r) = 9$ is (i).
Consider $G \cong {}^2E_6(q)$. By [LSS] there exists a subgroup of type $\mathrm{SL}_2(q) \circ \mathrm{SU}_6(q)$. Therefore $d_q(r) \in \{8, 12, 18\}$.

Suppose $d_q(r) = 12$. By [LSS] there exists a subgroup of type $({}^3D_4(q) \circ \frac{q^2-q+1}{q+1,3})$, therefore x is in the big connected subgroup, except $q = 2$ in case (viii).

Suppose $d_q(r) = 8$. By [LSS] there exists a subgroup of type $\Omega_{10}^-(q) \times \frac{q+1}{(q+1,3)}$. We use 4.31 for the connected components of this group, if $\frac{q+1}{(q+1,3)}$ is a 2-power. We get x centralized by a subgroup of size $(q-1)\frac{q+1}{(q+1,3)}$. This is a 2-power, iff $q = 2, 3$ or $q = 5$ by 3.2.

The case $d_q(r) = 9$ is (ii).

Consider $G \cong E_7(q)$. By [LSS] there exists a subgroup of type $\mathrm{SL}_2(q) \circ \Omega_{12}^+(q)$. This gives $d_q(r) \in \{7, 9, 12, 14, 18\}$. By [LSS] there exists a subgroup of type $\mathrm{PSL}_2(q^3) \times {}^3D_4(q)$, which shows x in the big connected component for $d_q(r) = 12$.
A subgroup of type $\mathrm{PSL}_2(q^7)$ gives parts of (iv) and (v) for $d_q(r) \in \{7, 14\}$.
Subgroups of type $E_6(q) \circ (q-1)$ and ${}^2E_6(q) \circ (q+1)$ complete (iv) and (v).

For the existence of these subgroups we use [LSS].

Consider $G \cong E_8(q)$. By [LSS] there exists a subgroup of type $\mathrm{SL}_2(q) \circ E_7(q)$. This gives $d_q(r) \in \{15, 20, 24, 30\}$. So we have (vi) or $d_q(r) = 20$. By [LSS] there exists a subgroup of type $\mathrm{SU}_5(q^2)$, which contains a torus of type $\frac{q^{10}+1}{q^2+1}$ and has a nontrivial center, if $5 \mid q^2 + 1$. This gives (vii). $\square$

5 Special centralizers

We later have to consider centralizers of elements of order 3 and 5 in the classical groups over GF(2).

Lemma 5.1 *Let $G \cong \mathrm{SL}_n(2)$, $\mathrm{Sp}_n(2)$, $\Omega_n^{\pm}(2)$ for $n \geq 2$, $x \in G$, $o(x) = 3$ or 5, V the natural n-dimensional* GF(2) *module for G.*

Then $V = U_0 \oplus U_1 \oplus ... \oplus U_k$ with $U_0 = C_V(x)$ and U_i irreducible for $i > 0$. Moreover in the symplectic and orthogonal case, the direct summands can be choosen in such a way, that $U_i \perp U_j$ for $i \neq j$ and the U_i are nondegenerate.

Proof. By coprime action the module splits into a direct sum of irreducibles. So suppose we have a nontrivial symplectic or quadratic form.
We use induction on $\dim V$. By coprime action we have $[V, x] \perp C_V(x) = U_0$, so $C_V(x) = 0$ by induction.

Now $o(x)$ determines the minimal polynomial of x uniquely:
It is $x^2 + x + 1$ for $o(x) = 3$ and $x^4 + x^3 + x^2 + x + 1$ for $o(x) = 5$.

Let U be some irreducible x-submodule of $V = [V, x]$. As U is irreducible, either $U \cap U^\perp = 0$ and U is nondegenerate or $U \cap U^\perp = U$, so U is totally singular.
Now $U^\perp/U$ is a x-module, but the extension splits over U, as x acts semisimple. So there exists an x-invariant complement $W \leq U^\perp$, which is nondegenerate as $U^\perp = U \perp W$.
By induction $W = 0$, else we can produce the U_i from proper subspaces W and $W^\perp$.

Therefore $\dim V = 4$ for $o(x) = 3$ and $\dim V = 8$ for $o(x) = 5$.
By inspection of the groups $\Omega_4^{\pm}(2)$, $\mathrm{Sp}_4(2)$, $\Omega_8^{\pm}(2)$ and $\mathrm{Sp}_8(2)$, see [ATLAS], these groups contain at most one class of fixed point free elements of order 3 resp. 5, except in case of $\Omega_8^+(2)$. In this case, there are three classes of elements of order 5, which are transitively permuted by $\mathrm{Out}(G) \cong \Sigma_3$. In particular there is one class of elements of order 5, with $C_V(x) \neq 0$ and two fixed point free classes. One of them comes from the $O_4^-(2) \perp O_4^-(2)$-decomposition, so for this element we have the above decomposition. But the other class is an image under the graph automorphism of order 2, which preserves the module and form of G, so we get a decomposition in this case too.
Therefore there are irreducible and nondegenerate subspaces $U_1, U_2 \leq V$ with $V = U_1 \perp U_2$. $\square$

We get the following corollaries from basic representation theory:

Corollary 5.2 *Let* $G \cong \mathrm{GL}_n(2)$ *and* $x, y \in G$ *with* $o(x) = 3, o(y) = 5$, $m = \dim[V, x], k = \dim[V, y]$.
Then $C_G(x) \cong \mathrm{GL}_{m/2}(4) \times \mathrm{GL}_{n-m}(2)$ *and* $C_G(y) \cong \mathrm{GL}_{k/4}(16) \times \mathrm{GL}_{n-k}(2)$.

Corollary 5.3 *Let* $G \cong \mathrm{Sp}_n(2)$ *and* $x, y \in G$ *with* $o(x) = 3, o(y) = 5$, $m = \dim[V, x], k = \dim[V, y]$.
Then $C_G(x) \cong \mathrm{GU}_{m/2}(2) \times \mathrm{Sp}_{n-m}(2)$ *and* $C_G(y) \cong \mathrm{GU}_{k/4}(4) \times \mathrm{Sp}_{n-k}(2)$.

In case of the orthogonal groups we formulate a weaker statement to avoid difficulties with automorphisms.

Corollary 5.4 *Let* $G \cong \Omega_n^+(2)$ *and* $x, y \in G$ *with* $o(x) = 3, o(y) = 5$, $m = \dim[V, x], k = \dim[V, y]$.
Then $O^2(C_G(x)) \cong (\mathrm{GU}_{m/2}(2))' \times \Omega_{n-m}^{\varepsilon_1}(2)$ *and* $O^2(C_G(y)) \cong \mathrm{GU}_{k/4}(4) \times \Omega_{n-k}^{\varepsilon_2}(2)$ *with* $\varepsilon_1 = (-1)^{m/2}$ *and* $\varepsilon_2 = (-1)^{k/4}$.

Corollary 5.5 *Let* $G \cong \Omega_n^-(2)$ *and* $x, y \in G$ *with* $o(x) = 3, o(y) = 5$, $m = \dim[V, x], k = \dim[V, y]$.
Then $O^2(C_G(x)) \cong (\mathrm{GU}_{m/2}(2))' \times \Omega_{n-m}^{\varepsilon_1}(2)$ *and* $O^2(C_G(y)) \cong \mathrm{GU}_{k/4}(4) \times \Omega_{n-k}^{\varepsilon_2}(2)$ *with* $\varepsilon_1 = (-1)^{1+m/2}$ *and* $\varepsilon_2 = (-1)^{1+k/4}$.

We now consider elements of order 3 and 5 in the groups $\mathrm{PGU}_n(2)$.
The situation is a bit more complicated. Let V be the natural n-dimensional GF(4)-module of $\mathrm{GU}_n(2)$ and $\omega \in \mathrm{GF}(4)$ with $\omega^2 + \omega + 1 = 0$.

Lemma 5.6 *Let* $x \in \mathrm{GU}_n(2)$*, such that* $\overline{x} \in \mathrm{PGU}_n(2)$ *has order 3. Then either*

(i) $o(x) = 3$ *and* x *is diagonalizable.*
The eigenspaces to 1*,* ω *and* ω^2 *are nondegenerate.*

(ii) $o(x) = 9$*,* $x^3 \in Z(\mathrm{GU}_n(2))$ *and* x *has minimal polynomial* $x^3 - \omega$ *or* $x^3 + \omega$*.*
There exist subspaces $U_1, ..., U_k$*,* $\dim U_i = 3$ *and* $V = U_1 \perp U_2 ... \perp U_k$*,* $n = 3k$ *and the* U_i *are nondegenerate.*

Proof. Consider case (i) and let $u, v \in V$ eigenvectors to different eigenvalues λ, μ. Then $(u, v) = (u, v)^x = (u^x, v^x) = (\lambda u, \mu v) = \lambda\overline{\mu}(u, v)$. If $\lambda \neq \mu$ and $\lambda, \mu \in \{1, \omega, \omega^2\}$ this implies $(u, v) = 0$, so the eigenspaces to different eigenvalues are orthogonal. As x is diagonalizable, V is the sum of these eigenspaces, so each one is nondegenerate.
Consider now case (ii). As $1 \neq x^3 \in Z(\mathrm{GU}_n(2))$, there are only the two choices $x^3 = \omega Id$ or $x^3 = w^2 Id$ with Id the identity matrix. Therefore the minimal polynomial is one of the two choices. As it is irreducible we have n a multiple of 3, so $\mathrm{SU}_n(2)$ already contains $Z(\mathrm{GU}_n(2))$. Notice, that in this case there are such elements, which come from the embedding $\mathrm{GU}_{n/3}(2^3).3 \leq \mathrm{GU}_n(2)$, so there is a conjugacy class of elements, which satisfies (ii). We show, that this subspace decomposition exists in general, by induction over n:
Let U be some irreducible x-submodule of V. Then either $U \cap U^\perp = 0$ or $U \leq U^\perp$, as U is irreducible. If $U \cap U^\perp = 0$, we may proceed by induction on $U^\perp$.
If $\dim U^\perp > 3$, $U^\perp$ has an x-invariant complement to W to U, as x acts semisimple. Then we may proceed by induction on both W and $W^\perp$, as $W \cap W^\perp = 0$.

So $\dim V = 6$. By [ATLAS] there exists a unique conjugacy class of elements of order 9 in $\mathrm{GU}_6(2)$ with the property $x^3 \in Z(\mathrm{GU}_6(2))$. It is class $3G$. But there is a class, which allows a decomposition into an orthogonal sum of two nondegenerate subspaces, so the statement is proven. □

Corollary 5.7 *Let $x \in G = \mathrm{GU}_n(2)$ with $x^3 \in Z(\mathrm{GU}_n(2))$. Then one of the following holds.*

(i) $o(x) = 3$. *Then* $C_G(x) \cong \mathrm{GU}_{n_1}(2) \times \mathrm{GU}_{n_2}(2) \times \mathrm{GU}_{n_3}(2)$ *with* $n = n_1 + n_2 + n_3$.

(ii) $o(x) = 9$. *Then* $C_G(x) \cong \mathrm{GU}_{n/3}(8)$.

We now consider elements of order 5.
Notice, that $\frac{x^5+1}{x+1} = x^4 + x^3 + x^2 + x + 1 = (x^2 + \omega x + 1)(x^2 + \omega^2 x + 1)$.

Lemma 5.8 *Let $x \in G = \mathrm{GU}_n(2)$ with $o(x) = 5$. Then there exist x-invariant subspaces $U, X_i, Y_i, i \in \{1..k\}$, $n = \dim U + 4k$, with*

- $U = C_V(x)$
- $\dim X_i = 2 = \dim Y_i$,
- *x is irreducible on X_i and Y_i,*
- *x has on X_i the minimal polynomial $x^2 + \omega x + 1$,*
- *x has on Y_i the minimal polynomial $x^2 + \omega^2 x + 1$,*
- $X_i \leq X_i^{\perp}$ *and* $Y_i \leq Y_i^{\perp}$,
- $(X_i \oplus Y_i) \cap (X_i \oplus Y_i)^{\perp} = 0$,
- $V = U \perp (X_1 \oplus Y_1) \perp ... \perp (X_k \oplus Y_k)$

Proof. The proof proceeds by induction on $\dim V$. So we may assume, that $\dim U = \dim C_V(x) = 0$.
Let X be some irreducible x-submodule, so x is 2-dimensional. Then $X \leq X^{\perp}$, as otherwise $X \cap X^{\perp} = 0$, but $|\mathrm{GU}_3(2)|$ is not divisible by 5.
Let W be an x-invariant complement to X in $X^{\perp}$. Then W is nondegenerate, so if $W \neq 0$ the result holds by induction on both W and $W^{\perp}$, so on V.
If $W = 0$, then $G = \mathrm{GU}_4(2)$ and $|\mathrm{GU}_4(2)|_5 = 5$. An easy callculation shows the statement in this case. □

Corollary 5.9 *Let $x \in G = \mathrm{GU}_n(2)$, $o(x) = 5$. Then $C_G(x) \cong \mathrm{GU}_k(2) \times \mathrm{GU}_{(n-k)/4}(4)$ with $k = \dim C_V(x)$.*

Proof. This is a consequence of 5.8. □

6 Results on loop folders

If not otherwise explictely defined, $\overline{G}$ is defined as $\overline{G} := G/O_2(G)$ and for subsets $X \subseteq G$ we denote with $\overline{X}$ the image of X under the natural homomorphism from G to $\overline{G}$.

6.1 Classic facts from loop theory

The following arguments can be found already in [Asch] or [Hei], we just split them up for better quotation later on. These results should be well known in loop theory. We don't give references, as the statements are presented here in the not so widely used language of loop folders. Furthermore most of the statements have very elementary proofs, which a reference may hide.
In addition parts of these statements can be seen as exercises, to make the reader familiar with terms and arguments used throughout this paper.

Lemma 6.1 *In a Bol loop, the order of every element divides the loop order. Therefore a Bol loop of exponent 2 has even size or size 1.*
If (G, H, K) is a loop folder to a Bol loop of exponent 2, then $|G : H|$ is 1 or even.

For a proof the reader should be aware, that we defined only 'order 2 of an element'.
What the general problem is and where the Bol property comes into play, for all these questions we point at the basic theory of loops, see for instance [Bruck].

Lemma 6.2 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. A subgroup $U \leq G$ gives rise to a subloop, iff $U = (U \cap H)(U \cap K)$, the subloop folder being $(U, U \cap H, U \cap K)$, the size of the subloop being $|U : U \cap H|$.*
In particular overgroups of H and of $\langle K\rangle$ satisfy this condition.

Hint: we did not define the term subloop folder.

Lemma 6.3 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. Let $a \in K$, $h \in H$ and $g \in G$. If $(h^g)^a = (h^g)^{-1}$, then $h^2 = 1$.*

Proof. Suppose $h^{ga} = (h^g)^a = (h^g)^{-1} = (h^{-1})^g$. Let $b = gag^{-1} \in K$. Then $h^b = h^{-1}$ or $[h, b] = h^{-2} \in H$. But $[h, b] = h^{-1}bhb = b^h b \in KK$. Since $KK \cap H = 1$ by the loop folder property, $h^2 = 1$. □

Lemma 6.4 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. Then $O_{2'}(G) \leq C_H(\langle K\rangle)$.*
If (G, H, K) is a faithful loop folder, then $O_{2'}(G) = 1$.
If (G, H, K) is a loop envelope, then $O_{2'}(G) \leq Z(G) \cap H$.

Proof. $O_{2'}(G)H$ gives rise to a subloop by 6.2, but $|O_{2'}(G)H : H|$ is odd, so by 6.1, $|O_{2'}(G)H : H| = 1$. By 6.3 then $[\langle K\rangle, O_{2'}(G)] = 1$. □

Lemma 6.5 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2 and $U \leq G$ with $H \leq U$. Then $|G : U|$ is even or 1.*

Proof. Assume $|G : U|$ to be odd. Then U contains a Sylow-2-subgroup of G, so every element of K is conjugate to some element of $U \cap K$. Then $|\{k^g : k \in K \cap U, g \in G\}| \leq 1 + (|U : H| - 1)|G : U| = 1 + |G : H| - |G : U|$. Since $|G : H| = |K| = |\{k^g : k \in K \cap U, g \in G\}|$ this forces $|G : U| = 1$. $\square$

Corollary 6.6 *H is a 2-group iff G is a 2-group.*

Proof. If H is a 2-group, then H is a contained in 2-Sylow M of G, so by 6.5 $|G : M| = 1$ and G is a 2-group. $\square$

Corollary 6.7 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. Then $O_{2,2'}(G)H = O_2(G)H$.*

Proof. $O_2(G)H$ is of odd index in $O_{2,2'}(G)H$, so the statement is a consequence of 6.5 and 6.2. $\square$

Lemma 6.8 *Let (G, H, K) be a faithful loop envelope to a soluble Bol loop L of exponent 2. Then $|L| = |G : H|$ is a 2-power.*

Proof. In a soluble Bol loop of exponent 2 we find a sequence of subloops $L = L_1 > L_2 > \cdots L_k = 1$ with $|L_{i+1}| = 2|L_i|$. $\square$

Lemma 6.9 *Let (G, H, K) be a loop envelope to a Bol loop of exponent 2 and $|G : H|$ be a 2-power. Then G is a 2-group.*

Proof. As $|L| = |G : H|$ is a 2-power, H contains Sylow subgroups for all odd primes p. But then the product of any two elements of K has to be of 2-power order: else we may find some elements k_1, k_2, such that k_1k_2 has odd prime order, which is inverted by k_1. By 6.3 this is not possible. By the Baer-Suzuki theorem then $\langle K\rangle = G$ is a 2-group. $\square$

Theorem 3 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. Assume G is soluble. Then $\langle K\rangle \leq O_2(G)$ is a 2-group.*

Proof. Let $\overline{G} = G/O_2(G)$. By 6.7, $F^*(\overline{G}) = F(\overline{G}) \leq \overline{H}$. Let $k \in K$. If k acts nontrivially on $F(\overline{G})$, it inverts some element of odd prime order p in $F(\overline{G})$. By 3.5, k inverts some element of order p in the preimage of $F(\overline{G})$, but H contains a Sylow-p-subgroup of that preimage. By 6.3 we get a contradiction. Therefore elements of K act trivially on $F(\overline{G})$, but since $C_{\overline{G}}(F(\overline{G})) \leq Z(F(\overline{G}))$, $k \in O_2(G)$, so $\langle K\rangle \leq O_2(G)$. $\square$

Lemma 6.10 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2,$N \trianglelefteq G$ with $N \leq H$ and $\overline{G} = G/N$. Then $(\overline{G}, \overline{H}, \overline{K})$ is a loop folder to the same loop.*

Proof. The loop folder property is clearly inherited to the factor group. The two loops are natural isomorphic from the definition of the loop: the multiplication depends only on the action of K on the H-cosets and N is in the kernel of this action. $\square$

Lemma 6.11 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. Let $\overline{G} = G/O_{2'}(G)$. Then $(\overline{G}, \overline{H}, \overline{K})$ is a loop folder to the same loop.*

Proof. By 6.4, $O_{2'}(G) \leq H$, so 6.10 gives the result. □

6.2 Universal covering group and loop embeddings

The following section was included to understand, if Y is a subloop of X, how the group $\mathrm{RMult}(Y)$ is embedded in $\mathrm{RMult}(X)$. The result is, that $\mathrm{RMult}(X)$ in general contains only a central extension of $\mathrm{RMult}(Y)$ as a subgroup. The only, but crucial application of this section is 6.30.

We begin with a well known technical lemma from loop theory.

Lemma 6.12 *Let (G, H, K) be a loop envelope to a finite loop X. Then $H = \langle R(x)R(y)R(x \cdot y)^{-1} : x, y \in X \rangle$.*

Proof. Let $H_0 = \langle R(x)R(y)R(x \cdot y)^{-1} : x, y \in X \rangle \leq H$. Notice, that since H is the stabilizer in G of $1 \in X$ (in the permutation action of G on X), all generators fix $1 \in X$.
We show $G = H_0K$, so $|H : H_0| = 1$. As $G = \langle K \rangle$, any element of G is a finite product of elements of K. For $x \in G$, we show by induction over the minimal length of such a word, that $x = h_x k_x$ for some $h_x \in H_0$ and some $k_x \in K$: If the minimal length is less than 2, nothing is to show, as all those elements are in K. Else we can write $x = hk_1k_2$ with $k_1, k_2 \in K$ and $h \in H_0$. Let $k_1 = R(x)$ and $k_2 = R(y)$. Then $x = hR(x)R(y)R(x \cdot y)^{-1}R(x \cdot y) = hh_1R(x \cdot y)$ with $h_1 = R(x)R(y)R(x \cdot y)^{-1} \in H_0$ and $R(x \cdot y) \in K$. □

Notice, that the Bol identity

$$x \cdot ((y \cdot z) \cdot y) = ((x \cdot y) \cdot z) \cdot y$$

can be written using the right translations as

$$\rho((y \cdot z) \cdot y) = \rho(y)\rho(z)\rho(y).$$

Definition 6.13 *Let X be a Bol loop of exponent 2. Let $\hat{G} = \hat{G}(X)$ be the image of the free group $\mathcal{F}$ with free generators $\beta_x : x \in X$ with relation kernel generated by the following types of relations: $\beta_x^2 = 1$ for all $x \in X, x \neq 1$, $\beta_1 = 1$ and $\beta_x\beta_y\beta_x = \beta_{(x \cdot y) \cdot x}$ for all $x, y \in X$.*

Lemma 6.14 *If X is finite, then $\hat{G}$ is finite.*

Proof. The relations of type $\beta_x\beta_y\beta_x = \beta_{(x\cdot y)\cdot x}$ can be transformed into $\beta_x\beta_y = \beta_{(x\cdot y)\cdot x}\beta_x$, using $\beta_x^2 = 1$. In this way the translation can be used to reduce a word in the $\beta_x : x \in X$: If in such a word a generator β_x occurs twice, we can use these relations to moves the first of these generators next to the other generator. Then the relation $\beta_x\beta_x$ cancels out these two generators, and reduces the length of the word. Therefore the group $\hat{G}$ has only finitely many irreducible words, so it is of finite order. □

Lemma 6.15 *Let $\hat{H} = \langle \beta_x\beta_y\beta_{x\cdot y} : x, y \in X\rangle$ and $\hat{K} = \{\beta_x : x \in X\}$. Then $(\hat{G}, \hat{H}, \hat{K})$ is a loop envelope to X. It is even a universal loop envelope in the following sense: if there is another loop envelope (G, H, K) to X, then a group homomorphism $\gamma : \hat{G} \to G$ exists, which maps $\hat{H}$ to H and $\hat{K}$ to K.*

Proof. Using the generators of $\hat{H}$, every word in generators of $\hat{G}$ can be written as a product of an element of $\hat{H}$ and an element of $\hat{K}$, using the method as seen in the proof of 6.12. Notice, that $\mathrm{RMult}(X)$ is an image of $\hat{G}$, such that $\hat{H}$ maps to the group H of inner mappings of X. Therefore $\hat{H}$ has index exactly $|\hat{K}| = |X|$. From the relations of $\hat{G}$ it is clear, that $\hat{K}$ is a union of $\hat{G}$-conjugacy classes of involutions with 1, which forms a transversal to $\hat{H}$. Therefore $(\hat{G}, \hat{H}, \hat{K})$ is a loop envelope to a Bol loop of exponent 2. (It satisfies (1),(3),(4) and (5) of 2.3, but in general not (2).)
Let (G, H, K) be another loop envelope to X with bijection κ between the elements of K and the loop elements of X.
Define a map γ from $\hat{K}$ to K through $\kappa(\gamma(\beta_x)) = x$ for all $x \in X$.
We show, that this map extends to a group homomorphism from $\hat{G}$ to G, which maps $\hat{H}$ to H. But for the homomorphism property we just need, that the elements $\gamma(\beta_x)$ satisfy the relations for the β_x, as $\hat{G}$ was a quotient of a free group.
Surely $\gamma(\beta_x)\gamma(\beta_x) = 1$ for all $x \in X, x \neq 1$. But the identity $\gamma(\beta_x)\gamma(\beta_y)\gamma(\beta_x) = \gamma(\beta_{(x\cdot y)\cdot x})$ comes from the (right) Bol identity in X, written in (right) translations. So if (G, H, K) is a loop folder to X, the elements of K satisfy this identity.
As H is the stabilizer in G of the loop element $1 \in X$, it contains the elements $\gamma(\beta_x)\gamma(\beta_y)\gamma(\beta_{x\cdot y})$, as these elements stabilize $1 \in X$. By the transversal property of K then the image of $\hat{H}$ is H. □

Lemma 6.16 *$\hat{G}$ is a central extension of $\mathrm{RMult}(X)$, which is generated by involutions, so $O^{2'}(\hat{G}) = \hat{G}$. If X is soluble, then $\hat{G}$ is a 2-group.*

Proof. The relations of type $\beta_x\beta_y\beta_x = \beta_x^{\beta_y} = \beta_{(x\cdot y)\cdot x}$ describe the fusion of $\hat{G}$ of the set $\hat{K}$, so they define the permutation action of $\hat{G}$ on $\hat{K}$. Notice that we get the same permutation action of $\mathrm{RMult}(X)$ on the set $\{R(x) : x \in X\}$ by the Bol identity. As $\hat{G}$ is generated by $\hat{K}$ and $\mathrm{RMult}(X)$ is generated by $\{R(x) : x \in X\}$, we conclude:

$$\hat{G}/Z(\hat{G}) \cong \mathrm{RMult}(X)/Z(\mathrm{RMult}(X)),$$

as the kernel of the action is in the center of the group.
As $\mathrm{RMult}(X)$ is an image of $\hat{G}$, we see, that the kernel of this homomorphism is in the center of $\hat{G}$. □

Remark 6.17 *Let Y be a subloop of a loop X and (G, H, K) be a loop folder to X. Let $D_Y := \langle R(y) : y \in Y\rangle \leq G$. Then D_Y is an image of the universal group $\hat{G}(Y)$ as defined above: $(D_Y, D_Y \cap H, D_Y \cap K)$ is a loop folder to Y and 6.15 applies.*

6.3 Selected Aschbacher's results

We will later make heavy use of the following fact, which produces lots of subloops in a Bol loop of exponent 2. From knowledge on the structure of these subloops we get strong restrictions on the local group structure.

Lemma 6.18 *Let (G, H, K) be a loop folder to a Bol loop X of exponent 2.*

(i) Let $L \leq H$. Then $(N_G(L), N_H(L), C_K(L))$ and $(C_G(L), C_H(L), C_K(L))$ are loop folders to a (the same) subloop of X.

(ii) Let $H \leq U \leq G$. Then $(U, H, U \cap K)$ is a loop folder to subloop of X.

(iii) Let $U \leq G$ with $|U| \geq |U \cap H||U \cap K|$. Then $(U, U \cap H, U \cap K)$ is a loop folder to a subloop of X.

(iv) Let $U \leq G$ with $U = (U \cap H)(U \cap K)$. Then $(U, U \cap H, U \cap K)$ is a loop folder to a subloop of X.

Proof. (i) is (11.1)(4) of [Asch]. The main argument was, that Bol loops of exponent 2 are A_r-loops, so L acts as a group of automorphisms on X. The elements, which are fixed by every $l \in L$ form a subloop, which is the subloop in question. This is Aschbachers (4.3).
(ii) is immediate as each H-coset contains exactly one element of K.
(iii) and (iv) are (3.3) in [Asch]. These conditions more or less state, that each coset of $U \cap H$ in U contains at least one element of K. As $U \cap H$-cosets are contained in H-cosets, they contain at most one element of K. □

For quotation we repeat the following from Aschbachers paper [Asch]:

Definition 6.19 *An N-loop is a finite Bol loop of exponent 2 such that the enveloping group of X is not a 2-group, but for all proper sections S of X, the enveloping group of S is a 2-group.*

Aschbacher's main theorem stated:

Theorem 4 *Let X be a finite Bol loop of exponent 2 which is an N-loop. Let (G, H, K) be a faithful loop envelope to X, $J = O_2(X)$ and $G^* = G/J$. Then*

(1) $G^* \cong \mathrm{PGL}_2(q)$ with $q = 2^n + 1 \geq 5$,H^* is a Borel subgroup of G^* and K^* consists of the involutions in $G^* - F^*(G^*)$.

(2) $F^*(G) = J$.

(3) Let $n_0 = |K \cap J|$ and $n_1 = |K \cap aJ|$ for $a \in K - J$. Then n_0 is a power of 2, $n_0 = n_1 2^{n-1}$ and $|K| = (q + 1)n_0 = n_1 2^n(2^{n-1} + 1)$.

The following lemma is another formulation of Aschbachers [Asch] (12.5)(2), which was based on an idea of S.Heiss and/or G.Nagy.:

Lemma 6.20 *Let (G,H,K) be a loop folder to a Bol loop of exponent 2 and $N \trianglelefteq G$. Let $a_i, i \in \{1,...,r\}$ be representatives for the orbits of $\overline{G} = G/N$ on $\overline{K}^\sharp$, $m_i := |\{\overline{a_i}^{\overline{G}}\}|$, $n_i = |K \cap a_iN|$ and $n_0 := K \cap N$. Then*

$$|K| = n_0 + \sum_{i=1}^{r} n_i m_i.$$

Proof. Let $K_i := \{a \in K : \overline{a} \in \overline{a_i}^{\overline{G}}\}$ and $K_0 := K \cap N$. Then $\{K_i : i \in \{0,..,r\}\}$ is a partition of K with $|K_0| = n_0$ and $|K_i| = n_i m_i$ for $i \in \{1,..,r\}$. □

6.4 Additional results

The following is a corollary to 6.20.

Corollary 6.21 *Let (G,H,K) be a loop folder to a Bol loop of exponent 2 and $\overline{G} = G/O_2(G)$. Suppose $O_2(\overline{H}) = 1$ and there exists an odd prime p dividing $|\overline{G}|$, such that $m_i \equiv 0 \pmod p$ for all $i \in \{1,..,r\}$, with m_i as in 6.20 for $N = O_2(G)$. Then p does not divide $|K| = |G:H|$, so $\mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$.*

Proof. Since $O_2(H) \subseteq O_2(G)$, we have $O_2(O_2(G)H) = O_2(G)$. Now $(O_2(G)H, H, O_2(G)H \cap K)$ gives a soluble subloop folder by 6.9, as $|O_2(G)H : H|$ is a 2-power. Therefore $|O_2(G)H \cap K|$ is a 2-power, but $\langle O_2(G)H \cap K\rangle \leq O_2(O_2(G)H) = O_2(G)$, so $n_0 := |O_2(G) \cap K|$ is a 2-power. By 6.20 now p does not divide $|K|$. □

There exists a slight extension of the previous lemma:

Corollary 6.22 *Let (G,H,K) be a loop folder to a Bol loop of exponent 2 and $\overline{G} = G/O_2(G)$. Suppose $\overline{H} \cap \overline{K} = 1$ and there exists an odd prime p dividing $|\overline{G}|$, such that $m_i \equiv 0 \pmod p$ for all $i \in \{1,..,r\}$, with m_i as in 6.20 for $N = O_2(G)$. Then p does not divide $|K| = |G:H|$, so $\mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$.*

Proof. As seen in 6.21, we can show $O_2(G)H \cap K = O_2(G) \cap K$, since $\overline{H} \cap \overline{K} = 1$. The proof of 6.21 continues.
In theory, K may contain involutions, which map into $\overline{H}$. In this case $\overline{H} \cap \overline{K}$ is weakly closed in $\overline{H}$, as otherwise elements of K invert elements of odd order in H.
A $\overline{G}$-conjugacy class of involutions from $\overline{H} \cap \overline{K}$ has to lift in G to different conjugacy classes, one for the elements of K and one for the elements of H. □

Lemma 6.23 *Let (G,H,K) be a loop folder to a Bol loop of exponent 2. Let $x \in K, y \in G$ and $\overline{G} = G/O_2(G)$. If $\overline{y}$ has odd order and $\overline{y}^{\overline{x}} = \overline{y}^{-1}$, then for every $z \in G$: $\overline{y^z} \notin \overline{H}$, so $y \notin O_2(G)H$.*

Proof. Assume otherwise. Since $\langle \overline{y}, \overline{x}\rangle$ is a dihedral group with all involutions conjugate, we may assume w.l.o.g that $o(\overline{y})$ is some odd prime p, by replacing y with some suitable element from $\langle y\rangle$. Now x inverts some element of prime order p in $\overline{O_2(G)\langle y\rangle}$, by 3.5 then x inverts some element of prime order p in $O_2(G)\langle y\rangle$. But $O_2(G)\langle y\rangle \leq O_2(G)H$ and H contains a p-Sylow-subgroup of

$O_2(G)H$. So x inverts some element of odd order, which is conjugate into H, a contradiction to 6.3. $\square$

Corollary 6.24 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. Let $\overline{C}$ be a component of $\overline{G} = G/O_2(G)$. If $\overline{C} \leq \overline{H}$, then $[\overline{C}, \overline{\langle K \rangle}] = 1$ and $\overline{C} \cap \overline{\langle K \rangle} \leq Z(\overline{C})$.*

Proof. Let $x \in K$. If $\overline{x} \notin O_2(\langle x, \overline{C} \rangle)$), by the Baer-Suzuki theorem some $y \in G$ exists with $\overline{y} \in \overline{H}$, $\overline{y}^{\overline{x}} = \overline{y}^{-1}$, $o(\overline{y})$ odd. But by 6.23 this is impossible.
So $[\overline{\langle K \rangle}, \overline{C}] = 1$. Since $[\overline{C}, \overline{C}] = \overline{C} \neq 1$, $\overline{C} \not\leq \langle K \rangle$. Since $\overline{\langle K \rangle} \trianglelefteq G$, $\overline{C} \cap \overline{\langle K \rangle} \trianglelefteq \overline{C}$, so $\overline{C} \cap \overline{\langle K \rangle} \leq Z(\overline{C})$. $\square$

Corollary 6.25 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. If $F^*(\overline{G}) = F(\overline{G})$, then $\overline{G} = \overline{H}$.*

Proof. We have $F(\overline{G}) \leq \overline{H}$ by 6.7. By 6.23, no element of $\overline{K}$ acts nontrivially on $F(\overline{G})$. Therefore $\overline{\langle K \rangle} \leq C_{\overline{G}}(F(\overline{G})) \leq Z(F(\overline{K}))$, so $\overline{\langle K \rangle} = 1$ and $\overline{G} = \overline{H}$. $\square$

In a nonsoluble loop therefore $\overline{G}$ has components. The next lemma shows a strategy, how to get rid of the center of $E(\overline{G})$.

Lemma 6.26 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2. Then a loop folder $(\hat{G}, \hat{H}, \hat{K})$ to a Bol loop of exponent 2 exists with $\overline{\hat{G}} \cong \overline{G}/Z(E(\overline{G}))$ and $|\overline{G} : \overline{H}| = |\overline{\hat{G}} : \overline{\hat{H}}|$.*

Proof. By 6.7 $O_{2,2'}(G)H = O_2(G)H$, so $Z(E(\overline{G})) \leq \overline{H}$. Let $Z \leq H$ with $\overline{Z} = Z(E(\overline{G}))$, but $|Z|$ odd. By a Frattini argument now $G = O_2(G)N_G(Z)$, Using Dedekind's identity, we get $O_2(G)H = O_2(G)(N_G(Z) \cap H) = O_2(G)N_H(Z)$. Now $G/O_2(G) \cong N_G(Z)/N_{O_2(G)}(Z)$, with $O_2(G)H$ mapping to the image of $N_H(Z)$ in $N_G(Z)/N_{O_2(G)}(Z)$.
Notice $N_{O_2(G)}(Z) = C_{O_2(G)}(Z) \leq C_G(Z)$ as $[N_{O_2(G)}(Z), Z] \leq O_2(G) \cap Z = 1$.
Let $G_1 := N_G(Z)$, $H_1 := N_H(Z)$ and $K_1 := K \cap N_G(Z) = K \cap C_G(Z)$. Now $O_2(G_1) = O_2(G) \cap G_1 = C_{O_2(G)}(Z)$, since $O_2(G_1/C_{O_2(G)}(Z)) = 1$. Remember (G_1, H_1, K_1) is a subloop folder by 6.18(i).
From the above isomorphism $\overline{G} = G/O_2(G) \cong N_G(Z)/N_{O_2(G)}(Z) = G_1/O_2(G_1) = \overline{G_1}$ we conclude:
$|\overline{G_1} : \overline{H_1}| = |\overline{G} : \overline{H}|$. But now $Z \trianglelefteq G_1$, even $Z \leq O_{2'}(G_1)$. Using 6.11 we get the loop folder $(\hat{G}, \hat{H}, \hat{K})$ in $\hat{G} = G_1/Z$. Since $Z \leq H_1$, we have $|\overline{G} : \overline{H}| = |G_1 : O_2(G_1)H_1| = |\hat{G} : O_2(\hat{G})\hat{H}| = |\overline{\hat{G}} : \overline{\hat{H}}|$. Notice, that $F^*(G_1)$ covers $F^*(G_2)$, since $Z = Z(E(G_1)) \leq \Phi(E(G_1)) \leq \Phi(G_1)$. As $\overline{G} \cong \overline{G_1}$ and $\overline{\hat{G}} \cong \overline{G_1}/\overline{Z}$, we have $\overline{\hat{G}} \cong \overline{G}/Z(E(\overline{G}))$. $\square$

The following lemma is useful for soluble subloops, as it makes quite a lot of elements of $\overline{H}$ visible in local subgroups.

Lemma 6.27 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2, $\overline{G} = G/O_2(G)$ and $U \leq G$ be a subgroup with the following properties:*

(i) $U = (U \cap H)(U \cap K)$, *so* U *is a group to a subloop.*

(ii) $O_2(U) \cap O^2(U) \leq O_2(G)$ *or equivalently* $[O_2(\overline{U}), O^2(\overline{U})] = 1$.

(iii) $\langle U \cap K \rangle \leq O_2(U)$, *so the subloop to* U *is soluble.*

Then $O^2(U) \leq O_2(G)H$ *or equivalently* $O^2(\overline{U}) \leq \overline{H}$.

Proof. Let $u \in U$ be of odd order. We can write $u = hk$ with $h \in H \cap U$ and $k \in K \cap U$ by (i).
Now $k \in \langle K \cap U \rangle \leq O_2(U)$ by (iii). By (ii) we have $[u, k] \in [O^2(U), O_2(U)] \leq O_2(G)$. Looking at $\overline{G} = G/O_2(G)$, we have $\overline{u}$ of odd order commuting with $\overline{k}$ of order 1 or 2. But this gives a contradiction if $\overline{k}$ has order 2, so $\overline{k} = 1$, so $k \in O_2(G)$ and $u \in HO_2(G)$. □

There exists a generalization to nonsoluble subloops:

Lemma 6.28 *Let* (G, H, K) *be a loop folder to a Bol loop of exponent 2,* $\overline{G} = G/O_2(G)$ *and* $D := \langle K \rangle$. *Then* $O^2(C_{\overline{G}}(\overline{D})) \leq \overline{H}$.

Proof. Let $x \in G$ be of odd order, such that $[\overline{D}, \overline{x}] = 1$. We can write $x = hk$ with $h \in H, k \in K$. As $k \in O_2(G)D$, $[\overline{k}, \overline{x}] = 1$. As $\overline{x} = \overline{hk}$, $[\overline{h}, \overline{k}] = 1$, so $\overline{k} = 1$ as $\overline{x}$ has odd order. Therefore $\overline{x} = \overline{h} \in \overline{H}$. □

Definition 6.29 *A Bol loop* L *of exponent 2 is called a* $2N$**-loop**, *iff* L *is not soluble, but every proper subloop is soluble.*

Remark: We introduced this term, since it allows us some ignorance: We don't have to care, whether the loop itself is simple or not. There may or may not exist nonsplit extensions of soluble subloops with N-loops.

Lemma 6.30 *Let* (G, H, K) *be a loop envelope to a* $2N$*-loop* L.
Then

- $C_G(O_2(G)) \leq O_2(G)$,
- $\overline{G} \cong \mathrm{PGL}_2(q)$ *and* $q = 9$ *or* $q \geq 5$ *is a Fermat prime,*
- $|G : O_2(G)H| = q + 1$ *and*
- $\overline{K}$ *consists of 1 and all involutions of* $\mathrm{PGL}_2(q)$ *outside* $\mathrm{PSL}_2(q)$.
- $O_2(G) = (O_2(G) \cap H)(O_2(G) \cap K)$

Proof. Let L_1, L_2 be normal proper subloops. These subloops are soluble by definition of the $2N$-loop. Notice, that L_1L_2 is another soluble normal subloop, thus a proper subloop too.
Therefore there exists a biggest proper normal subloop L_0, which is soluble. The quotient L/L_0 then is an N-loop as defined in 6.19. Let $D := \langle R(x) : x \in L_0 \rangle \leq G$. Then $D \leq O_2(G)$ and G/D is a loop envelope to an N-loop. If we manage to prove the statement for $(\tilde{G}, \tilde{H}, \tilde{K})$ with $\tilde{G} = G/D$, the statement

holds for (G, H, K), so we may assume $D = 1$.
The structure of a faithful loop envelope to an N-loop was described in Theorem 4, which implies the statement, together with 3.2(i).
If (G, H, K) is nonfaithful, $\mathrm{core}_G(H) \le Z(G)$: if $h^k \in H$, then $h^{-1}h^k = k^h k \in KK \cap H = 1$, therefore $[\mathrm{core}_G(H), \langle K\rangle] = 1$, but $\langle K\rangle = G$. Let $Z := O_{2'}(Z(G))$. Then $(\tilde{G}, \tilde{H}, \tilde{K})$ is a faithful loop envelope to an N-loop by 6.11, so we can apply Theorem 4. Let $\overline{G} = G/O_2(G)$. Then $\overline{G}$ is a central extension of $\mathrm{PGL}_2(q)$ with $\overline{Z}$ still contained in the group generated by $\overline{K}$. Thus $q = 9$ and $|Z| = 3$, as this is the only case of nontrivial odd order Schur multiplier of the groups in question. (The r-part of the Schur multiplier of a perfect group is nontrivial only for noncyclic Sylow-r-subgroups. The unique noncyclic case $q = 9$ actually results in a Schur multiplier $\mathbb{Z}_3$ for $\mathrm{Alt}_6 = \mathrm{PSL}_2(9)$.)
However in this case, involutions outside $\mathrm{PSL}_2(9)$ invert Z. This is visible using the embedding of $3\ \mathrm{Alt}_6$ into $\mathrm{SL}_3(4)$, see [ATLAS], p.23 for the action of $L_3(4)$-automorphisms on the Schur multiplier. This contradicts 6.23, so $Z = 1$.
The factorization $O_2(G) = (O_2(G) \cap H)(O_2(G) \cap K)$ can be seen as follows:
We have $O_2(G)H = H(O_2(G)H \cap K)$ by 6.2. Let $k \in K \cap O_2(G)H$. As $\overline{H}$ does not contain involutions from $\mathrm{PGL}_2(q)$ outside $\mathrm{PSL}_2(q)$,since the Sylow-2-subgroup of $\overline{H}$ is cyclic, we have $\overline{k} = 1$, so $k \in O_2(G)$. Now $O_2(G)H \cap K = O_2(G) \cap K$, so each coset of $O_2(G) \cap H$ in $O_2(G)$ contains exactly one element, as $|K \cap O_2(G)| = |O_2(G) : O_2(G) \cap H| = |HO_2(G) : O_2(G)| = |K \cap HO_2(G)|$.
□

Lemma 6.31 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2 with $G \ne O_2(G)H$. Then some subgroup $U \le G$ exists with*

- $U = (U \cap K)(U \cap H), U = \langle U \cap K\rangle$.
- *The loop to $(U, U \cap H, U \cap K)$ is a $2N$-loop, so*
- $F^*(U) = O_2(U)$,
- $U/O_2(U) \cong \mathrm{PGL}_2(q)$ *for $q \ge 5$ a Fermat prime or $q = 9$,*
- $|U : O_2(U)(U \cap H)| = q + 1$ *and*
- $\overline{K \cap U}$ *consists of 1 and all involutions of* $\mathrm{PGL}_2(q)$ *outside* $\mathrm{PSL}_2(q)$.
- *There exist elements of order $\frac{q+1}{2}$ in U inverted by elements of $K \cap U$.*
- *There exist elements $h \in U \cap H \cap G^{(\infty)}$ of order 3 in case $q = 9$ or q else.*
- *In particular $G^{(\infty)}$ contains a section isomorphic to* $\mathrm{PSL}_2(q)$.

Proof. We can find a subgroup U recursively: If the loop is nonsoluble, but every subloop is soluble, the loop is itself a $2N$-loop. Else we can find a proper nonsoluble subloop, which contains a $2N$-loop.
We may further assume, that $U = \langle U \cap K\rangle$. Then 6.30 describes the structure of U, which implies the statements. □

7 Reduction to $G/O_2(G)$ almost simple

If not explicitely defined otherwise, $\overline{G} = G/O_2(G)$ and for subsets $X \subseteq G$, $\overline{X}$ is the image of the natural homomorphism from G onto $\overline{G}$.

Definition 7.1 *Let S be a finite nonabelian simple group. Let $\mathcal{L}_S$ be the class of all Bol loops X of exponent 2, such that to X a loop folder (G_X, H_X, K_X) exists with $F^*(G_X/O_2(G_X)) \cong S$.*
A prime $p, p > 2$ is called **passive against** *S, if for all $X \in \mathcal{L}_S$: $p \nmid |X|$. (p may itself not divide $|S|$.)*
The smallest passive prime $p \in \pi(S)$ is called the **anchor prime** *of S. It is the smallest odd prime $p \in \pi(S)$ with the property:*
For every $X \in \mathcal{L}_S$: $p \nmid |G_X : H_X| = |X|$, so $\mathrm{Syl}_p(H_X) \subseteq \mathrm{Syl}_p(G_X)$.
This prime may not exist, if there are no passive primes in $\pi(S) - \{2\}$.
The finite nonabelian simple group S is called **passive**, *iff every odd prime $p \in \pi(S)$ is passive.*
Equivalently: For every $X \in \mathcal{L}_S$: X is soluble or equivalently: If (G, H, K) is any loop folder to a Bol loop of exponent 2 with $F^*(\overline{G}) \cong S$, then $G = O_2(G)H$.
The equivalence of these conditions follows from the $2N$-loop embedding 6.31: The $2N$-loop embedding states, that $\overline{G}^{(\infty)}$ contains elements of order either 3 or 5, which are products of two elements in K, so any $2N$-loop embedding prevents the primes 3 or 5 from being passive.
Therefore $2N$-loop embedding gives, that S is passive, iff both primes 3 and 5 are passive against S.

Remark 7.2 *The anchor prime to a finite nonabelian simple group may not exist. Its existence will be established later by classifiying the non-passive finite simple groups, using the classification of finite simple groups.*
If S is passive, then S has an anchor prime, usually 3, except in case of the Suzuki groups ${}^2B_2(q)$, where it is 5.

Lemma 7.3 *Let $S \cong \mathrm{PSL}_2(q)$ for $q \geq 5$ a Fermat prime. Then either q or 3 is the anchor prime of S.*

Proof. From the 2N-loop embedding, 6.31, we get always a Sylow-q-subgroup into H. For $q = 5$ the existence of examples ensures, that $q = 5$ is the smallest such prime. In the other cases there may be no examples of N-loops for the corresponding q, so $\mathrm{PSL}_2(q)$ is passive. Then $q = 3$ is the anchor prime. If examples exist, the anchor prime is q. □

Lemma 7.4 *Let $S \cong \mathrm{PSL}_2(9) \cong \mathrm{Alt}_6$. Then $p = 3$ is the anchor prime.*

Proof. Let (G, H, K) be a loop folder to a Bol loop of exponent 2 with $F^*(\overline{G}) \cong S$. If $G = O_2(G)H$, then H contains a Sylow-3-subgroup of G.
By 6.31 and Dixons theorem we can only embedd 2N-loops for $q = 5$ or $q = 9$. The case $q = 9$ implies, that H contains a Sylow-3-subgroup of G.
Otherwise suppose H contains elements of order 5. These elements are inverted by inner involutions of Alt_6 and (if in $\overline{G}$ existing) involutions of $\mathrm{PGL}_2(9)$ outside $\mathrm{PSL}_2(9)$. Therefore $\overline{K}$ can consist only of the 1-element, the 15 transpositions of

Σ_6 and the 15 involutions of Σ_6, which are a product of three commuting transpositions. Therefore $|F^*(\overline{G}) \cap \overline{H}| \geq 12$, so from the list of maximal subgroups of Alt_6 we conclude, that H contains elements of order 3. But the centralizer of an element of order 3 contains a Sylow-3-subgroup of G and is soluble, so H contains a Sylow-3-subgroup of G. By definition now 3 is the anchor prime to Alt_6. □

Definition 7.5 *Let (G, H, K) be a loop folder to Bol loop of exponent 2 and C a component of $\overline{G} = G/O_2(G)$. An* **anchor group** *A of C is a subgroup of $C \cap \overline{H}$ with $A \in \mathrm{Syl}_p(C)$ for the anchor prime p of $C/Z(C)$.*

Proposition 7.6 *Let (G, H, K) be a loop folder to a Bol loop of exponent 2 and suppose every nonabelian simple section of G has an anchor prime.*
Then every component of $\overline{G}$ has an anchor group.

Proof. The proof proceeds by induction on $|G|$.

(1): $O_{2'}(G) = 1$.
By induction on $G/O_{2'}(G)$, the statement holds for the loop folder from 6.11, but since $O_{2'}(G) \leq H$ by 6.4 then the statement holds in G too.

(2): $F(\overline{G}) = 1$.
By 6.7 we have $F(\overline{G}) \leq \overline{H}$. If $\overline{x} \in F(\overline{G})$ for some element $x \in H$ of odd prime order, then $(C_G(x), C_H(x), C_K(x))$ gives a subloop folder by 6.18(i). Since $O_{2'}(G) \neq 1$ by (1), $C_G(x)$ is a proper subgroup.
Now $C_G(x)$ covers $C_{\overline{G}}(\overline{x})$, which contains $E(\overline{G})$. Therefore anchor groups of components of $C_G(x)/O_2(C_G(x))$, which exist by induction, lift to anchor groups of $\overline{G}$.

(3): $E(\overline{G})$ contains more than one component.
Else $\overline{G}$ has a unique component, which has an anchor prime p by assumption. By definition of the anchor prime therefore an anchor group exists.

(4): If $C \cap \overline{H}$ contains elements of odd order for some component C of $\overline{G}$, then anchor groups for all components exists.
Let x be such an element. Then $C_G(x)$ covers all but the component C. By induction we get anchor groups for the components of $C_G(x)/O_2(C_G(x))$. But these lift to anchor groups for the components of $\overline{G}$, other than C. Since we have more than one component, we can use some element z of odd prime order in one of these anchor groups to get the anchor group to C by induction on $C_G(z)$, which covers C.

(5): All components of $\overline{G}$ are pairwise isomorphic.
Suppose $\overline{G}$ has nonisomorphic components C, D. Let $C_1, D_1 \leq \overline{G}$ be the products of all components isomorphic to C resp. D.
We claim $C_1\overline{H} \neq D_1\overline{H}$ or $C_1\overline{H} = \overline{H} = D_1\overline{H}$:

Suppose $C_1\overline{H} \neq \overline{H} \neq D_1\overline{H}$. For a group X let $r_C(X)$ be the number of composition factors of X isomorphic to C.

If $\overline{H} \cap C_1$ contains $r_C(C_1)$ composition factors isomorphic to C, then $C_1 = \overline{H} \cap C_1$, so $C_1\overline{H} = \overline{H}$, therefore $r_C(\overline{H} \cap C_1) < r_C(C_1) \leq r_C(\overline{G})$.
Now $\overline{G}/C_1 \cong \overline{H}/(\overline{H} \cap C_1)$. Therefore $r_C(\overline{G}) - r_C(C_1) = r_C(\overline{H}) - r_C(\overline{H} \cap C_1)$.
We conclude: $r_C(\overline{G}) > r_C(\overline{H})$. But this gives $D_1\overline{H} \neq C_1\overline{H}$.

Now at least one of $C_1\overline{H}$, $D_1\overline{H}$ is a proper subgroup. Suppose $|C_1\overline{H}| < |\overline{G}|$. Let B be the preimage of $C_1\overline{H}$. Since $H \leq B$, $(B, H, B \cap K)$ gives a loop folder to a subloop. By induction we get an anchor group A to C. By (4) we get now anchor groups for all components of $\overline{G}$.

(6): $\overline{H} \cap E(\overline{G})$ is a 2-group.
Otherwise let $\overline{x} \in \overline{H} \cap E(\overline{G})$ be of odd prime order p. We can write $\overline{x}$ uniquely as $\overline{x} = \overline{x_1}\,\overline{x_2} \cdots \overline{x_k}$ with $\overline{x_i} \in C_i$, $C_1, ..., C_k$ the components of $\overline{G}$. By (4), $\overline{x_i} \neq 1$ for every i, as otherwise $|C_G(x)| < |G|$, $\overline{C_G(x)}$ contains a component of $\overline{G}$ and we get anchor groups of $\overline{C_G(x)}$ and $\overline{G}$ by (4).
Now $C_{E(\overline{G})}(\overline{x})$ is the direct product of the $C_{C_i}(\overline{x_i})$. In particular $\langle \overline{x_1}, \overline{x_2}, ..., \overline{x_k} \rangle \leq O_p(C_{E(\overline{G})}(\overline{x})) \leq O_p(C_{\overline{G}}(\overline{x}))$. Let x be some preimage of $\overline{x}$ of order p.
Since $C_G(x)$ covers $C_{\overline{G}}(\overline{x})$, we have $O_p(C_{\overline{G}}(\overline{x}))$ covered by $O_{2,2'}(C_G(x))$. By 6.7, we may choose therefore preimages of the $\overline{x_i}$ in H.
By (4) we now get anchor primes for all components of $\overline{G}$.

Let $h \in H$ be of odd prime order p. Such an element exists by 6.31.

(7): $\overline{h}$ normalizes every component of $\overline{G}$.
Otherwise let C be a component with $C^{\overline{h}} \neq C$ and $D = CC^{\overline{h}} \cdots C^{\overline{h}^{p-1}}$, the closure of C under $\overline{h}$. Now $C_D(\overline{h}) = \{cc^{\overline{h}} \cdots c^{\overline{h}^{p-1}} : c \in C\} \cong C$.
By 6.18(i), $C_G(h)$ is a group to a subloop. Notice, that $C_D(\overline{h})$ maps to a component of $C_G(h)/O_2(C_G(h))$:
D is subnormal in $\overline{G}$, so $C_D(\overline{h})$ is subnormal in $C_{\overline{G}}(\overline{h})$, but $C_G(h)$ covers $C_{\overline{G}}(\overline{h})$.
By induction, we get an anchor group A to $C_D(\overline{h})$. But now $A \leq D \leq E(\overline{G}) \cap \overline{H}$ contains elements of odd order contrary to (6).

(8): We get anchor groups for all components of $\overline{G}$:
We use 6.31 to get elements $h \in H$ of odd prime order p, with the property: the normal closure N_h of $\overline{h}$ in $\overline{G}$ is nonsoluble.
Let G_1 be the subgroup of G consisting of all elements, which normalize every component of $\overline{G}$. Notice, that the preimage E of $E(\overline{G})$ is contained in G_1. But using the Schreier-conjecture, we get that G_1/E is soluble. By (7) we have $h \in G_1$. Therefore $N_h \leq G_1$. Since N_h is nonsoluble, and $h \in N_h^{(\infty)} \leq E$, we have a contradiction to (6). □

Consequences:

Lemma 7.7 *Let (G, H, K) be a loop envelope to a Bol loop of exponent 2 and every nonabelian simple section of G has an anchor prime.*
Then every element x of K normalizes every component C of $\overline{G}$.
In particular a component of $\overline{G}$ is either normal in $\langle \overline{K} \rangle$ or contained in $\overline{H}$.

Proof. Let $x \in K$ and C be a component of $\overline{G}$. Assume $C^x \neq C$. Let A, B be anchor groups to the components C and C^x respectively. These exist by 7.6. As C and C^x are isomorphic, the corresponding anchor primes p_1, p_2 are equal. In particular $AB \in \mathrm{Syl}_{p_1}(CC^x)$. Let $y \in A$ be of order p_1. We may choose $y \notin C^x$, as p_1 is odd, so not every element of order p of A is in $Z(C) \geq C \cap C^x$. Then x inverts the element $y^{-1}y^x$, which is of order p_1, thus conjugate to some element of $AB \leq \overline{H}$. This is a contradiction to 6.23.
So $[C, \langle\overline{K}\rangle] \leq C \cap \langle\overline{K}\rangle$. Therefore either $C \trianglelefteq \langle\overline{K}\rangle$ or $[C, \langle\overline{K}\rangle] = 1$. In the later case let $c \in C$ be of odd order. We can write $c = \overline{kh}$ with $k \in K$, $h \in H$. As $\overline{k}$ commutes with c, $\overline{k} = 1$, so $c \in \overline{H}$. As $C = O^2(C)$, $C \leq \overline{H}$. □

Now we proof the following theorem which then togehter with Theorem 1 implies Theorem 2.

Theorem 5 *Let (G, H, K) be a loop envelope of a Bol loop of exponent 2 and assume that every nonabelian simple section of G is either passive or isomorphic to $\mathrm{PSL}_2(q)$ for $q = 9$ or $q \geq 5$ a Fermat prime. Then the following holds.*

(a) $\overline{G} := G/O_2(G) \cong D_1 \times D_2 \times \cdots \times D_k$ *for some non-negative integer* k

(b) $D_i \cong PGL_2(q_i)$ *for* $q_i \geq 5$ *a Fermat prime or* $q_i = 9$

(c) $D_i \cap HO_2(G)/O_2(G) \cong q_i : (q_i - 1)$ *is a Borel subgroup in* D_i *of index* $q_i + 1$

(d) $F^*(G) = O_2(G)$

(e) $\overline{K}$ *is the set of involutions in* $\overline{G} \setminus \overline{G}'$

We use induction on the order of G.

As $\overline{G} = \langle\overline{K}\rangle$, but no element of $\overline{K}$ acts on $F(\overline{G})$ nontrivially by 6.23, $F(\overline{G}) \leq Z(\langle\overline{K}\rangle)$, so $F(\overline{G}) \leq E(\overline{G})$.

Now $O_{2'}(\overline{G}) = 1$: If $O_{2'}(G) \neq 1$, then $O_{2'}(G) \leq H$, so by 6.11 the theorem holds on $G/O_{2'}(G)$. Now G is a central extension of $G/O_{2'}(G)$ with $\overline{G/O_{2'}(G)}$ a direct product of groups of isomorphism type $\mathrm{PGL}_2(q)$ for $q = 9$ or $q \geq 5$ a Fermat prime.
If the extension is not perfect on $E(G/O_{2'}(G))$, G has a factor group of odd index, a contradiction to $G = \langle K\rangle$. But the Schur multipliers of the components have no odd part, except for components of type Alt_6. In that case however $\mathrm{PGL}_2(9)$-involutions (which are present in $\overline{K}$) invert the center, as described in [ATLAS], p.23. This gives a contradiction to 6.3.
Else we can find a subgroup $Z \leq H$ of odd order, such that $\overline{Z} \leq Z(E(\overline{G}))$. By 6.23 then $\overline{Z} \leq Z(\overline{G})$. Now $C_G(Z)$ is a proper subgroup with $(C_G(Z), C_H(Z), C_K(Z))$ being a subloop folder and $G = O_2(G)C_G(Z)$.
By induction on $\langle C_G(Z) \cap K\rangle$, $O_{2'}(\langle C_G(Z) \cap K\rangle) = 1$, so $Z \not\leq \langle C_G(Z) \cap K\rangle$. But this produces a subgroup of odd index in G, a contradiction to the assumption $G = \langle K\rangle$.

If now $\overline{G}$ has a unique component, this component is either passive or of type $L_2(q)$ for $q = 9$ or $q \geq 5$ a Fermat prime.

If the component is passive, $\overline{H} = \overline{G}$, a contradiction to 6.24.
We get an anchor group A of $F^*(\overline{G})$ by 7.6.
We also use 6.31 to get a subgroup G_0 to a 2N-subloop with $G_0/O_2(G_0) \cong F^*(\overline{G})$ or $F^*(\overline{G_0}) \cong \mathrm{Alt}_5$ and $F^*(\overline{G}) \cong \mathrm{Alt}_6$. In that case, $|\overline{G} : \overline{H}|$ is a 2-power, as $A \leq \overline{H}$ contains a Sylow-3-subgroup of $\overline{G}$ and a Sylow-5-subgroup from $\overline{H} \cap \overline{G_0}$. Then $\langle K\rangle \leq O_2(G)$ by 6.9, a contradiction to $\langle K\rangle = G$.
We conclude, that $\overline{G} = \overline{G_0}$ or $q = 9$ and $|\overline{G} : \overline{G_0}| = 2$, so $\overline{G} \cong \mathrm{Aut}(\mathrm{Alt}_6)$. This case leads to a contradiction, as $\overline{K}$ consists only of 1 and maybe the 36 involutions from $\mathrm{PGL}_2(9)$ outside $\mathrm{PSL}_2(9)$, as this are the only involutions not inverting elements of order 3, but then $\overline{K} < \overline{G}$.
In particular we have $|\overline{G} : \overline{H}| = q+1$, as already $|G_0 : O_2(G_0)(H \cap G_0)| = q+1$ by 6.30. Furthermore the subgroup G_0 actually shows, that $O_2(G) = F^*(G)$.

If $\overline{G}$ has two components C_1, C_2, we get anchor groups $A_i \leq C_i$ by 7.6. By 7.7, both C_i are normal in $\overline{G}$. Let $B_i \leq H$ of odd order with $\overline{B_i} = A_i$. We can use induction on $G_i := \langle C_G(B_i) \cap K\rangle$ by use of 6.18(i) or repeated use in case of Alt_6-components (the only case, such that B_i is not cyclic).
As the theorem holds on G_i, we get the factorization of $\overline{G}$ into the subgroups D_i and the fact, that $|D_i : D_i \cap \overline{H}| = q_i + 1$ from G_i. For each component of $\overline{G}$ we get a D_i containing that component.
In particular no passive components occure and $F^*(G) = O_2(G)$. □

8 Passive simple groups: general arguments

We give here some general arguments involving both certain simple groups and arguments on loop folders. These arguments are used in the next section to show, that almost all finite simple groups are passive.
In this section (G, H, K) is a loop folder to a Bol loop of exponent 2, $\overline{G} = G/O_2(G)$, $F^*(\overline{G}) \cong S$ with S some finite simple nonabelian group, $S \leq T \leq \mathrm{Aut}(S)$ with $\overline{G} \cong T$ and G_0 the preimage of $F^*(\overline{G})$.

8.1 An assumption and consequences

Lemma 8.1 *We may assume $G = \langle K\rangle$, so $\overline{G} \cong T$ and T/S are generated by involutions.*

Proof. This is 6.2:
Let $g \in \langle K\rangle$. Then there exist $h \in H, k \in K$ with $g = hk$. As $k \in \langle K\rangle$, $h \in \langle K\rangle \cap H$. Therefore $\langle K\rangle = KH_0$ with $H_0 = \langle K\rangle \cap H$, so $(\langle K\rangle, H \cap \langle K\rangle, K)$ is a subloop folder to (G, H, K). As $|G : H| = |K| = |\langle K\rangle : H_0|$, this is a subloop folder to the same loop. □

This has consequences on the structure of T/S:

By the famous Theorem of Steinberg on the structure of $\mathrm{Aut}(S)$, (Theorem 2.5.1 in [GLS3]), every automorphism of S is a product of an inner, diagonal, field and graph automorphism. Moreover Theorem 2.5.12 in [GLS3] gives a detailed description of $\mathrm{Aut}(S)$:

Aut(S) is a semidirect product of a normal subgroup $\mathrm{InnDiag}(S) \le \mathrm{Aut}(S)$ with a subgroup $\Phi\Gamma$. InnDiag(S) is the subgroup consisting of inner and diagonal automorphisms, while $\Phi\Gamma$, is a product of a cyclic group Φ (inducing field automorphisms) with a supplement Γ, such that $\Phi\Gamma/\Phi$ is a group of automorphisms of the Dynkin diagram.
By Theorem 2.5.12(e), if the group is untwisted and the Dynkin diagram contains only roots of one length, then $\Phi\Gamma = \Phi \times \Gamma$ with Γ the full automorphism group of the Dynkin diagram. If the group is untwisted, but the Dynkin diagram contains roots of different length and a graph automorphism of order 2, (so the group is $B_2(q), F_4(q)$ in characteristic 2 or $G_2(q)$ in characteristic 3) then $\Phi\Gamma$ is cyclic, with a generator in Γ, which squares to a Frobenius automorphism of GF(q) generating Φ.
If the group is twisted, then $\Gamma = 1$.
We will use definition 2.5.13 of [GLS3] for the terms **field**, **graph-field** and **graph** automorphism.

Lemma 8.2 *Let S be a group of Lie type in characteristic p.
If T is generated by involutions, then $T/(T \cap \mathrm{InnDiag}(S))$ is isomorphic to $1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$, Σ_3 or $\mathbb{Z}_2 \times \Sigma_3$. By 8.1 we may assume this. In particular:*

(1) T does not contain field automorphisms of order bigger than two

(2) In case $S \cong B_2(q)$, $F_4(q)$, or $G_2(q)$, $|T : T \cap \mathrm{InnDiag}(S)| \le 2$.

(3) $|T : S|_2 \le 4$, if S is a group of Lie type in characteristic 2 or $|\mathrm{InnDiag}(S) : S|$ is odd.

Proof. This is a consequence of Theorem of 2.5.12 of [GLS3]. □

We now establish some consequences in even characteristic:

Lemma 8.3 *Let S be a group of Lie type in characteristic 2 and $X \le T$ with $O_2(X) \cap S = 1$.
Then $|O_2(X)| \le 4$,$O_2(X) \le Z(X)$ and $O_2(X) \cap O^2(X) = 1 = O_2(X) \cap O^2(T)$.*

Proof. By 8.2 we know T/S. In particular we see, that $|O^2(T) : S|$ is odd, while $|O_2(X)S : S| = |O_2(X)|$. Therefore $|O^2(T)O_2(X)| = |O^2(T)||O_2(X)|$, so $O^2(T) \cap O_2(X) = 1$.From 8.2(3) we get $|O_2(X)| \le 4$.
We can write $X = O^2(X)P$ for for some $P \in \mathrm{Syl}_2(X)$. Then $|PO^2(T) : O^2(T)| \le 4$, so $(PO^2(T))' \le O^2(T)$. As $X \le O^2(T)P$ we get $[X, O_2(X)] \le (PO^2(T))' \cap O_2(X) \le O^2(T) \cap O_2(X) = 1$. □

Corollary 8.4 *Let S be group of Lie type in characteristic 2 and $U \le G$ a subgroup to a soluble subloop. If $\overline{U}$ is reductive, so $O_2(\overline{U}) \cap \overline{G_0} = 1$, then $O^2(\overline{U}) \le \overline{H}$.*

Proof. By 8.3 we get $O_2(U) \cap O^2(U) \le O_2(G)$. Now 6.27 gives the statement. □

Standard examples, where 8.4 may be applied, are centralizers of elements of odd order in $\overline{H} \cap \overline{G_0}$, as centralizers of semisimple elements in S are reductive.

8.2 Centralizers

The main connection between the local structure of loops and local subgroups of almost simple groups is 6.18(i).
Given a subgroup $1 \neq L \leq H$ of odd order, we have $C_G(L)$ covered by $C_{\overline{G}}(\overline{L})$ due to coprime action. As $C_G(L)$ does not cover $\overline{G}$, we may apply Theorem 5 on $\langle C_G(L) \cap K \rangle$. On the other hand we know from the local structure of simple groups, how $C_{\overline{G}}(\overline{L})$ looks like. Putting things together, we often can identify $C_{\overline{H}}(\overline{L})$ within $C_{\overline{G}}(\overline{L})$, without even knowing $\overline{H}$ completely.
We give here some lemmata, based on this idea.

Lemma 8.5 *Let $S \cong \mathrm{Alt}_n$ for $n \geq 7$ and $x \in H$ be of odd prime order p. Let k be the number of fixed points of $\overline{x}$ in the natural action of $\overline{G}$ on n points. If $k \neq 5$, then $|C_G(x) : C_H(x)|$ is a 2-power. If $k \notin \{4, 5\}$, then $O^2(C_{\overline{G}}(\overline{x})) \leq \overline{H}$. Remember that $O_p(C_{\overline{G}}(\overline{x})) \leq \overline{H}$ by 6.7. If $k = 5$, then $|C_G(x)O_2(G) : C_H(x)O_2(G)| \in \{1, 6\}$. In any case, $\overline{H}$ contains p-cycles from $O_p(C_{\overline{G}}(\overline{x}))$.*

Proof. The structure of $C_{\overline{G}}(\overline{x})$ is well known, as elements commuting with $\overline{x}$ permute the cycles of $\overline{x}$ and act on the k fixed points.
So we may apply the structure description of Theorem 5 on $C_G(x)$, which is an extension of $O_2(G) \cap C_G(x)$ by $C_{\overline{G}}(\overline{x})$. In particular $C_G(x)/O_2(C_G(x))$ has no subnormal $\mathrm{PGL}_2(q)$ for $q > 5$. There may be a subnormal $\mathrm{PSL}_2(9) \cong \mathrm{Alt}_6$, but the outer involution is missing in $C_{\overline{G}}(\overline{x})$, seen as a subgroup of Σ_n.
Therefore the subloop to $C_G(x)$ is soluble, if $k \neq 5$, as there is no subnormal Σ_5 in this case.
If $k = 5$, there may be a subnormal Σ_5, acting on the 5 fixed points of $\overline{x}$. In this case Theorem 5 decribes the structure of $\langle \overline{K} \cap C_{\overline{G}}(\overline{x})$ and its intersection with $\overline{H}$.
If $k \neq 4, 5$, then $O_2(C_{\overline{G}}(\overline{x})) = 1$ and the subloop to $C_G(x)$ is soluble, so by 6.27 we have $O^2(C_{\overline{G}}(\overline{x})) \leq \overline{H}$.
We could even determine, which elements end up in $\overline{H}$ in cases $k = 4$ and $k = 5$, but have no use for it.
By 6.7, $\overline{H}$ contains $O_p(C_{\overline{G}}(\overline{x}))$, so in particular p-cycles. $\square$

Lemma 8.6 *Let S be sporadic and $x \in H$ be of odd prime order p. Then $|C_G(x) : C_H(x)|$ is not divisible by p, unless maybe (p, S) is one of $(3, M_{23})$, $(3, HS)$ or $(5, Suz)$.*

Proof. This is a consequence of the list of centralizers of elements of prime orders in sporadic groups in [GLS3] and Theorem 5:
In case $p = 3$ we have to check, which centralizers of elements of order 3 contain components of type $\mathrm{PSL}_2(r)$ for r some Fermat prime, $r \geq 5$.
In case $p = 5$ we have to check, that centralizers of elements of order 5 do not contain components of type Alt_6.
In the cases listed above, there are elements of order 3 resp. 5, such that the corresponding centralizers may even contain subnormal subgroups isomorphic

to $\mathrm{PGL}_2(5)$ resp. $\mathrm{PGL}_2(9)$, depending on the presence of outer automorphisms of S in T. $\square$

Lemma 8.7 *Let S be a group of Lie type in odd characteristic p and $x \in H$ be of odd prime order r. If $r = p$, then $|C_G(x) : C_H(x)|$ is not divisible by p. If $r \neq p$, then $\overline{H} \cap \overline{G_0}$ contains elements of order p or $C_{\overline{G}}(\overline{x})$ is soluble or both.*

Proof. By 8.2, $\overline{x}$ is either innerdiagonal or $S \cong D_4(q)$ and $\overline{x}$ is a graph or graph-field automorphism of order 3.
In case $\overline{x}$ innerdiagonal we have the cases $\overline{x}$ unipotent ($r = p$) or semisimple ($r \neq p$).

In the semisimple case we use Theorem 4.2.2 of [GLS3] for the description of $C_{\overline{G}}(\overline{x})$. In particular nonsoluble composition factors of $C_{\overline{G}}(\overline{x})$ are groups of Lie type in characteristic p. Therefore, if $\langle C_G(x) \cap K\rangle \not\leq O_2(C_G(x))$, then by Theorem 5 and induction, $\overline{G_0} \cap \overline{H}$ contains elements of order p. Recall, that this may happen only, if $C_G(x)$ is nonsoluble by Theorem 3.

If $\overline{x}$ is unipotent, so $r = p$ and $x \in G_0$, then we may apply the Borel-Tits Theorem on $C_{\overline{G}}(\overline{x})$, (Theorem 3.1.3 and Corollary 3.1.4 of [GLS3]).

In the exceptional case of $D_4(q)$, the centralizer of a graph or graph-field automorphism of order 3 is decribed by Proposition 4.9.1 and 4.9.2 of [GLS3] and listed in 4.7.3 in [GLS3]. $C_{\overline{G}}(\overline{x})$ contains a subnormal ${}^3D_4(q^{1/3})$ or a $G_2(q)$, which by induction is passive, so the subloop to $C_G(x)$ is soluble, so $\overline{G_0} \cap \overline{H}$ contains elements of order p. $\square$

Lemma 8.8 *Let S be a group of Lie type in even characteristic with q the field parameter of S. (If S is defined relative to a field extension, q is the size of the smaller field.)*
Let $x \in H$ be of odd prime order r.
If $q \geq 8$, then $C_G(x)$ gives a soluble subloop.
If $q = 4$, then $\pi(|C_G(x) : C_H(x)|) \subseteq \{2, 3\}$, so the subloop to $C_G(x)$ may not be soluble.
If $q = 2$, then $\pi(|C_G(x) : C_H(x)|) \subseteq \{2, 3, 5\}$, so again the subloop to $C_G(x)$ may not be soluble.
If the subloop to $C_G(x)$ is soluble, then $O^2(C_{\overline{G}}(\overline{x})) \leq \overline{H}$.

Proof. By 8.2 we may assume, that T does not contain field automorphisms of odd order. Therefore automorphisms of S of odd order are either innerdiagonal, so semisimple or are graph or graph-field automorphisms of order 3 in case of $S \cong D_4(q)$.
For these automorphisms we refer to Propositions 4.9.1 and 4.9.2 as well as 4.7.3 of [GLS3]. In that case $C_{\overline{G}}(\overline{x})$ is reductive and contains a unique component (isomorphic to $G_2(q)$ or ${}^3D_4(q^{1/3})$), which is passive by induction, so the subloop to $C_G(x)$ is soluble.
The centralizers of semisimple elements are reductive. Moreover the structure of the centralizers is described by Theorem 4.2.2 of [GLS3]. In particular the nonsoluble composition factors come from components, which are groups of Lie type in characteristic 2 and defined over field extensions of $\mathrm{GF}(q)$. The only

nonpassive components, which may arise, are $\mathrm{Sp}_4(2)' \cong \mathrm{Alt}_6 \cong \mathrm{PSL}_2(9)$ and $\mathrm{PSL}_2(4) \cong \mathrm{Alt}_5 \cong \mathrm{PSL}_2(5)$. As $\mathrm{Sp}_4(2)'$ is a group defined over $\mathrm{GF}(2)$, it does not arise if $q > 2$. As $\mathrm{PSL}_2(4)$ is a group defined over $\mathrm{GF}(4)$, it does not arise as a component, if $q > 4$. This is the reason for the case division $q > 4$, $q = 4$ and $q = 2$.
Now Theorem 5 gives solubility of the subloop for $q > 4$, as no such component occurs. In case $q = 4$ it gives, that $|C_G(x) : C_H(x)|$ is a 2-power times a 3-power, as only $\mathrm{PSL}_2(4)$-components may be not passive. Finally in case $q = 2$ it gives, that $\pi(|C_G(x) : C_H(x)|) \subseteq \{2, 3, 5\}$, as there may occure $\mathrm{PSL}_2(4)$- or $\mathrm{Sp}_4(2)'$-components, but no other non-passive components.

Notice, that in any case $C_{\overline{G}}(\overline{x})$ is reductive, so we may use 8.4, if the subloop is soluble. □

Corollary 8.9 *Let S be a group of Lie type in characteristic 2, as in 8.8. For $q > 4$ we have $O^2(C_{\overline{G}}(\overline{x})) \leq \overline{H}$, so if $\overline{x}, \overline{y} \in \overline{G}$ are elements of odd order with $\overline{x} \in \overline{H}$, then $\overline{y} \in \overline{H}$.*
For $q = 4$ we have either FS_3-property or $\overline{H}$ contains elements of order 15. Furthermore either $5 \in \pi(H)$ or with $\overline{x} \in \overline{H}$ of odd prime order the full connected component $\mathcal{C}_{\overline{x}}$ of $\overline{x}$ in $\Gamma_{\mathcal{O}}$ to S is contained in $\overline{H}$.
For $q = 2$ we have either both FS_3 and FS_5-property or $\overline{H}$ contains elements of order 15.

Proof. For $q > 4$, subloops to centralizers of elements of odd prime order are soluble. Then the statement is 8.4 together with the fact, that centralizers of semisimple elements (or outer automorphisms of order 3 in case of $D_4(q)$) are reductive, by Theorems 4.2.2, 4.9.1, 4.9.2 and 4.7.3 of [GLS3].
For $q = 4$, how can FS_3-property fail? Only, if there is some element $x \in H$, $o(x) = 3$, such that $C_G(x)$ gives a nonsoluble subloop in G, so $C_{\overline{G}}(\overline{x})$ contains a subnormal $\mathrm{PSL}_2(4)$. In that case the size of the subloop is a 2-power times a 3-power, so $C_H(x)$ contains elements of order 5, so $\overline{H}$ contains elements of order 15.
If $\overline{H}$ contains no elements of order 5, the subloops to $C_G(x)$ for $x \in H$ of odd order are soluble, so by 8.8 $\mathcal{C}_{\overline{x}} \subseteq \overline{H}$.
For $q = 2$ there is also the possibility for the FS_5-property to fail: There may exist some element $x \in H$, $o(x) = 5$, such that the subloop to $C_G(x)$ is nonsoluble, but the size of the loop is divisible by 5. So some elements of order 5 are commutators of elements of $\overline{K}$, and $\overline{H}$ cannot contain a Sylow-5-subgroup of $\overline{G}$. In that case $C_{\overline{G}}(\overline{x})$ contains a subnormal $\mathrm{Sp}_4(2)'$, so $C_H(x)$ contains elements of order 15. □

8.3 The property FS_p

Recall the class $\mathcal{L}_S$ from 7.1. We have to generalize this concept slightly to our group T to avoid difficulties:

Definition 8.10 *We denote the class ℓ_T of Bol loops X of exponent 2, for which a loop folder (G_X, H_X, K_X) exists with $G_X/O_2(G_X) \cong T$.*

Let $p \in \pi(T), p > 2$. The class ℓ_T has the **property** *FS_p, iff for all $X \in \ell_T$: Either $|X|_p = |T|_p$ or $|X|_p = 1$.*

Lemma 8.11 *The class ℓ_T has property FS_p, iff for every loop folder to a Bol loop of exponent 2 with $G/O_2(G) \cong T$: $p \nmid (|H|, |G:H|)$, so $\mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$ or $\mathrm{Syl}_p(H) = 1$. (And FS_p stands for 'full Sylow-p'.)*

Proof. Suppose (G, H, K) is a loop folder to a Bol loop X of exponent 2 with the property: If $p \in \pi(H)$, then $p \nmid |G:H|$.
Then either $p \in \pi(H)$ and $p \nmid |X| = |G:H|$ or $p \notin \pi(H)$, so $|X|_p = |G:H|_p = |G|_p = |T|_p$. If every loop folder (G, H, K) with $G/O_2(G) \cong T$ has the above property, then the class ℓ_T has property FS_p.
The converse statement is immediate from the definition. Notice, that the property $(|H_X|, |G_X : H_X|)_p = 1$ depends only on the isomorphism type of X, not on the particular loop folder (G_X, H_X, K_X) to X. □

The reason for defining this property FS_p is, that it can be established from the p-local structure of T in many cases, and has powerful applications.

Lemma 8.12 *The class ℓ_T has property FS_p, if any $\overline{x} \in T$, $o(\overline{x}) = p$ satisfies one of:*

- *(0) $p \nmid |C_{\overline{G}}(\overline{x}) : C_{\overline{H}}(\overline{x})|$.*
- *(1) $C_T(\overline{x})$ is soluble.*
- *(2) $F^*(C_T(\overline{x})) = O_p(C_T(\overline{x}))$ for $p > 2$.*
- *(3) $C_T(\overline{x})/O_2(C_T(\overline{x}))$ has only passive components.*
- *(4) $C_T(\overline{x})/O_2(C_T(\overline{x}))$ has no subnormal $\mathrm{PGL}_2(q)$ for $q = 9$ or a Fermat prime $q \geq 5$ with $p|q+1$.*

Proof. The general argument in all cases is the same:
Suppose $p \in \pi(H)$. We will show, that each of (1)-(4) implies (0). Once this is established, H contains a Sylow-p-subgroup of G for the following reason:
Every element $x \in G$ of order p is centralized by some element y of order p with $y \in \Omega_1(Z(Y))$ for some $Y \in \mathrm{Syl}_p(G)$ with $x \in Y$.
If $x \in H$, then by (0), some G-conjugate z of y is in H. Using (0) on $C_G(z)$ we get a Sylow-p-subgroup of G into H.
Notice, that $C_T(\overline{x})$ is covered by $C_G(x)$ for $x \in H$ some preimage of $\overline{x}$ of order p, due to coprime action. This enables to establish (0) from information of $\overline{G}$ only:

By 6.18, $C_G(x)$ gives a subloop folder, so we can use inductive arguments on $C_G(x)$.
In case (1), if $C_T(x)$ is soluble, then $C_G(x)$ is soluble, so by Theorem 3 $|C_G(x) : C_H(x)|$ is a 2-power and we have (0).
In case (2) we have $|C_G(x) : C_H(x)|_{2'} = 1$ by 6.25.
In case (3) we use Theorem 5 to establish, that $|C_G(x) : C_H(x)|$ is a 2-power, as only passive components show up, so $\langle K \cap C_G(x) \rangle \trianglelefteq C_G(x)$ has to be a 2-group.

Case (4) gives the most powerful criterion: The condition on $C_T(\overline{x})$ gives a condition on the structure of $C_G(x)$. Using the factorization $C_G(x) = C_H(x)\langle C_G(x) \cap K\rangle$ and Theorem 5 for the structure description of $\langle C_G(x) \cap K\rangle$, we see that $|O_2(G)C_G(x) : O_2(G)C_H(x)|$ is a product of integers $q_i + 1$ for $q_i - 1 \geq 4$ a 2-power. But the condition (4) on $C_T(\overline{x})$ ensures, that none of $q_i + 1$ is divisible by p, so $|C_G(x) : C_H(x)|$ is indeed not divisible by p. □

We now establish the FS_p-property in certain cases of the classification of finite simple groups. Notice, that a critical point may arise from the existence of outer automorphisms of S of odd order, as we may have to establish condition (0) for such elements too. This is one reason for the assumption, which led to 8.2.

Lemma 8.13 *Let S be an alternating group and $S \leq T \leq \mathrm{Aut}(S)$ and $p > 3, p \in \pi(T)$. Then ℓ_T has property FS_p.*

Proof. By 8.5 we have condition (4) of 8.12 for $p > 3$. □

Lemma 8.14 *If S is sporadic, $S \leq T \leq \mathrm{Aut}(S)$ and $p > 2$, then ℓ_T has property FS_p, unless (p, S) is one of $(3, M_{23})$, $(3, HS)$ or $(5, Suz)$.*

Proof. We can use condition (4) of 8.12 by 8.6. □

Lemma 8.15 *If S is a group of Lie type in characteristic p, $p > 2$ and $S \leq T \leq \mathrm{Aut}(S)$, then ℓ_T has property FS_p.*

Proof. By 8.7 we have either condition (2) or condition (3) of 8.12. □

This fact implies later, that in odd characteristic p and Lie rank at most two, $\overline{G} = \overline{H}$, if $p \in \pi(H)$, see 8.20.
But before this we continue with groups of Lie type in characteristic 2.

Lemma 8.16 *Let S be a group of Lie type in characteristic 2, defined over the field with q elements. (In case the group is defined relative to a field extension, q refers to the smaller field.) Let $S \leq T \leq \mathrm{Aut}(S)$.*
If $q > 4$, then ℓ_T has property FS_p for every prime $p > 2$.
If $q = 4$, then ℓ_T has property FS_p for every prime $p > 3$.
If $q = 2$, then ℓ_T has property FS_p for every prime $p > 5$.

Proof. This is a consequence of 8.8, which enables condition (4) of 8.12 under the given restrictions. □

8.4 Terminal elements

One strategy in the generic case (where simple groups are big enough), is the identification of terminal elements.
Plainly, an element $\overline{x} \in \overline{G}$ is terminal, if $\overline{x} \in \overline{H}$ implies $\overline{G} = \overline{H}$.
This property can sometimes established from the structure of $C_T(\overline{x})$ together

with the structure of T. We will give here some examples, which we will use in Section 6.
We may need a little lemma:

Lemma 8.17 *Assume $\overline{G_0} \leq \overline{H}$. Then $\overline{G} = \overline{H}$.*

Proof. Assume otherwise, so the loop to G is nonsoluble. We get a contradiction from 6.31:
If the loop to G is nonsoluble, then $\overline{G_0}$ contains elements of odd order, which are not in $\overline{H}$, as they are commutators of elements of $\overline{K}$. □

Lemma 8.18 *Let $S \cong \mathrm{Alt}_n$ for $n \geq 9$. If $\overline{H}$ contains a 3-cycle $\overline{x}$, then $\overline{H} = \overline{G}$.*

Proof. Let $x \in H$ be of order 3, a preimage of $\overline{x}$. By 8.5 we have, that $C_G(x)$ gives a soluble subloop. Moreover $O_2(C_{\overline{G}}(\overline{x})) = 1$, so by 6.27 we have $O^2(C_{\overline{G}}(\overline{x})) \leq \overline{H}$. In particular $\overline{H}$ contains with $\overline{x}$ all 3-cycles, which commute with $\overline{x}$. As the commuting graph of 3-cycles is connected (see 3.7), we have $\overline{G_0} \leq \overline{H}$, which implies $\overline{G} = \overline{H}$ by 8.17. □

Lemma 8.19 *Let S be a group of Lie type in odd characteristic p. Assume the (twisted) Lie rank is not 1 (so S is not of type A_1, ${}^2A_2$ or ${}^2G_2$) and $\overline{H}$ contains elements of order dividing p.*
Then $\overline{G} = \overline{H}$.

Proof. By 8.15, ℓ_T has property FS_p. Since $\overline{H}$ contains elements of order p, $\overline{H}$ contains a Sylow-p-subgroup of $\overline{G}$.
Since the (twisted) Lie rank of S is not 1,we may find subgroups $V_1, V_2 \leq P$, such that the normalizers of $\overline{V_i}$ in $\overline{G_0}$ contain different parabolic subgroups of $\overline{G_0}$, which together generate $\overline{G_0}$.
By 6.18(1) and 6.25, $\overline{H}$ covers these parabolic subgroups, so $\overline{H}$ covers $\overline{G_0}$. By 8.17 we have $\overline{G} = \overline{H}$. □

Corollary 8.20 *Let S be a group of Lie type in odd characteristic p. Assume, the (twisted) Lie rank is not 1 and let $r \in \pi(T), r > 2$. Then ℓ_T has property FS_r.*
Suppose $\overline{H} \neq \overline{G}$. Let $x \in H \cap G_0$ of odd prime order. Then $\overline{x}$ is not in the big connected component of $\Gamma_{\mathcal{O}}$.

Proof. Let $\overline{x} \in \overline{H}$ be of order r. By 8.7 either $\overline{H} \cap \overline{G_0}$ contains elements of order p or $C_{\overline{G}}(\overline{x})$ is soluble. In the first case $\overline{H} = \overline{G}$ by 8.19, so $\overline{H}$ contains a Sylow-r-subgroup and we have (0) of criterion 8.12, while in the second case we may use (1) of 8.12 to get property FS_r.

Notice, that $H \cap G_0$ contains elements of odd order by 6.31.
Unfortunately we cannot use 6.27 or 6.28, as we have no control about $O_2(C_G(x))$.
Suppose $\overline{x}$ is in the big connected component of $\Gamma_{\mathcal{O}}$.
Let $\pi = (\overline{x_i})$, $i \in \{1, .., k\}$ be a path of shortest length in $\Gamma_{\mathcal{O}}$ from some element

$\overline{x_1} \in \overline{H}$ of odd prime order to some element $\overline{k}$ of order p.
Suppose $s = o(\overline{x_1}) = o(\overline{x_2})$.
By FS_s-property, H contains a Sylow-s-subgroup of G, so we may find a $\overline{g} \in \overline{G}$, such that $\langle \overline{x_1}, \overline{x_2} \rangle^{\overline{g}} \leq \overline{H}$. As $\overline{x_2}^{\overline{g}} \in \overline{H}$, we get a shorter path by dropping $\overline{x_1}^{\overline{g}}$ from $\pi^{\overline{g}}$.
Suppose $s = o(\overline{x_1}) \neq o(\overline{x_2}) = t$. Choose $x_1 \in H$ in the preimage of $\overline{x_1}$. Recall, that the subloop to $C_G(x_1)$ is soluble. Furthermore $C_G(x_1)$ covers $C_{\overline{G}}(\overline{x_1})$ by coprime action, so $C_G(x_1)$ contains elements of order t. As $|C_G(x_1) : C_H(x_1)|$ is a 2-power, $t \in \pi(H)$ and by FS_t-property, H contains a Sylow-t-subgroup of G.
Therefore some $\overline{g} \in \overline{G}$ exists with $\overline{x_2}^{\overline{g}} \in \overline{H}$. We get again a shorter path from $\pi^{\overline{g}}$ by dropping $\overline{x_1}^{\overline{g}}$. Consequently the path consists of $\overline{x_1}$ only, so $\overline{H}$ contains elements of order p and $\overline{H} = \overline{G}$. □

Lemma 8.21 *Let S be a group of Lie type in characteristic 2 and $x \in H \cap G_0$ of odd order $r > 1$. Assume, that the commuting graph of $\overline{x}^{\overline{G_0}}$ in $\overline{G_0}$ is connected and the subloop to $C_G(x)$ is soluble.*
Then $\overline{H} = \overline{G}$.

Proof. By 8.4 we have $O^2(C_{\overline{G}}(\overline{y})) \leq \overline{H}$ for $y = x$ and conjugates of x, which are contained in $\overline{H}$. Therefore with $\overline{x} \in \overline{H}$ all $\overline{G_0}$-conjugates of $\overline{x}$, which commute with $\overline{x}$, are in $\overline{H}$ too. Then $\overline{H}$ contains $\overline{G_0}$, so by 8.17, $\overline{H} = \overline{G}$. □

Once certain elements are established as being terminal, we can classify 'isolated elements'. An element $x \in \overline{G}$ is called inductive, if $\overline{x} \in \overline{H}$ implies, that $\overline{y} \in \overline{H}$ for $\overline{y}$ either a terminal element or an inductive element.
Elements, which are neither terminal or inductive are called isolated. In the characteristic 2-case with $q > 4$, inductive elements are simply elements from the same connected component in $\Gamma_{\mathcal{O}}$, while isolated elements come from the small connected components.

8.5 Other recurring arguments

The following lemma is often used in case of cyclic groups $\overline{L} \leq \overline{H}$ to get additional primes into $\overline{H}$.

Lemma 8.22 *Given $1 \neq \overline{L} \leq \overline{H}$ with $|\overline{L}|$ odd, then $|N_{\overline{G}}(\overline{L}) : C_{\overline{G}}(\overline{L})|_{2'}$ divides $|\overline{H}|$.*

Proof. Let $L \leq H$ be a preimage of $\overline{L}$ with $|L| = |\overline{L}|$. By 6.18(i), both $(N_G(L), N_H(L), C_K(L))$ and $(C_G(L), C_H(L), C_K(L))$ are subloop folders. As $\langle C_K(L) \rangle \leq C_G(L)$ and $N_G(L) = N_H(L)\langle C_K(L) \rangle$ we have that $|N_G(L) : C_G(L)|$ divides $|H|$.
By coprime action we have $|N_{\overline{G}}(\overline{L})|_{2'} = |N_G(L)|_{2'}$, $|C_{\overline{G}}(\overline{L})|_{2'}| = |C_G(L)|_{2'}$ and $|H|_{2'} = |\overline{H}|_{2'}$, which implies the lemma. □

9 Passive simple groups: the classification

In this section (G, H, K) is a loop folder to a Bol loop of exponent 2, $\overline{G} = G/O_2(G)$, $F^*(\overline{G}) \cong S$ with S some finite simple nonabelian group, $S \leq T \leq \mathrm{Aut}(S)$ with $\overline{G} \cong T$ and G_0 the preimage of $F^*(\overline{G})$.
Remember, that as a starting point by 6.31, $\overline{H} \cap \overline{G_0}$ contains nontrivial elements of odd order.

The goal of this section is to prove Theorem 1.

9.1 The groups $\mathrm{PSL}_2(q)$

Lemma 9.1 *Let $S \cong \mathrm{PSL}_2(r), r > 3$. If $\overline{G} \neq \overline{H}$, then $\overline{G} \cong \mathrm{PGL}_2(q)$ for $q = 9$ or q a Fermat prime or $\overline{G} \cong \mathrm{P\Gamma L}_2(9)$, $|G : O_2(G)H| = q + 1$ and $\overline{K}$ consists of 1 and all involutions of $\mathrm{PGL}_2(q)$ outside $\mathrm{PSL}_2(q)$.*

Proof. Let $r = p^e$ with p a prime. Suppose first p is odd. By 6.31 we get some element x of odd order into $\overline{H} \cap \overline{G}^{(\infty)}$. If x is a p'-element, x is contained in some torus of size $\frac{r-1}{2}$ or $\frac{r+1}{2}$.
Remember, that $\mathrm{Aut}(\mathrm{PSL}_2(r))$ has the following types of involutions: those in $\mathrm{PSL}_2(r)$, those in $\mathrm{PGL}_2(r)$ outside $\mathrm{PSL}_2(r)$ and possibly field automorphisms of order 2.
The first two types of involutions invert both tori, so invert some conjugate of x. So by 6.23 K cannot contain involutions of $\mathrm{PGL}_2(r)$, so consists of 1 and field automorphisms only. Since field automorphism act nontrivially on a Sylow-p-subgroup, in this case $\overline{H}$ is a p'-group. We can now estimate the size of $\overline{K}$ and $|\overline{G} : \overline{H}|$: Let $r = s^2$. Then $|K| \leq 1 + s(s^2 + 1)$. On the other hand $|G : H| \geq \frac{1}{2}s^2(s^2 - 1)$. This gives a contradiction since $s \geq 3$. So $\overline{K}^\sharp$ cannot consist of field automorphisms only or contain p'-elements of odd order.
So x is a p-element. Since $C_G(x)$ is soluble, but contains a Sylow-p-subgroup P of G we may assume by Theorem 3, that $P \leq H$. The Borel subgroup $N_{\overline{G}}(\overline{P})$ is then covered by $N_G(P)$ and $|G : O_2(G)H| = r + 1$.
As $\overline{H}$ does not contain p'-elements of odd order, $r - 1$ is a 2-power. Notice, that in the case of $\overline{G} = \mathrm{Aut}(\mathrm{Alt}_6) = \mathrm{P\Gamma L}_2(9)$ we still get $|\overline{G} : \overline{H}| = r + 1$, since the normalizer of a Sylow-3-subgroup of $\overline{G}$ has index 10 and is contained in $\overline{H}$. Furthermore, in this case $\overline{K} \subseteq \mathrm{PGL}_2(q)$ since the other involutions are in Σ_6 and invert elements of order 3, which now cannot be in $\overline{K}$ by 6.23.
So let r be even, so $r \geq 4$. There are only two types of involutions, field automorphisms and inner automorphisms. Inner automorphisms invert conjugates of all elements of odd order, so cannot be in $\overline{K}$.
Field automorphisms act on a torus of size $r-1$ inside some invariant Borel subgroup, so $\overline{H}$ has to be the normalizer of a torus of size $r+1$. Calculation as in the case r odd gives: $|\overline{K}| \leq 1 + s(s^2+1)$ and $|\overline{G} : \overline{H}| \geq \frac{1}{2}s^2(s^2 - 1)$, a contradiction for $s \geq 4$. The case $s = 2$ was already handled as $\mathrm{Alt}_5 \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$. □

9.2 The alternating groups

Lemma 9.2 *Let $S \cong \mathrm{Alt}_n$ for $n \geq 7$. Then $\overline{G} = \overline{H}$.*

Proof. Remember, that ℓ_T has property FS_p for $p \geq 5$. Furthermore 8.5 turns out to be useful.
Let $n = 7$. If $O_2(\overline{H}) = 1$, by 6.21, $7 \in \pi(H)$. By 8.22, $7 \in \pi(H)$ implies $3 \in \pi(H)$, which implies 3-cycles in $\overline{H}$ by 8.5 and therefore a full Sylow-3-subgroup. No proper maximal subgroup of $\overline{G}$ exists with this property by [ATLAS], so $\overline{H} = \overline{G}$ in this case.
So $O_2(\overline{H}) \neq 1$. If $\overline{H}$ contains elements of order 3, then a full Sylow-3-subgroup by 8.5. A full Sylow-3-subgroup of Alt_7 does not normalize any 2-subgroup of Σ_7 by [ATLAS], a contradiction. Again elements of order 7 in $\overline{H}$ imply elements of order 3 in $\overline{H}$ by 8.22. So by 6.30, $\overline{H}$ is a $\{2, 5\}$-group with index at least $2 \cdot 3^2 \cdot 7 = 126$. But elements of order 5 in Σ_7 are inverted by involutions, which are products of two or 3 commuting cycles, so we get $|\overline{K}| \leq 1 + 21 < |\overline{G} : \overline{H}| = 126$ by 6.23, a contradiction.

Let $n = 8$. By 6.21 we get $7 \in \pi(\overline{H})$ or $O_2(\overline{H}) \neq 1$.
If $7 \in \pi(H)$, then $3 \in \pi(H)$ and $\overline{H}$ contains elements of order 3, which are the product of two 3-cycles. By 8.5 then $\overline{H}$ contains 3-cycles, so a Sylow-3-subgroup. By [ATLAS] this implies, that $\overline{H}$ contains a subgroup isomorphic to Alt_7, in which case $|\overline{G} : \overline{H}|$ is a 2-power, which implies, that the loop is soluble, so $\overline{H} = \overline{G}$.
So $O_2(\overline{H}) \neq 1$ and $7 \notin \pi(H)$. If H contains some element of order 3, which is a product of two 3-cycles, H contains a Sylow-3-subgroup of $\overline{G}$. So $\overline{H}$ contains 3-cycles. If $\overline{H}$ contains 3-cycles, the centralizer of a 3-cycle contains Alt_5, so $\overline{H}$ contains elements of order 5. Conversely if $\overline{H}$ contains elements of order 5, its centralizer contains a normal 3-group generated by a 3-cycle, so $\overline{H}$ contains 3-cycles. So $\overline{H}$ contains a subgroup of order 15 and $O_2(\overline{H}) \neq 1$. No such proper subgroup $\overline{H}$ of $\Sigma_8 = \mathrm{Aut}(\mathrm{Alt}_8)$ exists.

Finally for $n \geq 9$ let $X \leq H$ be a p-group for some odd prime p. By 8.5 we have p-cycles in $\overline{H}$. Furthermore for $p > 3$ we have a full Sylow-p-subgroup in $\overline{H}$ by 8.13, while for $p = 3$ we have $\overline{H} = \overline{G}$ by 8.18. If $n - p \geq 6$, the centralizer of a p-cycle has a component of degree at least 6, so this component ends up in $\overline{H}$, and contains 3-cycles.
If $n - p = 5$, the index $|C_{\overline{G}}(\overline{x}) : C_{\overline{H}}(\overline{x})|$ may be 6, but $C_{\overline{H}}(\overline{x})$ contains elements of order 5. Now 5-cycles in $\overline{H}$ imply 3-cycles in $\overline{H}$ for $n \geq 11$, but also for $n = 9$. In case $n = 10$, $\overline{H}$ contains a Sylow-5-subgroup, so $O_2(\overline{H}) = 1$ and by 6.21 $3 \in \pi(H)$.
If $n - p = 3$ or $n - p = 4$, we get 3-cycles into $\overline{H}$, since the centralizer is soluble.

This leaves $n = p$, $n = p + 1$ or $n = p + 2$ for a prime p. Now if p is not a Fermat prime, we get another odd prime r dividing $p - 1$ by 8.22.
Therefore p is a Fermat prime $p \geq 17$ and p is the unique odd prime dividing $|H|$.
If $n = p$, then $\overline{H}$ is the normalizer of a Sylow-p-subgroup, which is a maximal subgroup of $\overline{G}$. By 6.31 we need a $\mathrm{PGL}_2(p)$ in $\overline{G}$ for a nonsoluble loop. As the permutation degree of $\mathrm{PGL}_2(p)$ is $p + 1$, we get $\overline{H} = \overline{G}$.
If $n > p$, $\overline{H}$ cannot act transitively, so $\overline{H}$ is contained in the stabilizer of the orbit decomposition. This stabilizer leads to a subloop by 6.2. By induction however $\overline{H}$ contains elements of order 3.
Since our arguments in this last case are based on the N-loop theorem of As-

chbacher, it should be mentioned, that the direct approach of counting the involutions in $\overline{G}$ and comparing their number with the index of $\overline{H}$ was Aschbachers original argument. $\square$

9.3 The sporadic groups

Lemma 9.3 *The sporadic simple groups are passive.*

Proof. By 6.6, $\overline{H}$ is not a 2-group. By 3.8 and 8.22 we may assume, that $\overline{H}$ contains an element of order p with p a Fermat prime, so $p = 3, 5$ or 17.
If $\overline{G}$ has only one class of involutions, the embedding of a 2N-loop by 6.31 shows, that involutions from this class invert some element of odd order in $\overline{H}$, a contradiction to 6.23. Therefore $\overline{H} = \overline{G}$. For this reason M_{11},J_1, M_{23}, Ly and Th are passive.
Remember, that we have FS_p-property except (p, S) is one of $(3, M_{23})$, $(3, HS)$ or $(5, Suz)$ by 8.14.
We use the character tables in [ATLAS] as provided in GAP for calculation of structure constants. Specifically we calculated, which classes of involutions invert elements from classes of Fermat prime order.
For the structure of centralizers of elements we use without further reference only the informations from [ATLAS] in the list of maximal subgroups as well as the size of the centralizer from the character tables.

In case of M_{12}, structure constant calculations show, that $\overline{H}$ does not contain elements of classes $3B$ or $5A$, as these classes are inverted by all classes of involutions. By FS_3 and FS_5-property, $\overline{H}$ does not contain elements of order 3 or 5.

In case of M_{22}, elements from all conjugacy classes of involutions invert class $3A$, so $\overline{H}$ does not contain elements of order 3. From the list of maximal subgroups we conclude, that $\overline{H}$ is contained in a maximal subgroup $\overline{M}$ of type $2^5 : \Sigma_5$. All other classes of maximal subgroups imply elements of order 3 in $\overline{H}$ by Theorem 5. Furthermore $\overline{K}$ consists of class $1A$ and $2B$, as elements from $2A$ and $2C$ invert elements of class $5A$. Now $|\overline{G} : \overline{H}| \geq 2 \cdot 3^2 \cdot 7 \cdot 11 = 1386 > |\overline{K}| = 1 + 330$, a contradiction.

In case of J_2, we get the following implications for containement in $\overline{H}$:
We have FS_3 and FS_5-property. Furthermore H-intersection with $3A$ implies intersection with $5AB$, while 5-elements in H imply 3-elements in H from the normalizer of a Sylow-5-subgroup.
Among maximal subgroups picked up by $\overline{H}$ are the normalizer of a $3A$-cyclic group and the normalizer of a Sylow-5-subgroup. Therefore $\overline{H} = \overline{G}$.

In case of HS, elements from all classes of involutions invert elements from class $3A$, so $\overline{H}$ does not contain elements of order 3. (There is no class $3B$).
So $\overline{H}$ contains a Sylow-5-subgroup by FS_5-property.
From a structure constant calculation we conclude, that $\overline{K}$ consists of classes $1A$ and $2C$.
A maximal subgroup containing a Sylow-5-subgroup is of type $U_3(5).2$ or the normalizer of the Sylow-5-subgroup (in $HS.2$). By 6.2 and Theorem 5, $\overline{H}$ is a

$\{2,5\}$-group. Now $|\overline{G} : \overline{H}| \geq 2 \cdot 3^2 \cdot 7 \cdot 11 = 1386 > |\overline{K}| = 1 + 1100$ gives a contradiction.

In case of J_3 we get from structure constant calculation, that $\overline{H}$ does not contain elements of order 3.
Elements of order 5 in $\overline{H}$ imply elements of order 3 in $\overline{H}$, since the centralizer of 5-elements is soluble of size 30.
As $J_3.2$ does not involve a $\mathrm{PGL}_2(17)$, only a $\mathrm{PSL}_2(17)$, we conclude $\overline{H} = \overline{G}$ by 6.31.

In case of M_{24} we get by structure constant calculations, that $\overline{H}$ does not contain any elements of order 3 or 5, so $\overline{H} = \overline{G}$.

In case of McL, we can calculate that elements from class $2B$ (outer involutions) invert elements from all classes of elements of order 3 and 5. So $\overline{K}$ does not contain outer involutions. But this gives a contradiction as in the case of M_{11}, J_1 etc. as above.

In case of He, structure constant calculations show, that $\overline{H}$ does not contain elements of order 3.
If $\overline{H}$ contains elements of order 5, it contains a Sylow-5-subgroup. From the shape of the normalizer of a Sylow-5-subgroup we conclude, that then $\overline{H}$ contains elements of order 3. From the list of maximal subgroups we conclude, that $\mathrm{PSL}_2(17)$ is not involved in $\overline{G}$, so by 6.31 we have $\overline{H} = \overline{G}$.

In case of Ru, structure constant calculations show, that $\overline{H}$ contains no elements of order 3 or 5, so $\overline{H} = \overline{G}$.

In case of Suz, structure constant calculations show, that $\overline{H}$ does not contain elements from class $3C$ or $5B$. By FS_3-property $\overline{H}$ does not contain elements of order 3. Furthermore $\overline{H}$ contains elements of order 3, if it contains elements of class $5A$, as the $5A$-centralizer may involve a $\mathrm{PGL}_2(9)$.
Therefore $\overline{H} = \overline{G}$.

In case of $O'N$, structure constant calculations show, that $\overline{H}$ does not contain elements of order 3 or 5.

In case of Co_3, structure constant calculations show, that $\overline{H}$ does not contain elements of classes $3B$, $3C$ or $5B$, so by FS_p-property no elements of order 3 or 5.

In case of Co_2, structure constant calculations show, that $\overline{H}$ does not contain elements of class $3B$, so by FS_3-property no elements of order 3.
Elements of class $5B$ in $\overline{H}$ imply elements of class $5A$ in $\overline{H}$, while the later imply elements of order 3 in $\overline{H}$.

In case of Fi_{22}, elements of order 5 in $\overline{H}$ imply, that $\overline{H}$ contains a Sylow-5-subgroup of $\overline{G}$. From the list of maximal subgroups we conclude, that $\overline{H}$ is contained in a subgroup of type $O_8^+(2).\Sigma_3$, ${}^2F_4(2)'$,Σ_{10} or the corresponding maximal subgroups in $Fi_{22} : 2$. By Theorem 5 therefore $\overline{H}$ is among one of these groups. Calculations of structure constants gives the bound $|\overline{K}| \leq 65287$,

if $\overline{H}$ contains elements of order 5.
Therefore $\overline{H}$ contains $\overline{M} \cong O_8^+(2).\Sigma_3$. This is a contradiction to FS_3-property, as the group in question does not contain a Sylow-3-subgroup of G. In particular we find in $\overline{H}$ a subgroup $\overline{P}$ of order 3^6, such that $N_{\overline{H}}(\overline{P}) \not\leq \overline{M}$.
Therefore $\overline{H}$ does not contain elements of order 5 or $\overline{H} = \overline{G}$.
Elements of class $3A$ in $\overline{H}$ imply elements of order 5 in $\overline{H}$, so $\overline{H} = \overline{G}$. By FS_3-property then $\overline{H}$ does not contain elements of order 3.

In case of HN, structure constant calculations show, that $\overline{H}$ does not contain elements from classes $3A, 5A$ or $5D$. By FS_3 and FS_5-property then $\overline{H} = \overline{G}$.

In case of Fi_{23}, structure constant calculations show, that $\overline{H}$ does not contain elements of class $3A$, so by FS_3-property no elements of order 3. FS_5-property implies $K = 1A \cup 2A$, so $|K| = 31672$. But then $\overline{H}$ contains elements of order 3. So $\overline{H}$ does not contain elements of order 3 or 5.
If $\overline{H}$ contains elements of order 17, then structure constant calculations show $|\overline{K}| \leq 55614277$, but the index of a $\{2,17\}$-subgroup is at least $2 \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23 = 1835304921450$.

In case of Co_1, structure constant calculations show, that $\overline{H}$ does not contain elements of classes $3B, 3D$ or $5B$, so by FS_p-property no elements of order 3 or 5.

In case of J_4, structure constant calculations show, that $\overline{H}$ does not contain elements of order 3 or 5.

In case of Fi'_{24}, structure constant calculations show, that $\overline{H}$ does not contain elements of class $3A$, so no elements of order 3 by FS_3-property.
Structure constant calculations also show, if $\overline{H}$ contains elements of order 5, $\overline{K}$ consists of $1A$ and $2C$ only, so $|\overline{K}| = 1 + 306936$. In that case $\overline{H}$ contains a subgroup isomorphic to Fi_{23}, so contains elements of order 3.
Now $\overline{H}$ is a $\{2,17\}$-group, of index at most 4860791965. (Bound obtained from structure constant calculations.) But $|\overline{G} : \overline{H}| \geq 2 \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 23 \cdot 29 = 70415143921272150$, a contradiction.

In case of B, structure constant calculations show, that $\overline{H}$ does not contain elements of class $3A$, so by FS_3-property no elements of order 3.
Elements of classes $5A$ or $5B$ in $\overline{H}$ imply elements of order 3 in $\overline{H}$, so by FS_5-property $\overline{H}$ does not contain elements of order 5.
Again $\overline{H}$ is a $\{2,17\}$-subgroup. Structure constant calculations show $|\overline{K}| \leq 11721020628376$, but $|\overline{G} : \overline{H}| \geq \frac{|B|}{2^{40} \cdot 17} = 222279514364689031250$.

Finally in case M, structure constant calculations show, that $\overline{H}$ does not contain elements of classes $3A, 3C$ or $5A$, so no elements of order 3 or 5 by FS_p-property.
Elements of class $17A$ in $\overline{H}$ imply elements of order 3 in $\overline{H}$. □

9.4 Groups of Lie type in odd characteristic

In this section S is a simple group of Lie type in characteristic $p > 2$. Let $q = p^f$ be the field parameter of S, q odd in this paragraph. (If S is defined relative to a field extension, q is the size of the smaller field.)
Before using 8.20 we should handle the cases $S \cong \mathrm{PSU}_3(q)$ and $S \cong {}^2G_2(q)$:

Lemma 9.4 *Let $S \cong \mathrm{PSU}_3(q)$ for q odd. Then $\overline{G} = \overline{H}$.*

Proof. Let $d := (q+1, 3)$ and $s \in \pi(H)$ with s odd. By 6.30 such an s exists. Let $\overline{x} \in \overline{H}$ be of order s. If s divides $\frac{q^2-q+1}{d}$, then we get $3 \in \pi(H)$ by 8.22 and 3 divides $(q-1)q(q+1)$. So we may assume from the start, that s divides $q-1$, q or $q+1$.

Notice, that $\mathrm{Aut}(S)$ has only two classes of involutions: inner involutions of S with centralizer $\frac{q+1}{d} \cdot \mathrm{SL}_2(q) : 2$ and outer involutions with centralizer $O_3(q) \times \mathbb{Z}_2 \cong \mathrm{PGL}_2(q) \times \mathbb{Z}_2$. (See 4.5.1 of [GLS3] for details.)
Remember that by 8.2 T/S is a subgroup of Σ_3 not of order 3.
Let $i \in \mathrm{Aut}(S)$ be an outer involution with $C = C_S(i) \cong \mathrm{PGL}_2(q)$. As S has only one class of involutions, elements of order p, $q+1$ and $q-1$ are inverted by inner involutions of S. As $\mathrm{Aut}(S)$ has only one class of outer involutions, also outer involutions invert elements of order p, $q+1$ and $q-1$.

By 8.15, $\overline{H}$ does not contain any elements of order p, as otherwise $\overline{H}$ would contain a Sylow-p-subgroup of $\overline{G}$ and some element of $\overline{K}$ would invert some element of $\overline{H}$ of odd order p.

If s divides $q-1$, then $C_{\overline{G}}(\overline{x})$ is soluble, so ℓ_T has property FS_s, and as in case $s = p$ we get a contradiction.

This leaves the case, that s divides $q+1$. Then either $C_{\overline{G}}(\overline{x})$ is soluble or contains a unique $\mathrm{SL}_2(q)$-component. The later case would imply elements of order p in $\overline{H}$. So $C_{\overline{G}}(\overline{x})$ is soluble. If the Sylow-s-subgroup is abelian, $\overline{H}$ contains a Sylow-s-subgroup, so elements, which are inverted by some involution of K, a contradiction to 6.23.

The only remaining prossibility is $s = 3$ and $3|q+1$. But then $N_{\overline{G}}(O_3(C_{\overline{G}}(\overline{x}))) \leq \overline{H}$ and $\overline{H}$ contains a Sylow-3-subgroup of $\overline{G_0}$, which is again a contradiction, as every involution of $\mathrm{Aut}(S)$ inverts some element of order 3 in S. □

Lemma 9.5 *Let $S \cong {}^2G_2(q)$. Then $\overline{G} = \overline{H}$.*

Proof. Remember that $\mathrm{Aut}(S)$ does not contain outer involutions and only one class of inner involutions. Then 6.31 gives a contradiction, as $\overline{H}$ contains involutions, which invert nontrivial elements of odd order in $\overline{H}$. □

Let $\overline{x} \in \overline{H} \cap \overline{G_0}$ be some element of odd prime order r. By 8.20 we may assume that $r \neq p$ and $C_{\overline{G}}(\overline{x})$ is a soluble p'-group as otherwise $\overline{H} = \overline{G}$.

Lemma 9.6 *Let $S \cong \mathrm{PSL}_3(q)$ for q odd. Then $\overline{H} = \overline{G}$.*

Proof. We handle the two cases separately: q a square and q not a square.
If q is not a square, then by Theorem 4.5.1 of [GLS3] $\mathrm{Aut}(S)$ has only two classes of involutions, inner involutions and graph automorphisms of order 2, which centralize a $\mathrm{PGL}_2(q) \cong O_3(q)$ in S.
We see, that inner involutions invert elements of order p, $q-1$ and $q+1$. But in the direct product $\mathrm{PGL}_2(q) \times \mathbb{Z}_2$, which is isomorphic to the centralizer of graph automorphism j, the products ij with i an inner involution of S, $i \in \mathrm{PGL}_2(q)$, all are outer involutions, but invert the same elements as i. Therefore any involution of $\mathrm{Aut}(S)$ inverts some elements of order p,$q-1$ and $q+1$. By 6.23 and FS_r-property, $\overline{H}$ is therefore not divisible by $p, q-1$ or $q+1$. By 6.31 $\overline{H} \cap \overline{G_0}$ contains elements of odd order r, so r divides q^2+q+1. Now the centralizer of an element $\overline{x} \in \overline{H}$ of order r is cyclic and by 8.22 we get $3 \in \pi(H)$, a contradiction as 3 divides $(q-1)q(q+1)$.
If $q = q_0^2$ is a square, we have four classes of involutions: in addition to inner involutions and graph automorphisms we get field automorphisms and graph-field automorphisms into $\mathrm{Aut}(S)$. Field automorphisms centralize a $\mathrm{PSL}_3(q_0)$, while graph-field automorphisms centralize a $\mathrm{PSU}_3(q_0)$. Notice, that 3 divides $(q_0-1)q_0(q_0+1)$. Let $r \in \pi(H)$, r odd. We show, that $r \nmid q_0^2 - 1$:
As in case q not a square, graph and inner involutions invert elements of order $q - 1 = q_0^2 - 1$.
But also field and graph field involutions invert elements of order $q_0 - 1$ and $q_0 + 1$ with the same argument as in the case q not a square. Notice, that the involutions i and ij with $[i,j] = 1$, i an inner involution and j an outer involution, are in the same S-coset. By Theorem 4.9.1 of [GLS3] these involutions are conjugate.
By 6.23 and FS_r-property, $\overline{K}$ does not contain involutions of $\overline{G}$, so $\overline{H} = \overline{G}$, if $r \mid q_0^2 - 1 = q - 1$.
If $r \mid q^2 + q + 1 = (q_0^2 + q_0 + 1)(q_0^2 - q_0 + 1)$, then by 8.22 we have $3 \in \pi(H)$, but $3 \mid (q_0-1)q_0(q_0+1) \mid (q-1)q$, so $\overline{H} = \overline{G}$ in this case.
So remains $r \mid q_0^2+1 = q+1$. Then K consists of 1 and field and/or graph-field involutions. The index $|\overline{G} : \overline{H}|$ is divisible by 2, $q^3 = q_0^6$ and $q^2+q+1 = q_0^4+q_0^2+1$. On the other hand $|\overline{K}| \leq 1+q_0^3(q_0^2+1)(q_0^3-1)+q_0^3(q_0^2+1)(q_0^3+1) = 1+2q_0^6(q_0^2+1)$, which gives the contradiction $|\overline{K}| < |\overline{G} : \overline{H}|$. □

Lemma 9.7 *Let $S \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$ with $n \geq 4$, q odd. Then $\overline{H} = \overline{G}$.*

Proof. By 8.20, we may assume, that $\overline{x}$ is not in the big connected component of $\Gamma_{\mathcal{O}}$. We use 4.23, 4.24, 4.26 and 4.27 for a list of small connected components.

In the cases (ii) of 4.23 and 4.26 we may use 8.22 to get $3 \in \pi(H)$, but elements of order 3 are in the big connected component.

In the cases (i) of 4.23 and (i) of 4.26 we determine the the structure of maximal subgroups $\overline{M}$ of $\overline{G}$. By Theorem 5 we conclude that either $3 \in \pi(H)$ or $p \in \pi(H)$ with elements of order 3 in the big connected component.

In cases (ii) of 4.24 and (ii) of 4.24, $n-1$ is a prime. We use 8.22 to get $n - 1 \in \pi(H)$. But elements of order $n-1$ are in the big connected component, as $d_q(n-1) | n-2$.

In cases (iii) of 4.24 and (iii) of 4.27 n is a prime. By 8.22, $n \in \pi(H)$. We have $d_q(n) \mid n-1$. Now elements of order n are either in the big connected

component or in the small connected component from (i). Together with the cases (i) of 4.24 and 4.27 this gives either $\pi(H) \subseteq \{2,5,11\}$ in $\mathrm{PSL}_5(3)$ or $\pi(H) \subseteq \{2,5,61\}$ in $\mathrm{PSU}_5(3)$.
In $\mathrm{PSL}_5(3)$, if $11 \in \pi(H)$, then $\overline{H}$ contains a torus normalizer $11^2 : 5$, which is a maximal subgroup of $\overline{G_0}$.
In $\mathrm{PSU}_5(3)$, if $61 \in \pi(H)$, then $\overline{H}$ contains a torus normalizer $61 : 5$, which is a maximal subgroup of $\overline{G_0}$.
In both cases we get a contradiction to 6.31, as elements of order 5 are inverted in $\overline{H} \cap \overline{G_0}$.

So $\overline{H}$ is a $\{2,5\}$-group. We use the list of maximal subgroups in [KL] to determine the possible $\overline{M}$ containing $\overline{H}$. In almost all cases then Theorem 5 produces additional primes into $\pi(H)$. The only remaining case is a subgroup of type $4^5 : \Sigma_5$ in $\mathrm{PSU}_5(3)$.
In this case elements of order 5 are inverted by involutions from all but one conjugacy class, a class of length 4941. Since $|\overline{G} : \overline{H}| \geq 2 \cdot 3^{10} \cdot 7 \cdot 61$, $\overline{H} = \overline{G}$ in this case. □

Lemma 9.8 *Let $S \cong \mathrm{PSp}_{2n}(q)$ for $n \geq 2$ or $\mathrm{P\Omega}_{2n+1}(q)$ for $n \geq 3$ with q odd. Then $\overline{H} = \overline{G}$.*

Proof. By 8.20, we may assume, that $\overline{x}$ is not in the big connected component of $\Gamma_{\mathcal{O}}$. We use 4.28, 4.29 and 4.32 for a list of small connected components.
In the cases (ii),(iii) of 4.29 and (i),(ii) of 4.32, n is a prime and we use 8.22 to get $n \in \pi(H)$. As $d_q(n)|n-1$, we have elements of order n in the big connected component, so $\overline{H} = \overline{G}$.
In the remaining cases n is a 2-power and $r \mid q^n+1$. We determine the structure of possible maximal subgroups $\overline{M}$, which contain $\overline{H}$, using the list in [KL].

In the symplectic case we get candidates: $\overline{M_1} \cong \mathrm{Sp}_n(q^2).2$ in class $\mathcal{C}_3$, $\overline{M_2} \cong \mathrm{GU}_n(q)$ in class $\mathcal{C}_3$, $\overline{M_3} \cong 2^{1+2m}.O^-_{2m}(2)$ in class $\mathcal{C}_6$, or $\overline{M_4}$ in class $\mathcal{S}$ with $F^*(\overline{M_4})$ a simple group.
By 6.2 and Theorem 5, $\overline{M_1}$ and $\overline{M_2}$ imply $p \in \overline{H}$.
In case $\overline{M_3}$ we have $2n = 2^m$ and the largest prime dividing $|O^-_{2m}(2)|$ is bounded by $2^m+1 = 2n+1$. On the other hand, for each odd prime r dividing $|H|$ we have $d_q(r) = 2n$, so $r \geq 2n+1$. Therefore $\overline{M_3}$ contains the torus iff $\frac{q^n+1}{2} = 2n+1$, which holds only for $q = 3$, $n = 2$.
In case M_4, if $F^*(\overline{M})$ is passive and not a Suzuki group, then $3 \in \pi(H)$ and elements of order 3 are in the big connected component. If $\overline{M_4}$ is a Suzuki group, then $5 \in \pi(H)$, but by Landazuri-Seitz, $n \geq 4$, so elements of order 5 are in the big connected component.
Remains $F^*(\overline{M}) = \mathrm{PSL}_2(q_1)$ for $q_1 = 9$ or q_1 a Fermat prime. As then $\overline{H} \cap \overline{M}_4$ contains a Borel subgroup, we have $\frac{q^n+1}{2} = q_1$.
Suppose $q_1 = 2^e + 1$ with e even. Then $\frac{q^n+1}{2} = 2^e + 1$ gives $q^n - 1 = 2^{e+1}$, so $q = 3, n = 2, e = 2$, which is again the special case of $\mathrm{Sp}_4(3)$.

In case $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ we still have to exclude the case, that $\overline{H}$ is a $\{2,5\}$-group. We use information from [ATLAS].
The only possible subgroup $\overline{M}$ is of shape $2^4 : \Sigma_5$ and of index 27. This gives a candidate for $\overline{H}$ of index $6 \cdot 27$.

Involutions not inverting elements of order 5 are in class 2A and 2C, so $|\overline{K}| \leq 1 + 45 + 36 = 82$. As $|\overline{G} : \overline{H}| \geq 2 \cdot 3^4 = 2 \cdot 81$ we get a contradiction.

In the orthogonal case we get a subgroup of type $O_1(q) \perp O_{2n}^-(q)$ and maybe maximal subgroups in class S. From Theorem 5 we conclude, that $\overline{H}$ contains other elements of odd order, so $\overline{H} = \overline{G}$ or a maximal subgroup is of type $\mathrm{PGL}_2(q_1)$ for $q_1 \geq 257$ a Fermat prime. (Since $n \geq 3$, $\frac{q^n+1}{2}$ is at least $\frac{3^4+1}{2} = 41$.)
Now $q_1 = 2^e + 1$ for e even, so if $\frac{q^n+1}{2} = 2^e + 1$, then $q^n - 1 = 2^e$, which happens only for $q = 3$, $n = 2$, a case which is excluded by $n \geq 3$. □

Lemma 9.9 *Let $S \cong \mathrm{P}\Omega_{2n}^+(q)$ or $\mathrm{P}\Omega_{2n}^-(q)$ for $n \geq 4$, q odd. Then $\overline{H} = \overline{G}$.*

Proof. By 8.20, we may assume, that $\overline{x}$ is not in the big connected component of $\Gamma_{\mathcal{O}}$. We use 4.30 and 4.31 for a list of small connected components.
In cases (i) of 4.30 and 4.31, n is a prime and by 8.22, $n \in \pi(H)$. From the list of small connected components, we conclude, that n is in the big connected component.
In cases (ii),(iii) of 4.30 and (iii) of 4.31, $n - 1$ is a prime and $n - 1 \in \pi(H)$ by 8.22. Again $n - 1$ is in the big connected component.
In the remaining cases we use the list of maximal subgroups in [KL] for possible maximal subgroup $\overline{M}$ containing $\overline{H}$. The following observations eliminate some maximal subgroups: In both cases (iv), as $d_q(r) = 2n - 2$, either $r = 2n - 1$ or $r \geq 2(2n - 2) + 1 = 4n - 1$. If $r = 2n - 1$, then r is a Fermat prime. If $\frac{q^{n-1}+1}{2} = 2n - 1$, then $q = 3$ and $n = 3$ contrary to $n \geq 4$.
In case (ii) of 4.31 a similiar arguments works.
Therefore, if $\overline{M}$ is not in class $\mathcal{S}$, we have:
In case (iv) of 4.30, $\overline{M}$ is of type $O_2^-(q) \perp O_{2n-2}^-(q)$ in class $\mathcal{C}_1$ or a subgroup of type $O_n(q^2)$ in class $\mathcal{C}_3$.
In case (iv) of 4.31, $\overline{M}$ is of type $O_2^+(q) \perp O_{2n-2}^-(q)$ in class $\mathcal{C}_1$, a parabolic subgroup of type P_1 also from class $\mathcal{C}_1$, a subgroup of type $\mathrm{GU}_n(q)$ in class $\mathcal{C}_3$ or a subgroup of type $O_n(q^2)$ also from $\mathcal{C}_3$.
In case (ii) of 4.31, $\overline{M}$ is of type $O_n^-(q^2)$ in class $\mathcal{C}_3$.
By 6.2 and Theorem 5 then either $3 \in \pi(H)$ or $5 \in \pi(H)$ with elements of order 3 and 5 in the big connected component or $\overline{M}$ is of type $\mathrm{PSL}_2(q_1)$ for a Fermat prime $q_1 \geq \frac{3^4+1}{2} = 41$, so $q_1 \geq 257$. We get a contradiction as a faithful representation of $\mathrm{PSL}_2(q_1)$ has degree at least $\frac{q_1-1}{2}$, so $\frac{q_1-1}{2} \leq 2n$, but $q_1 \geq \frac{q^{2n-2}+1}{2}$.
□

Lemma 9.10 *Let S isomorphic to one of $G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ for q odd. Then $\overline{H} = \overline{G}$.*

Proof. By 8.20, we may assume, that $\overline{x}$ is not in the big connected component of $\Gamma_{\mathcal{O}}$. We use 4.33, 4.34, 4.36 and 4.37 for a list of small connected components. In every case, the exceptions come from self centralizing tori T in S. If $|N_S(T) : T|$ is not a 2-power, we get $s \in \pi(H)$ for s some odd prime divisor of $|N_S(T) : T|$. Notice, that $s \in \{3, 5, 7\}$ and in all these cases the big connected component contains elements of order s, so $\overline{H} = \overline{G}$.

So the following cases of $(S, d_q(r))$ remain:
$({}^3D_4(q), 12)$, $(F_4(q), 8)$, $(E_6(3), 8)$, $(E_6(7), 8)$, $({}^2E_6(3), 8)$ or $({}^2E_6(7), 8)$. In these cases we have either $O_2(\overline{H}) = 1$ or $\neq 1$. If $O_2(\overline{H}) = 1$, the prime p itself satisfies the prerequisites of 6.21, so by 6.21 and 8.20: $\overline{G} = \overline{H}$.

Else $O_2(\overline{H}) \neq 1$, so $\overline{H}$ is a 2-local subgroup of $\overline{G}$. Now [CLSS] and [LSS] give a list of maximal subgroups, which contain all maximal local subgroups except centralizers of outer automorphisms. The structure of centralizers of outer involutions in these cases is described in [GLS3]. So for any $\overline{H}$ we know the structure of at least one maximal subgroup $\overline{M}$ containing $\overline{H}$.
By 6.2, we can use Theorem 5 on maximal subgroups $\overline{M}$, which contain $\overline{H}$. As a result we get elements of order 3 into $\overline{H}$, so by 8.20 $\overline{H} = \overline{G}$.

Notice, that in case of $\overline{G} \cong {}^3D_4(q)$ the list of maximal subgroups actually produces the torus normalizer (of type $q^4 - q^2 + 1 : 4$) itself as the unique maximal subgroup containing the torus. In that case we conclude, that $O_2(\overline{H}) = 1$, since outer involutions are field automorphisms and act on the torus nontrivially. □

9.5 Groups of Lie type in even characteristic

The main arguments used in this sections are 8.16, 8.21, 8.4 and the results on the commuting graph $\Gamma_{\mathcal{O}}$ and special centralizers from Section 2.

We first handle groups of low rank, with subcases $q = 2$, $q = 4$ and $q > 4$ and later the generic case with subcases $q \geq 4$ and $q = 2$.

9.5.1 Low rank

Recall, that we handled already $\mathrm{PSL}_2(q)$ in 9.1 and that $Suz(q) = {}^2B_2(q)$ is passive due to $2N$-loop embedding,6.31.

In this section we handle the cases S of type $\mathrm{PSL}_3(q)$, $\mathrm{PSL}_4(q)$, $\mathrm{PSU}_3(q)$, $\mathrm{PSU}_4(q)$, $\mathrm{Sp}_4(q)$, $G_2(q)$, ${}^3D_4(q)$, and ${}^2F_4(q)$.
Let $S \leq T \leq \mathrm{Aut}(S)$ and $G/O_2(G) \cong T$. Remember the FS_p-property from 8.16. We will make the case division $q = 2$, $q = 4$ and $q > 4$.

Lemma 9.11 *If $q = 2$, then $\overline{H} = \overline{G}$.*

Proof. In case $q = 2$ we can already exclude some groups for the following reasons:
$\mathrm{PSL}_3(2)$ because of $2N$-Loop-embedding, 6.31,
$\mathrm{PSL}_4(2)$ because of the isomorphism with Alt_8,
$\mathrm{PSU}_3(2)$ because the group is soluble,
$\mathrm{PSU}_4(2)$ because of the isomorphism with $\mathrm{PSp}_4(3)$,
$\mathrm{Sp}_4(2)'$ because of the isomorphism with $\mathrm{Alt}_6 \cong \mathrm{PSL}_2(9)$,
$G_2(2)'$ because of the isomorphism with $U_3(3)$ and $2N$-Loop-embedding, 6.31.

The groups ${}^3D_4(2)$ and ${}^2F_4(2)'$ are ATLAS-groups, so we can use the information from [ATLAS]: By $2N$-Loop-embedding and the list of maximal subgroups we conclude, that if $S \cong {}^3D_4(2)$, then $\overline{H} = \overline{G}$.

For the Tits group ${}^2F_4(2)$ we can establish the FS_p-property for all primes $p > 2$ by 8.12(1).
Notice, that ${}^2F_4(2) = \mathrm{Aut}({}^2F_4(2)')$ is not generated by involutions. From the list of maximal subgroups in [ATLAS] we conclude, that $\overline{M}$, a maximal subgroup of $\overline{G}$, which contains $\overline{H}$ is isomorphic to $\mathrm{PSL}_3(3).2$. This implies $\overline{H} = \overline{M}$ and $O_2(\overline{H}) = 1$. By 6.21 we get a contradiction as the length of both classes $2A$ and $2B$ is divisible by 5, so $\overline{H}$ has to contain a Sylow-5-subgroup of $\overline{G}$ too. $\square$

Remember, that for $q = 4$ we have already the FS_p-property for all odd primes $p > 3$.

Lemma 9.12 *If $q = 4$, then $\overline{G} = \overline{H}$ or S of type $L_3(4)$.*

Proof. Let $S \cong \mathrm{PSL}_4(4)$. Notice, that if $5 \in \pi(H)$, then $\overline{H}$ contains a Sylow-5-subgroup. The normalizer of a 5-Sylow-subgroup contains elements of order 3, while there exist elements of order $85 = 5 \cdot 17$ in G, so then $|H|$ is divisible by $3 \cdot 5^2 \cdot 17$, and contains subgroups of type $5^2 : 3$ and 5×17. No such proper subgroup exists by the list of maximal subgroups in [KL]. If $5 \notin \pi(H)$, then also $17 \notin \pi(H)$ and elements of order 3 in $\overline{H}$ do not commute with elements of order 5 in $\overline{G}$. (Otherwise by the structure of the centralizers of elements of order 3, $5 \in \pi(H)$ by Theorem 5.) No such elements of order 3 exist. Elements of order 7 in $\overline{H}$ imply elements of order 3 in $\overline{H}$ both by the centralizer of elements of order 7 and the normalizer of subgroups of order 7. We now get a contradiction to 6.31.

Let $S \cong \mathrm{PSU}_3(4)$. We use notation of p.30 of [ATLAS]. If $3 \in \pi(H)$, then $5 \in \pi(H)$ as the Centralizer of a $3A$-element is cyclic of order 15. Furthermore the centralizer of a $5ABC-$ or D-element does not involve a $\mathrm{P\Gamma L}_2(4)$, only a $\mathrm{PSL}_2(4)$. So for $x \in \overline{H}$ of order 5 we have $O^2(C_{\overline{G}}(\overline{x})) \leq \overline{H}$ by 8.4. The normalizer of a Sylow-5-subgroup is a maximal subgroup of type $5^2 : \Sigma_3$. Therefore $\overline{H} = \overline{G}$, if $\overline{H}$ contains elements of order 3 or 5. Elements of order 13 in $\overline{H}$ imply elements of order 3 in $\overline{H}$.

Let $S \cong \mathrm{PSU}_4(4)$. We calculated some centralizer data using MAGMA:
Elements of order 17 in $\overline{H}$ imply elements of order 3 in $\overline{H}$, as $\overline{G}$ contains a $\mathrm{GL}_2(16)$, so the centralizer of an element of order 17 is soluble and contains elements of order 3.
Elements of order 13 in $\overline{H}$ imply elements of order 3 in $\overline{H}$ by 8.22 as already visible in $\mathrm{PSU}_3(4)$.
Elements of order 3 in $\overline{H}$ imply elements of order 5 in $\overline{H}$:
There are two classes of subgroups of order 3 with components $\mathrm{PSL}_2(16)$ and $\mathrm{PSL}_2(4)$ respectively in their centralizer. In both cases $3 \in \pi(H)$ implies, that $5 \in \pi(H)$.
If $5 \in \pi(H)$, by FS_5-property a Sylow-5-subgroup of $\overline{G}$ is already in $\overline{H}$. There are elements of order 5 in that Sylow-subgroup, whose centralizer has shape $5 \times \mathrm{PSU}_3(4)$. For these elements we can use 8.4 to get the component $\mathrm{PSU}_3(4)$

into $\overline{H}$. Calculation reveals, that there are so many elements of that type in a Sylow-5-subgroup, that $\overline{H} = \overline{G}$.

Let $S \cong Sp_4(4)$. Elements of order 3 in $\overline{H}$ imply $5 \in \pi(H)$, $5 \in \pi(H)$ or $17 \in \pi(H)$ implies $O_2(\overline{H}) = 1$. $O_2(\overline{H}) = 1$ implies that $\overline{H}$ contains Sylow-5- and Sylow-17-subgroups of $\overline{G}$ in $\overline{H}$ by 6.21. This implies $\overline{H} = \overline{G}$ by [ATLAS].

Let $S \cong G_2(4)$. We calculate centralizers with MAGMA, using the 12-dimensional representation of $G_2(4).2$ over $GF(2)$. In particular the subloops to centralizers of elements of order 5 are soluble, as not sections $\mathrm{P\Gamma L}_2(4)$ are involved, (only $\mathrm{PSL}_2(4)$.) So if $\overline{x} \in \overline{H}$ is of order 5, then $O^2(C_{\overline{G}}(\overline{x})) \leq \overline{H}$ by 8.4. Then $\overline{H} = \overline{G}$, as $\overline{H}$ does not only contains a $\mathrm{PSU}_3(4)$, but also subgroups of type $5 \times A_5$ from both conjugacy classes, while $\mathrm{PSU}_3(4)$ contains only one such class.
Elements of order 3 in $\overline{H}$ imply elements of order 5 in H by the centralizer structure, while elements of order 7 or 13 in $\overline{H}$ imply elements of order 3 in $\overline{H}$ by 8.22.

Let $S \cong {}^3D_4(4)$. If $3 \in \pi(H)$, then $5 \in \pi(H)$: Either FS_3-property fails on a subnormal $\mathrm{P\Gamma L}_2(4)$ in a centralizer of an element of order 3 or FS_3-property holds. The first case implies $5 \in \pi(H)$, while the second case implies elements of order 3 in $\overline{H}$ with centralizer shape $(7 \times \mathrm{SL}_3(4)).3$, so again $5 \in \pi(H)$.
By $2N$-loop embedding 6.31 either $3 \in \pi(H)$ or $5 \in \pi(H)$, so $5 \in \pi(H)$ and $\overline{H}$ contains a Sylow-5-subgroup of $\overline{G}$. As there are centralizers of elements of order 5 of shape $5 \times \mathrm{PSL}_2(64)$, we have $3, 7, 13 \in \pi(H)$, so $\overline{H}$ contains Sylow-subgroups for the primes $5, 7$ and 13. From the list of maximal subgroups of [K3D4] we conclude, that $\overline{H} = \overline{G}$. $\square$

Lemma 9.13 *Let $S \cong \mathrm{PSL}_3(4)$. Then $\overline{G} = \overline{H}$.*

Proof. This group needs special treatment due to the exception of Zsygmondy's theorem and the fact, that $q - 1 = (3, q - 1)$.
We use Atlas-notation for the conjugacy classes, see [ATLAS],p. 23.
Conjugacy classes of odd prime order are $3A, 5AB$ and $7AB$, of each odd prime order there is a unique conjugacy class of groups of that order in $S \cong \mathrm{PSL}_3(4)$.
$\mathrm{Aut}(\mathrm{PSL}_3(4))$ has involution conjugacy classes $2A, 2B, 2C$ and $2D$, which invert the following conjugacy classes of odd prime order:
$2A$ inverts elements from $3A$ and $5AB$.
$2B$ inverts elements from $3A$ and $7AB$.
$2C$ inverts elements of all classes of 3-elements (including outer classes).
$2D$ inverts elements from $3A$, $5AB$ and $7AB$.
Therefore $\overline{K}$ does not include elements of class $2D$. As any involution inverts elements of class $3A$, $\overline{H} \cap \overline{G_0}$ does not contain elements of order 3.
So $\overline{H} \cap \overline{G_0}$ is a $\{2, 5\}$-group by the $2N$-loop embedding, 6.31. In particular maximal subgroups containing $\overline{H}$ have no Alt_6 or $\mathrm{PSL}_3(2)$-components and $\overline{K}$ does not contain involutions from $2A$ or $2D$.

Notice, that class $2B$ has length 280, while class $2C$ has length 120 or 360, depending on the presence of diagonal automorphisms of order 3. Class $2B$ is a class of graph-field automorphisms, while class $2C$ is a class of field automorphisms.

We now check $\overline{G}$ for possible maximal subgroups $\overline{M}$ containing $\overline{H}$.
Let $X_0 := \mathrm{Aut}(S) = L_3(4).D_{12}$. Calculations of maximal subgroups were done in MAGMA, using a 42-point representation of X_0.

X_0 has 8 classes of maximal subgroups:
$X_1 \cong \Sigma_3 \times \Sigma_5$, but if $\overline{M}$ is X_1, then $|\overline{G} : \overline{H}| \geq 6 \cdot |\overline{X_0} : \overline{X_1}| = 2016$, while $|\overline{K}| \leq 1 + 280 + 360 = 641$.
X_2, X_3, X_4 are soluble of sizes $2^2 \cdot 3^2 \cdot 7, 2^5 \cdot 3^3$ resp. $2^8 \cdot 3^2$.
$X_5 \cong \mathrm{PSL}_3(4).2^2$ is analyzed below,
$X_6 \cong \mathrm{PSL}_3(4).6$ contains involutions of classes $2A$ and $2B$, but is not generated by involutions. See $X_{5,7}$ below for $\mathrm{PSL}_3(4).2$ with $2B$-outer involutions.
$X_7 \cong \mathrm{PSL}_3(4).\Sigma_3$ with $2C$-outer involutions is analyzed below, but
$X_8 \cong \mathrm{PSL}_3(4).\Sigma_3$ with $2D$-outer involutions is out.

X_5 has 8 classes of maximal subgroups:
$X_{5,1} \cong \mathbb{Z}_2 \times \Sigma_5$, but $\overline{M}$ of type $X_{5,1}$ implies $|\overline{G} : \overline{H}| \geq 6|X_5 : X_{5,1}| = 6 \cdot 336 = 2016$, a contradiction as in case X_0.
$X_{5,2}$ and $X_{5,3}$ are soluble of sizes $2^5 \cdot 3^3$ and $2^8 \cdot 3$,
$X_{5,4} \cong \mathrm{Alt}_6.2^2$ and $X_{5,5} \cong \mathbb{Z}_2 \times \mathrm{PSL}_3(2).2$ have bad components,
$X_{5,6} \cong \mathrm{PSL}_3(4).2$ with $2B$-involutions is analyzed below,
$X_{5,7} \cong \mathrm{PSL}_3(4).2$ with $2C$-involutions is analyzed below, but
$X_{5,8} \cong \mathrm{PSL}_3(4).2$ with $2D$-involutions is out.

$X_{5,6} \cong \mathrm{PSL}_3(4).2$ with $2B$-involutions has 10 classes of maximal subgroups: $X_{5,6,1} \cong \Sigma_5$, three classes of $\mathrm{PSL}_3(2).2$, three classes of $\mathrm{Alt}_6.2$, soluble groups of sizes $2^4 \cdot 3^2$ and $2^7 \cdot 3$ and $S \cong \mathrm{PSL}_3(4)$ itself.
Only $X_{5,6,1}$ for $\overline{M}$ remains, but then $|\overline{G} : \overline{H}| \geq 6 \cdot |X_{5,6} : X_{5,6,1}| = 6 \cdot 336 = 2016$ gives a contradiction as before.

$X_{5,7} \cong \mathrm{PSL}_3(4).2$ with $2C$-involutions has 6 classes of maximal subgroups: a soluble group of size $2^4 \cdot 3^2$, a $\mathbb{Z}_2 \times \mathrm{PSL}_3(2)$, a $\mathrm{Alt}_6.2$, two classes $X_{5,7,4}$ and $X_{5,7,5}$ of shape $2^4 : \Sigma_5$ and S itself.
If $\overline{M}$ is of type $X_{5,7,4}$ or $X_{5,7,5}$, then $|\overline{G} : \overline{H}| \geq 6 \cdot |X_{5,7} : X_{5,7,4}| = 6 \cdot 21 = 126$, but $|\overline{K}| \leq 1 + 120 = 121$, as class $2C$ has size 120 in this subgroup.

X_7 with $2C$-involutions has 6 classes of maximal subgroups:
two soluble classes of subgroup sizes $2 \cdot 3^2 \cdot 7$ resp. $2^4 \cdot 3^3$,
two classes $X_{7,3}$ and $X_{7,4}$ of shape $2^4 : ((3 \times A_5) : 2)$ and two classes containing $\mathrm{PSL}_3(4)$: $\mathrm{PSL}_3(4).3$ and $\mathrm{PSL}_3(4).2 \cong X_{5,7}$.
Notice, that $\overline{M}$ of type $X_{7,3}$ or $X_{7,4}$ implies $3 \in \pi(H)$ by 6.7, but $2C$ involutions invert elements from all classes of 3-elements. $\square$

Lemma 9.14 *Let $S \cong \mathrm{PSL}_3(q)$, $\mathrm{PSL}_4(q)$, $\mathrm{PSU}_3(q)$, $\mathrm{PSU}_4(q)$, $\mathrm{Sp}_4(q)$, $G_2(q)$, ${}^3D_4(q)$ or ${}^2F_4(q)$ for $q > 4$. Then $\overline{G} = \overline{H}$.*

Proof. We use 8.9 as well as FS_r-property for $r > 2$. Further we use the discussion of the connected components of the commuting graph $\Gamma_{\mathcal{O}}$ together with 8.17.
Let $x \in H \cap G_0$ be an element of odd prime order r, which exists by 6.31.

In case of $S \cong \mathrm{PSL}_3(q)$, either $r \mid q^2 - 1$, or $r \mid \frac{q^2+q+1}{(q-1,3)}$.
If $r \mid q^2 - 1$, then $\overline{G_0} \subseteq \overline{H}$ by 4.22 and 8.9. Then $\overline{H} = \overline{G}$ by 8.17.
If $r \mid \frac{q^2+q+1}{(q-1,3)}$, then $3 \in \pi(H)$ by 8.22, but $3 \mid q^2 - 1$.

In case of $S \cong \mathrm{PSL}_4(q)$ the graph $\Gamma_{\mathcal{O}}$ is connected by 4.23, so $\overline{G_0} \subseteq \overline{H}$ by 8.9 and $\overline{H} = \overline{G}$ by 8.17.

In case of $S \cong \mathrm{PSU}_3(q)$, either $r \mid q^2 - 1$, or $r \mid \frac{q^2-q+1}{(q+1,3)}$.
If $r \mid q^2 - 1$, then $\overline{G_0} \subseteq \overline{H}$ by 4.25 and 8.9. Then $\overline{H} = \overline{G}$ by 8.17.
If $r \mid \frac{q^2-q+1}{(q+1,3)}$, then $3 \in \pi(H)$ by 8.22, but $3 \mid q^2 - 1$.

In case of $S \cong \mathrm{PSU}_4(q)$ the graph $\Gamma_{\mathcal{O}}$ is connected by 4.26, so $\overline{G_0} \subseteq \overline{H}$ by 8.9 and $\overline{H} = \overline{G}$ by 8.17.

In case of $S \cong \mathrm{Sp}_4(q)$ either $r \mid q^2 - 1$ or $r \mid q^2 + 1$. If $r \mid q^2 + 1$, then $O_2(\overline{H}) = 1$:
No prime divisor of $q^2 + 1$ divides the order of a parabolic subgroup, so there is no $\{2, r\}$-subgroup of $\overline{G_0}$. Furthermore the centralizer of an outer involution, which is either $\mathrm{Sp}_4(q^{1/2})$ or ${}^2B_2(q)$, does not contain a torus of size $q^2 + 1$, which is contained in $C_{\overline{H}}(\overline{x})$. Therefore we may apply 6.21. Notice, that neither parabolic subgroups nor subgroups of type $\mathrm{Sp}_4(q^{1/2})$ or ${}^2B_2(q)$ contain a Sylow-s-subgroups for primes s dividing $q + 1$.
Therefore the length of every conjugacy class of involutions is divisible by s, so $s \in \pi(H)$, but $s \mid q^2 - 1$.
If $r \mid q^2 - 1$, we have $\overline{G_0} \subseteq \overline{H}$ by 8.9 and 4.28, so $\overline{H} = \overline{G}$ by 8.17.

In case of $S \cong G_2(q)$ with $3 \mid q - \varepsilon$ for $\varepsilon \in \{+1, -1\}$, if $r \mid q^2 - \varepsilon q + 1$ then $3 \in \pi(H)$ by 8.22, and $3 \mid q^2 - 1$.
So we have $r \mid (q^2-1)(q^2+\varepsilon q+1)$ and $\overline{G_0} \subseteq \overline{H}$ by 8.9 and 4.33, so $\overline{H} = \overline{G}$ by 8.17.

In case of $S \cong {}^3D_4(q)$, either $r \mid q^4 - q^2 + 1$ or $r \mid q^6 - 1$.
Suppose $r \mid q^4 - q^2 + 1$. Then $\overline{H}$ contains a torus of size $q^4 - q^2 + 1$ from $C_H(x)$, as $C_G(x)$ is soluble. Therefore $O_2(\overline{H}) = 1$, so by 6.21 and the list of maximal subgroups of S in [K3D4], $s \in \pi(H)$ for s some prime divisor of $\frac{q^4+q^2+1}{3}$.
If $r \mid q^6 - 1$, then $\overline{G_0} \subseteq \overline{H}$ by 8.9 and 4.34, so $\overline{H} = \overline{G}$ by 8.17.

In case of $S \cong {}^2F_4(q)$, either $r \mid q^4 - q^2 + 1$ or $r \mid (q^4 - 1)(q^3 + 1)$.
If $r \mid q^4 - q^2 + 1$, then $3 \in \pi(H)$ by 8.22, as $C_H(x)$ contains the normalizer of a torus either of size $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$ or $q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$ and $3 \mid q + 1$.
If $r \mid (q^4 - 1)(q^3 + 1)$, then $\overline{G_0} \subseteq \overline{H}$ by 8.9 and 4.35, so $\overline{H} = \overline{G}$ by 8.17. □

9.5.2 The case $q \geq 4$

In case $q > 4$ we use 8.9 as well as FS_r-property for $r > 2$ by 8.16.
At this point the discussion of the connected components of the commuting graph $\Gamma_{\mathcal{O}}$ becomes essential.
The arguments in case $q = 4$ are not that different:

Notice, that we have FS_r-property for $r > 3$ by 8.16, in particular for $r = 5$. Recall from 8.9, that if FS_3-property fails, then $5 \in \pi(H)$, so $\overline{H}$ contains a full Sylow-5-subgroup of $\overline{G}$ while $q + 1 = 5$ for $q = 4$.

Lemma 9.15 *Let $S \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$ with $n \geq 5$, $\mathrm{Sp}_{2n}(q)$ with $n \geq 3$ or $\Omega^{\pm}_{2n}(q)$ for $n \geq 4$ and $q \geq 4$, q even. Then $\overline{G} = \overline{H}$.*

Proof. Let $x \in H \cap G_0$ be an element of odd prime order r, which exists by 6.31.

In case of $S \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$ we use 4.24 and 4.27. If $q = 4$, then by 4.10,4.13 and 8.21, we have $\overline{H} = \overline{G}$, if $5 \in \pi(H)$.
Then either $\overline{H} = \overline{G}$ by 8.9 and 8.17 or $C_{\overline{G}}(\overline{x})$ is a self centralizing torus, on which a prime $p = n$ resp. $p = n - 1$ acts. In that case p occurs in $\pi(H)$ by 8.22. Notice, that in S there is at most one exceptional self centralizing torus, which does not contain the prime p itself. Therefore elements of order p are in the big connected component and $\overline{H} = \overline{G}$.

In case of $S \cong \Omega^{+}_{2n}(q)$ or $S \cong \mathrm{Sp}_6(q)$ the graph $\Gamma_{\mathcal{O}}$ is connected by 4.30 and 4.29. If $q = 4$, then by 4.15,4.16 and 8.21, we have $\overline{H} = \overline{G}$, if $5 \in \pi(H)$.
So $\overline{H} = \overline{G}$ by 8.9 and 8.17.

In case of $S \cong \mathrm{Sp}_{2n}(q)$ or $\Omega^{-}_{2n}(q)$ for $n \geq 4$ we have either $\overline{H} = \overline{G}$ by 4.29 and 4.31 or n is a 2-power and $r \mid q^n + 1$.
If $q = 4$, again by 4.15,4.16 and 8.21, we have $\overline{H} = \overline{G}$, if $5 \in \pi(H)$.
We determine the isomorphism type of maximal subgroups $\overline{M}$ of $\overline{G}$, which contain $\overline{H}$.
Notice, that $\overline{H}_{2'} \geq 4^4 + 1 = 257$.
In the symplectic case we get by [KL] $F^*(\overline{M}) \cong \mathrm{Sp}_n(q^2)$ or $\Omega^{-}_{2n}(q)$ or $\overline{M}$ in class $\mathcal{S}$, while in the orthogonal case we have $F^*(M) \cong \Omega^{-}_n(q^2)$ or in class $\mathcal{S}$. So in any case $F^*(\overline{M})$ is a simple group. By 6.2, M is a group to a subloop, so we may use Theorem 5 on $\langle M \cap K \rangle$. If $F^*(\overline{M})$ is a passive group, then $3 \in \pi(H)$ or $5 \in \pi(H)$ with $15 \mid q^4 - 1$. So $\overline{H} = \overline{G}$ as $\overline{H}$ contains elements of odd order, which are in the big connected component.
Else $F^*(M)$ is a group $\mathrm{PSL}_2(p)$ for a Fermat prime $p \geq 257$ with $p = q^n + 1$. The minimal degree of a faithful representation of $\mathrm{PSL}_2(p)$ is $\frac{p-1}{2} = \frac{q^n}{2}$ by Landazuri-Seitz, but $\overline{G}$ has a faithful module in dimension $2n$. As $q \geq 4$, this is absurd. $\square$

Lemma 9.16 *Let $S \cong F_4(q)$ $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ for $q \geq 4$, q even. Then $\overline{G} = \overline{H}$.*

Proof. Let $x \in H$ be an element of odd prime order r, which exists by 6.31.

In case of $S \cong F_4(q)$ we use 4.36. If $q = 4$, by 4.20 and 8.21, we have $\overline{H} = \overline{G}$, if $5 \in \pi(H)$.
Then either x is in the big connected component and $\overline{H} = \overline{G}$ by 8.9 and 8.17 or $C_{\overline{G}}(\overline{x})$ is a self centralizing torus of size $q^4 + 1$ or $q^4 - q^2 + 1$.
The normalizer of the torus of size $q^4 - q^2 + 1$ contains elements of order 3,

as $\overline{G}$ contains a subgroup ${}^3D_4(q).3$ with an outer field automorphism of order 3. We can find this field automorphism acting on top of the torus, so by 8.22, $3 \in \pi(H)$. As elements of order 3 are in the big connected component, $\overline{H} = \overline{G}$ in this case.
Remains the torus of size q^4+1. Let $\overline{M}$ be a maximal subgroup of $\overline{G}$ containing $\overline{H}$. We can use Theorem 5 on $\langle M \cap K \rangle$ by 6.2. As we know, which elements of odd order occure in $\overline{H}$, there remains only the case, that $\overline{H} \cong \mathrm{PSL}_2(p)$ for p some Fermat prime with $p = q^4 + 1 \geq 4^4 + 1 = 257$. By the boundaries of Landazuri-Seitz, a faithful representation of $\mathrm{PSL}_2(p)$ has dimension at least $\frac{p-1}{2} \geq 128$, but S has a faithful representation in dimension 26, a contradiction.

In case $S \cong E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ we use 4.37.
If $q = 4$, by 4.21 and 8.21, we have $\overline{H} = \overline{G}$, if $5 \in \pi(H)$.
Then either x is in the big connected component and $\overline{H} = \overline{G}$ by 8.9 and 8.17 or $C_{\overline{G}}(\overline{x})$ is a self centralizing torus, on which elements of order 3 or 5 act non-trivially. By 8.22 then $3 \in \pi(H)$ or $5 \in \pi(H)$, but elements of order 3 or 5 are in the big connected component, so $\overline{H} = \overline{G}$. □

9.5.3 The case $q = 2$.

The remaining groups are $\mathrm{PSL}_n(2), \mathrm{PSU}_n(2)$ for $n \geq 5$, $\mathrm{Sp}_{2n}(2)$ for $n \geq 3$, $\Omega^{\pm}_{2n}(2)$ for $n \geq 4$, $F_4(2)$, $E_6(2)$, ${}^2E_6(2)$, $E_7(2)$ and $E_8(2)$.

These cases behaves differently from $q > 4$, if $3 \in \pi(H)$ or $5 \in \pi(H)$.
We use results on centralizer of elements of order 3 and 5, to overcome the exceptions of 8.9.

Lemma 9.17 *Let $S \cong \mathrm{PSL}_n(2)$ for $n \geq 5$. Then $\overline{H} = \overline{G}$.*

Proof. Suppose $3 \in \pi(H)$. Let V be the natural n-dimensional GF(2)-module for S.
By 4.10 and 8.21 there exists terminal elements $\bar{t}$ of order 3, which have $\dim[V, \bar{t}] = 2$. Therefore $\overline{H}$ does not contain a Sylow-3-subgroup of G, so FS_3-property fails. How can FS_3-property fail? From the structure of centralizers of semisimple elements and the structure of nonsoluble subloops we conclude as in 8.8, that some $\overline{y} \in \overline{H}, o(\overline{y}) = 3$ exists, such that $C_{\overline{G}}(\overline{y})$ contains a subnormal subgroup isomorphic to Σ_5. By 5.2, $C_{\overline{G}}(\overline{y}) \cong \mathrm{GL}_{m/2}(4) \times \mathrm{SL}_{n-m}(2)$ with $m := \dim[V, y]$ for V the natural n-dimensional GF(2)-module of G. In particular components of type $\mathrm{Alt}_5 \cong \mathrm{PSL}_2(4)$ occure only for $m = 4$. If $m \geq 6$, $\overline{H}$ covers the $\mathrm{SL}_{n-m}(2)$ acting on $C_V(\overline{y})$ by 6.28. We conclude, that then $\overline{H}$ contains elements, which are conjugate to $\bar{t}$, so $\overline{H} = \overline{G}$ for $n \geq 6$ and $3 \in \pi(H)$.

We show the statement now for $n = 5$:
We use the list of maximal subgroups and centralizer sizes in [ATLAS].
Let $\overline{M}$ be a maximal subgroup of $\overline{G}$, which contains $\overline{H}$.
By 6.2, M is a group to a subloop, so we may apply Theorem 5. We know, that $\overline{H} = \overline{G}$, if $\overline{H}$ contains a Sylow-3-subgroup or elements conjugate to $\bar{t}$.

But otherwise $\overline{H}$ contains elements of order 5,7 or 31.
$7 \in \pi(H)$ implies $\overline{H} = \overline{G}$, as the centralizer of elements of order 7 is soluble and contains elements conjugate to $\bar{t}$.
But $5 \in \pi(H)$ implies $3 \in \pi(H)$ from centralizers sizes and $31 \in \pi(H)$ implies $5 \in \pi(H)$ by 8.22.

So let $n \geq 6$ and $5 \in \pi(H)$. Let $\overline{x} \in \overline{H}$ be of order 5 and consider the action of $C_{\overline{G}}(\overline{x})$. Let $m := \dim[V, \overline{x}]$. By 5.2, $C_{\overline{G}}(\overline{x}) \cong \mathrm{GL}_{m/4}(16) \times \mathrm{SL}_{n-m}(2)$. We conclude, that $O_3(C_{\overline{G}}(\overline{x})) \neq 1$, so $3 \in \pi(H)$ by 6.7.

Finally let $n \geq 6$ and $3 \notin \pi(H) \not\ni 5$. From 4.24 we conclude, that either n or $n-1$ is a prime and $\overline{H}$ contains a torus of size $2^n - 1$ resp. $2^{n-1} - 1$. Then by 8.22 either $n \in \pi(H)$ or $n - 1 \in \pi(H)$, and n resp. $n-1$ are in the connected component containing all elements of order 3 and 5. We then get a contradiction, as this implies $3 \in \pi(H)$ or $5 \in \pi(H)$. □

Lemma 9.18 *Let $S \cong \mathrm{Sp}_n(2)$ for $n \geq 6$. Then $\overline{H} = \overline{G}$.*

Proof. Let V be the natural n-dimensional module of $\overline{G}$. Recall, that $\mathrm{Out}(G) = 1$.

Suppose $3 \in \pi(H)$. By 4.15 and 8.21 there exists a terminal element $\bar{t}$ of order 3, with $\dim[V, \bar{t}] = 2$. Therefore $\overline{H}$ does not contain a Sylow-3-subgroup of G, so FS_3-property fails. We conclude as in 8.8, that some $\overline{y} \in \overline{H}, o(\overline{y}) = 3$ exists, such that $C_{\overline{G}}(\overline{y})$ contains a subnormal subgroup isomorphic to Σ_5. Let $\overline{y} \in \overline{H}$ with $o(\overline{y}) = 3$. By 5.3, $C_{\overline{G}}(\overline{z}) \cong \mathrm{GU}_{m/2}(2) \times \mathrm{Sp}_{n-m}(2)$, so no such subnormal subgroup occurs and FS_3-property holds, so $3 \notin \pi(H)$.

Suppose $5 \in \pi(H)$. Let $\overline{x} \in \overline{H}$ be of order 5 and $m := \dim[V, x]$. By 5.3, $C_{\overline{G}}(\overline{y}) \cong \mathrm{GU}_{m/4}(4) \times \mathrm{Sp}_{n-m}(2)$. If $\dim C_V(x) > 0$, $\overline{H}$ covers the $\mathrm{Sp}_{n-m}(2)$-factor of $C_{\overline{G}}(\overline{y})$ by 6.28, so $3 \in \pi(H)$. If $m \geq 12$, then $\overline{H}$ covers the $\mathrm{GU}_{m/4}(4)$-factor too and $3 \in \pi(H)$. So $n = m = 8$, but then the centralizer has structure $\mathbb{Z}_5 \times \mathrm{Alt}_5$ and no subgroup Σ_5 occurs in this centralizer, so $3 \in \pi(H)$ in this case too.

If now $3 \notin \pi(H) \not\ni 5$, we use 4.29 for the connected components of $\Gamma_{\mathcal{O}}$, which do not contain elements of order 3 or 5.
Let $\overline{x} \in \overline{H}$ be an element of odd order. We conclude, that either n is 2-power and $o(x)$ divides $q^{n/2} + 1$ or $n = 2p$ for a prime p and n divides $2^p - 1$. In this last case $p \in \pi(H)$ by 8.22 and p is in the big connected component, so $\overline{H} = \overline{G}$. Let $\overline{M}$ be a maximal subgroup containing $\overline{H}$, so M is a group to a subloop by 6.2. We get by [KL], that $F^*(\overline{M}) \cong \mathrm{Sp}_{n/2}(q^2)$ or $\Omega_n^-(q)$ or $\overline{M}$ in class $\mathcal{S}$. In any case $F^*(\overline{M})$ is a simple group. If this group is passive, either $3 \in \pi(H)$ or $5 \in \pi(H)$ and $\overline{H} = \overline{G}$. Else $\overline{M} \cong \mathrm{PGL}_2(p)$ for a Fermat prime $p \geq 17$. We have $p \mid 2^{n/2} + 1$, G has an n-dimensional GF(2)-module, but the minimal representation degree of $\mathrm{PGL}_2(p)$ is $p - 1$. Now $2^{n/2} \leq n$, so $n \leq 4$, a contradiction. Notice, that the group $\mathrm{Sp}_8(2)$ actually has a maximal subgroup isomorphic to $\mathrm{PSL}_2(17)$, but no $\mathrm{PGL}_2(17)$. □

Lemma 9.19 *Let $S \cong \Omega_n^+(2)$ for $n \geq 8$. Then $\overline{H} = \overline{G}$.*

Proof. Let V be the natural n-dimensional module of $\overline{G}$. Recall, that $\mathrm{Out}(G) = \mathbb{Z}_2$ for $n \geq 10$ and Σ_3 for $n = 8$.

Suppose $3 \in \pi(H)$. By 4.16 and 8.21 there exists a terminal element $\bar{t}$ of order 3, which have $\dim[V, \bar{t}] = 2$. Therefore $\overline{H}$ does not contain a Sylow-3-subgroup of G, so FS_3-property fails. We conclude as in 8.8, that some $\overline{y} \in \overline{H}, o(\overline{y}) = 3$ exists, such that $C_{\overline{G}}(\overline{y})$ contains a subnormal subgroup isomorphic to Σ_5. Let $\overline{y} \in \overline{H}$ with $o(\overline{y}) = 3$. By 5.4, $O^2(C_{\overline{G}}(\overline{y})) \cong (\mathrm{GU}_{m/2}(2))' \times \Omega_{n-m}^{\varepsilon_1}(2)$ for $\varepsilon_1 = (-1)^{m/2}$, if $\overline{y}$ is an element of $\Omega_n^+(2)$. If $\overline{y}$ is outside of $\Omega_8^+(2)$, we use [ATLAS] for the structure of $O^2(C_{\overline{G}}(\overline{y}))$. In any case a subnormal Alt_5 exists only for $n - m = 4$, with $\varepsilon_1 = -1$.
In that case $\overline{H}$ covers the subgroup $\mathrm{GU}_{m/2}(2)$ by 6.28, so $\overline{H}$ contains an element, which is conjugate to $\overline{x}$ and $\overline{H} = \overline{G}$.
Therefore either $\overline{H} = \overline{G}$ or FS_3-property holds, so $3 \notin \pi(H)$.

Suppose $5 \in \pi(H)$. Let $\overline{x} \in \overline{H}$ be of order 5 and $m := \dim[V, \overline{x}]$. By 5.4, $C_{\overline{G}}(\overline{y}) \cong \mathrm{GU}_{m/4}(4) \times \Omega_{n-m}^{\varepsilon_2}(2)$ for $\varepsilon_2 = (-1)^{m/4}$.
Suppose $n - m \geq 6$. Then $\Omega_{n-m}^{\varepsilon_2}(2)$ is passive, so by 6.28, $3 \in \pi(H)$.
If $(n - m, \varepsilon_2) = (4, +1)$ or $(2, -1)$, for the same reason $3 \in \pi(H)$.
If $m \geq 12$, then $\overline{H}$ covers the $\mathrm{GU}_{m/4}(4)$-factor too by 6.28 and $3 \in \pi(H)$.
So $m \leq 8$ and $(n - m, \varepsilon_2) \in \{(0, +1) = (0, -1), (2, +1), (4, -1)\}$. This gives the groups $O_8^+(2)$ and $O_{10}^+(2)$.
In both cases $\overline{H}$ contains the normalizer of a Sylow-5-subgroup.
In case of $S \cong \Omega_{10}^+(2)$ we get $3 \in \pi(H)$: We check the list of maximal subgroup in [ATLAS] and use 6.2 with Theorem 5, to get $3 \in \pi(H)$.
If $S \cong \Omega_8^+(2)$, $3 \in \pi(H)$ by 8.22, if $\overline{G} \cong \Omega_8^+(2).\Sigma_3$, so $|\overline{G} : \overline{G_0}| \leq 2$.
Calculation of structure constants within $\Omega_8^+(2).2$ reveals, that $\overline{K}$ can contain in this case of the classes $2A$ and $2F$ (Notation as in [ATLAS]) only, as the other classes of involutions invert elements of order 5. Therefore $|\overline{K}| \leq 1+1575+120 = 1796$. On the other hand $|\overline{G} : \overline{H}| \geq 2 \cdot 3^5 \cdot 7 = 3402$, a contradiction.

If now $3 \notin \pi(H) \not\ni 5$, we may use 4.30 for the connected components of $\Gamma_{\mathcal{O}}$, which do not contain elements of order 3 or 5.
In particular there exists a prime p with $n = 2p$ or $n = 2p+2$ and the connected component contains elements of prime order r for all prime divisors r of $2^p - 1$. By 8.22, $r \in \pi(H)$ implies $p \in \pi(H)$, while p is in the connected component containing the elements of order 3 and 5. Therefore $\overline{H} = \overline{G}$. □

Lemma 9.20 *Let $S \cong \Omega_n^-(2)$ for $n \geq 8$. Then $\overline{H} = \overline{G}$.*

Proof. Let V be the natural n-dimensional module of $\overline{G}$. Recall, that $\mathrm{Out}(G) = \mathbb{Z}_2$.

Suppose $3 \in \pi(H)$. By 4.16 and 8.21 there exists terminal elements $\bar{t}$ of order 3, which have $\dim[V, \bar{t}] = 2$. Therefore $\overline{H}$ does not contain a Sylow-3-subgroup of G, so FS_3-property fails. We conclude as in 8.8, that some $\overline{y} \in \overline{H}, o(\overline{y}) = 3$ exists, such that $C_{\overline{G}}(\overline{y})$ contains a subnormal subgroup isomorphic to Σ_5. Let

$\overline{y} \in \overline{H}$ with $o(\overline{z}) = 3$. By 5.5, $O^2(C_{\overline{G}}(\overline{y})) \cong (\mathrm{GU}_{m/2}(2))' \times \Omega^{\varepsilon_1}_{n-m}(2)$ for $\varepsilon_1 = (-1)^{1+m/2}$. Therefore a subnormal Alt_5 exists only for $n - m = 4$, with $\varepsilon_1 = -1$.
In that case $\overline{H}$ covers the subgroup $\mathrm{GU}_{m/2}(2)$ by 6.28, so $\overline{H}$ contains an element, which is conjugate to $\overline{x}$ and $\overline{H} = \overline{G}$.
Therefore either $\overline{H} = \overline{G}$ or FS_3-property holds, so $3 \notin \pi(H)$.

Suppose $5 \in \pi(H)$. Let $\overline{x} \in \overline{H}$ be of order 5 and $m := \dim[V, x]$. By 5.5, $C_{\overline{G}}(\overline{y}) \cong \mathrm{GU}_{m/4}(4) \times \Omega^{\varepsilon_2}_{n-m}(2)$ for $\varepsilon_2 = (-1)^{1+m/4}$.
Suppose $n - m \geq 6$. Then $\Omega^{\varepsilon_2}_{n-m}(2)$ is passive, so by 6.28, $3 \in \pi(H)$.
If $(n - m, \varepsilon_2) = (4, +1)$ or $(2, -1)$, for the same reason $3 \in \pi(H)$.
If $m \geq 12$, then $\overline{H}$ covers the $\mathrm{GU}_{m/4}(4)$-factor too by 6.28 and $3 \in \pi(H)$.
The case $m = 4$ gives a contradiction as then $C_V(\overline{x})$ is a $O^+_4(2)$-space.
In case $m = 8$, $[V, \overline{x}]$ is a $O^+_8(2)$-space. Therefore $C_V(\overline{x})$ has to be an $O^-_4(2)$-space and $S \cong \Omega^-_{12}(2)$. But from the centralizer structure we conclude, that FS_5-property holds. So $\overline{H}$ contains a Sylow-5-subgroup of $\overline{G}$ and there are other elements of order 5 in $\overline{H}$ which imply $3 \in \pi(H)$.

If now $3 \notin \pi(H) \not\ni 5$, we may use 4.31 for the connected components of $\Gamma_{\mathcal{O}}$, which do not contain elements of order 3 or 5.
Then either n or $n - 2$ is a 2-power and the connected component contains elements of order r for primes r dividing $2^{n/2} + 1$ resp. $2^{n/2-1} + 1$.
Let $\overline{M}$ be a maximal subgroup of $\overline{G}$ containing $\overline{H}$. By [KL] we get $\overline{M}$ of type $O^-_{n/2}(4)$ or in class $\mathcal{S}$, if n is a 2-power. If $n-2$ is a 2-power, then $\overline{M}$ is of type $\mathrm{Sp}_{2n-2}(2)$, a parabolic of type $2^{n-2} : O^-_{n-2}(2)$ or $\overline{M}$ in class $\mathcal{S}$.
Using 6.2 and Theorem 5 we get $3 \in \pi(H)$, $5 \in \pi(H)$ or $\overline{M} \cong \mathrm{PGL}_2(p)$ for a Fermat prime p. Then $p = 2^{n/2}+1$ or $p = 2^{n/2-1}+1$ and $\mathrm{PGL}_2(p)$ has a faithful representation in degree at least $p - 1$, but $\overline{G}$ has an n-dimensional module, so $2^{n/2-1} \leq n$, which gives contradictions: either $p = 9$ or $n \geq 10$. □

Lemma 9.21 *Let $S \cong \mathrm{PSU}_n(2)$ for $n \geq 5$. Then $\overline{H} = \overline{G}$.*

Proof. Recall, that $\mathrm{Out}(S) \cong \mathbb{Z}_2$ or Σ_3 depending on whether n is divisible by 3.

Suppose $3 \in \pi(H)$. By 4.13 and 8.21 there are terminal elements of order 3, so $\overline{H}$ does not contain a Sylow-3-subgroup of $\overline{G}$, so FS_3-property fails. So there exists some $x \in \overline{H}$, $o(\overline{x}) = 3$, such that $C_{\overline{G}}(\overline{x})$ contains a subnormal $\Sigma_5 \cong \mathrm{P\Gamma L}_2(4)$.
We use 5.7 for the description of centralizers of elements of order 3. In particular we see, that no subnormal Σ_5 exists, so FS_3-property holds and $\overline{H} = \overline{G}$.

Suppose $5 \in \pi(H)$. By 5.9 and 8.12, property FS_5 holds, so $\overline{H}$ contains a Sylow-5-subgroup of $\overline{G}$.
In particular $\overline{H}$ contains an element $\overline{x}$ of order 5, such that $\dim[V, \tilde{x}] = 4$, for $\tilde{x}$ some preimage of $\overline{x}$ in $\mathrm{GU}_n(2)$ and V the natural $\mathrm{GF}(4)\mathrm{GU}_n(2)$-module. For this element, $O_3(C_{\overline{G}}(\overline{x})) \neq 1$, so $3 \in \pi(H)$ by 6.7.

So suppose $3 \notin \pi(H) \not\ni 5$. We use 4.27 for the connected components of Γ_O, which do not contain elements of order 3 or 5.

Let $\overline{x} \in \overline{H}$, $o(\overline{x})$ some odd prime r. By 8.9 we conclude, that a prime p exists with either $n = p$ or $n-1 = p$ and $r \mid \frac{2^p+1}{3}$. Then $\overline{H}$ contains a torus of size $\frac{2^p+1}{3}$, on which a subgroup of size p acts. By 8.22 then $p \in \pi(H)$, but p is in the connected component containing elements of order 3 and 5, so $3 \in \pi(H)$ or $5 \in \pi(H)$. $\square$

Lemma 9.22 *Let $S \cong F_4(2)$, $E_6(2)$ or ${}^2E_6(2)$. Then $\overline{H} = \overline{G}$.*

Proof. Case $S \cong F_4(2)$: We use [ATLAS]. By inspection of centralizers of 3-elements we get FS_3-property, so together with 4.20 and 8.21 we get $\overline{H} = \overline{G}$, if $3 \in \pi(H)$.
If $5 \in \pi(H)$ we get $3 \in \pi(H)$: The centralizer of a $5A$-element has structure $\mathbb{Z}_5 \times \mathrm{Sp}_4(2)$, so we get elements of order 3 into $\overline{H}$.
By 6.30 there remains only the case $17 \in \pi(H)$. Subgroups of order 17 are self centralizing. Since we cannot exclude the existence of a $\mathrm{PGL}_2(17)$, we count involutions. We get $|\overline{K}| \leq 96648112$, as elements of class $2C$ invert elements of order 17. On the other hand $\overline{H} \cong \mathbb{Z}_{17} : \mathbb{Z}_{16}$, so $|\overline{G} : \overline{H}| \geq 2^{19} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 = 6086629785600$.

Case $S \cong {}^2E_6(2)$. We use the character table of ${}^2E_6(2).3$ and ${}^2E_6(2).2$ provided by GAP.
We first establish the FS_3-property:
Let $y \in X = {}^2E_6(2).3$ be an element of order 3. We claim, that $C_X(y)$ does not contain a subnormal $\mathrm{Alt}_5 \cong \mathrm{PSL}_2(4)$. We show this, using the list of conjugacy classes and centralizer sizes of X, which we get from the character table: If $C_G(y)$ contains no elements of order 5 or elements of order 11 or 19, the statement is obvious:
Elements of order 11 or 19 do not commute with elements of order 5, but cannot permute components of type Alt_5 nontrivially.
Remains only one class of elements (class 3B), which contains a subgroup of order 3^8. As elements of order 5 commute in G with 3-groups of size at most 3^3, we have FS_3-property.
By 4.21 and 8.21, we have $\overline{H} = \overline{G}$, if $3 \in \pi(H)$.
If $5 \in \pi(H)$, the centralizer of a $5A$-element is $\mathbb{Z}_5 \times \mathrm{Alt}_8$, which is contained in a maximal subgroup $\Omega_{10}^-(2)$.
Therefore $5 \in \pi(H)$ implies $3 \in \pi(H)$.
By 6.31 there remains only the case $17 \in \pi(H)$. Subgroups of order 17 are self centralizing in ${}^2E_6(2)$, but not in ${}^2E_6(2).3$, so $|\overline{G} : \overline{G_0}| \leq 2$.
Let $\overline{M}$ be a maximal subgroup of $\overline{G}$ containing $\overline{H}$. If $O_2(\overline{M}) \neq 1$, we have either $\overline{M}$ parabolic or the centralizer of an outer involution. If $\overline{M}$ is a parabolic subgroup, then $3 \in \pi(H)$ by 6.2 and Theorem 5. Of the two outer classes of involutions only the class with centralizer $F_4(2)$ does not invert elements of order 17. So $\overline{M} \cong F_4(2)$ and $3 \in \pi(H)$ by 6.2 and Theorem 5. So $O_2(\overline{M}) = 1$ and we may use 6.21.
This shows $3 \in \pi(H)$ and $19 \in \pi(H)$, so $\overline{H} = \overline{G}$.

Case $S \cong E_6(2)$. Recall $\mathrm{Out}(S) \cong \mathbb{Z}_2$.
We show FS_3-property, using the character table of $E_6(2)$ as provided in GAP. Elements of class $3A$ and $3B$ commute with elements of order 31 resp. 17, while elements of of order 31 and 17 do not commute with elements of order 5, so the

centralizers of $3A$ and $3B$-elements do not contain subnormal $\mathrm{Alt}_5 = \mathrm{PSL}_2(4)$-subgroups.
Remains the centralizer of a $3C$-Element, which has size $2^9 \cdot 3^6 \cdot 5 \cdot 7$. As the centralizer of a $5A$-element has size $2^6 \cdot 3^4 \cdot 5^2$, we get a contradiction: A component of type $\mathrm{PSL}_2(5)$ or $\mathrm{SL}_2(5)$ would be normal in $C_S(3C)$, so a Sylow-3-subgroup of size 3^6 acts on it. This gives a contradiction, as the kernel of this action has size at most 3^4.
By 4.21 and 8.21, we have $\overline{H} = \overline{G}$, if $3 \in \pi(H)$.
If $5 \in \pi(H)$, then $3 \in \pi(H)$. From the existence of a Levi complement $\Omega_{10}^{+}(2)$, 5.4 and the centralizer size we conclude, that $C_S(5A) \cong \mathbb{Z}_5 \times U_4(2)$, so $5 \in \pi(H)$ implies $3 \in \pi(H)$.
By 6.31 there remains only the case $17 \in \pi(H)$, but this time $17 \in \pi(H)$ implies $3 \in \pi(H)$ by the centralizer size of $3 \cdot 17$. □

Lemma 9.23 *Let $S \cong E_7(2)$ or $E_8(2)$. Then $\overline{H} = \overline{G}$.*

Proof. Recall $\mathrm{Out}(S) = 1$.
Suppose $O_2(\overline{H}) \neq 1$. Then $\overline{H}$ is contained in a maximal parabolic, so by 6.2, some maximal parabolic $\overline{P}$ is a group to a subloop. We use Theorem 5 on it, to get this subloop soluble. In particular $\overline{H}$ contains a Sylow-3-subgroup of $\overline{P}$.
In 4.21 we showed the connectedness of a conjugacy class of elements of order 3, which has a centralizer of type $\mathbb{Z}_3 \times \Omega_{12}^{+}(2)$ resp. $\mathbb{Z}_3 \times E_7(2)$, so this conjugacy class is terminal by 8.21. But any maximal parabolic subgroup contains such elements, as these elements come from a $\mathrm{PSL}_2(2)$, which is generated by root subgroups $X_\alpha, X_{-\alpha}$. Therefore $\overline{H} = \overline{G}$, if $O_2(\overline{H}) \neq 1$.

So $O_2(\overline{H}) = 1$ and we would like to use 6.21.
As centralizers of involutions are contained in maximal parabolics, we see from [ATLAS], p. 219 and p.235, that 3 is a prime, for which we may apply 6.21. So $\overline{H}$ contains a Sylow-3-subgroup of $\overline{G}$.
Since we showed already, that there are terminal elements of order 3, $\overline{H} = \overline{G}$.
□

10 Conclusion

Let (G, H, K) be a faithful loop envelope to a Bol Loop X of exponent 2. By Theorem 5 and Theorem 1 we have $\overline{G} \cong D_1 \times D_2 \times ... \times D_e$ with $D_i \cong \mathrm{PGL}_2(q_i)$ for $q_i = 9$ or a Fermat prime with $q_i \geq 5$. Furthermore $D_i \cap \overline{H} =: B_i$ is a Borel subgroup of D_i. Let π_i be the projection of $\overline{G}$ onto D_i.

Lemma 10.1 *(i)* $\overline{H} = \prod_{i=1}^{e} B_i$.

(ii) If $\overline{k} \in \overline{K}$ and $1 \leq i \leq e$, then $\pi_i(\overline{k})$ is either 1 or an involution from $\mathrm{PGL}_2(q_i)$ outside $\mathrm{PSL}_2(q_i)$.

Proof. We have $B := \prod_{i=1}^{e} B_i \leq \overline{H}$ by Theorem 5.
This implies (ii), as within D_i only this type of involutions does not invert elements of odd order in B_i: Inner involutions of $\mathrm{PSL}_2(q_i)$ act nontrivially on the Borel subgroup, field automorphisms are not present.

Let $\overline{x} \in \overline{H}$, $o(\overline{x}) = r$ for some odd prime r and $\overline{x} = \prod_{i=1}^{e} \overline{x_i}$ with $\overline{x_i} \in D_i$.
Then for all i: $\overline{x_i} \in \overline{H}$:
As $\overline{G}$ is a direct product, $C_{\overline{G}}(\overline{x}) = \prod_{i=1}^{e} C_{D_i}(\overline{x_i})$. If $x \in H$, $o(x) = r$ is some preimage of $\overline{x}$, then $C_G(x)$ is a group to a subloop, which covers $C_{\overline{G}}(\overline{x})$. As all the $\overline{x_i}$ are contained in $O_r(C_{\overline{G}}(\overline{x}))$, but $O_{2,r}(C_G(x)) \cdot C_H(x) = O_2(C_G(x))C_H(x)$ by 6.7, there are preimages in H of order r for those $\overline{x_i}$ with $o(\overline{x_i}) = r$.
Let $L \leq \overline{G}$, with $B \leq L$ and $B \neq L$. Suppose $L = \overline{H}$. We will create a contradiction:
Consider the projections $\pi_i(L)$. There exists some i with $\pi_i(L) > B_i$. As B_i is maximal in D_i, $\pi_i(L) = D_i$. As we showed above, for all elements $\overline{x} \in \overline{H}$ of odd prime order, $\pi_j(\overline{x}) \leq \overline{H}$ for all j. This implies $D_i \leq L$.
Let $\overline{k} \in \overline{K}$ and consider $\pi_i(\overline{k})$. If $\pi_i(\overline{k}) \neq 1$, then $\pi_i(\overline{k})$ inverts some element of odd prime order in D_i by Baer-Suzuki. This gives a contradiction to 6.23. But if $\pi_i(\overline{k}) = 1$ for all $\overline{k} \in \overline{K}$, then we cannot have $\overline{G} = \langle \overline{K} \rangle$. So $L \neq \overline{H}$ and $B = \overline{H}$. □

We can now prove, that $O_2(G)$ is a group to a subloop:

Lemma 10.2 *$O_2(G)H \cap K = O_2(G) \cap K$ and $O_2(G) = (O_2(G) \cap H)(O_2(G) \cap K)$.*

Proof. By 10.1, $O_2(\overline{H}) = 1$. By 6.2 we have a subloop to $O_2(G)H$, which is soluble by 6.9. Therefore $\langle K \cap O_2(G)H \rangle \leq O_2(O_2(G)H) = O_2(G)$. This, together with 6.2, implies the statement. □

Now there are lots of other subloops: Let $I = \{1, 2, ..., e\}$ and for $J \subseteq I$ let G_J the preimage of $\prod_{j \in J} D_j$.

Lemma 10.3 *For any $J \subseteq I$, $G_J = (G_J \cap H)(G_J \cap K)$.*

Proof. For $J = \emptyset$ this is 10.2 and for $J = I$ this is the loop folder property.
Let $x \in G_J$ and $x = hk$ with $h \in H$, $k \in K$. Let $l \in I - J$. As $\pi_l(x) = 1$, we cannot have $\pi_l(\overline{k}) \neq 1$: Else by 10.1(ii), $\pi_l(\overline{k})$ is some involution of $\mathrm{PGL}_2(q_l)$ outside $\mathrm{PSL}_2(q_l)$. But $\pi_l(\overline{H}) = B_l$ and B_l contains only involutions from $\mathrm{PSL}_2(q_l)$.
So $\pi_l(\overline{k}) = 1$, thus $\pi_l(\overline{h}) = 1$ too. This implies the statement. □

Our next goal is to produce subloops to certain Sylow-2-subgroups P of G. Therefore we have to calculate $|P \cap K|$.

Lemma 10.4 *For $J \subseteq I$, $\overline{G}$ has a unique conjugacy class $\mathcal{C}_J$ with the property: For $t \in \mathcal{C}_J$: $\pi_i(t) = 1$ for $i \notin J$ and $\pi_i(t)$ is some involution of $\mathrm{PGL}_2(q_i)$ outside $\mathrm{PSL}_2(q_i)$ for $i \in J$. Moreover*

$$|\mathcal{C}_J| = \prod_{j \in J} q_j \frac{q_j - 1}{2}.$$

Proof. This is immediate from the structure of $\overline{G}$. Recall, that for q odd, the centralizer of an involution in $\mathrm{PGL}_2(q)$ is the normalizer of a torus of size either $q-1$ or $q+1$. In our case $q-1$ is divisible by 4, so inner involutions of $\mathrm{PSL}_2(q)$ have a centralizer of size $2(q-1)$ while outer involutions have centralizer size $2(q+1)$. □

Let $t \in \mathcal{C}_J$, We denote with $O_2(G)t$ the full preimage of t in G. By the previous lemma, the number $n_J := |O_2(G)t \cap K|$ is well defined and independent of the choice of $t \in \mathcal{C}_J$. Recall $n_\emptyset = |O_2(G) \cap K| = |O_2(G) : O_2(G) \cap H|$ by 10.2

Lemma 10.5

$$n_J = \frac{n_\emptyset \cdot 2^{|J|}}{\prod\limits_{j \in J}(q_j - 1)}$$

Proof. As G_J is a subloop by 10.3, we have $|G_J : G_J \cap H| = |G_J \cap K|$. As $|G_J : G_J \cap H| = |\overline{G_J} : \overline{G_J} \cap \overline{H}||O_2(G) : O_2(G) \cap H|$, we have

$$|G_J : G_J \cap H| = n_\emptyset \prod_{j \in J}(q_j + 1).$$

On the other hand

$$|G_J \cap K| = \sum_{L \subseteq J} n_L |\mathcal{C}_L|.$$

We therefore get a system of equations for the n_J. This is a special case of 6.20. Now the statement can be shown by induction on $|J|$. For example for $|J| = 1$ we get the equation $n_\emptyset(q_j + 1) = n_\emptyset + n_{\{j\}} \cdot q_j \frac{q_j - 1}{2}$, which gives $n_{\{j\}} = \frac{2n_\emptyset}{q_j - 1}$. In general we have:

$$n_\emptyset \prod_{j \in J}(q_j + 1) = \sum_{L \subseteq J} n_L \prod_{j \in L} q_j \frac{q_j - 1}{2}.$$

For $L \subseteq J$, $L \neq J$ we have the formula for n_L by induction. On the other hand for any numbers $q_j, j \in J$ the equation

$$\prod_{j \in J}(q_j + 1) = \sum_{L \subseteq J} \prod_{j \in L} q_j$$

holds. After some calculation this gives exactly the formula for n_J. □

Lemma 10.6 *Let $P \in \mathrm{Syl}_2(G)$. Then $|P \cap K| = 2^e n_\emptyset = |G : H|_2 = |X|_2$. If $P \cap O_2(G)H \in \mathrm{Syl}_2(O_2(G)H)$, then $P = (P \cap H)(P \cap K)$.*

Proof. We choose $P \in \mathrm{Syl}_2(G)$ with $P \cap O_2(G)H \in \mathrm{Syl}_2(O_2(G)H)$. As K is a G-normal subset, $|P \cap K|$ is independent of the choice of P.
Let $i \in I$ and consider $P_i = \pi_i(\overline{P}) \in \mathrm{Syl}_2(D_i)$. Then P_i is a dihedral group, $P_i \cap \overline{H}$ is a cyclical group of size $q_i - 1$. The other coset of $P_i \cap \overline{H}$ in P_i consists entirely of involutions, half of them involutions in $\mathrm{PSL}_2(q_i)$ and half of them outside of $\mathrm{PSL}_2(q_i)$. As all involutions outside of $\mathrm{PSL}_2(q_i)$ in $\mathrm{PGL}_2(q_i)$ are conjugate, we have

$$\pi_i(\overline{P}) \cap \overline{K} = 1 + \frac{q_i - 1}{2},$$

the summand 1 coming from the $1 \in \overline{K}$. This shows for $J \subseteq I$:

$$|\overline{P} \cap \mathcal{C}_J| = \prod_{j \in J} \frac{q_j - 1}{2}.$$

As

$$|P \cap K| = \sum_{J \subseteq I} n_J |\overline{P} \cap \mathcal{C}_J|,$$

this gives

$$|P \cap K| = \sum_{J \subseteq I} \frac{n_\emptyset 2^{|J|}}{\prod\limits_{j \in J} (q_j - 1)} \prod_{j \in J} \frac{q_j - 1}{2} = 2^{|I|} n_\emptyset = 2^e n_\emptyset.$$

By Dedekind we have $O_2(G)(P \cap H) = P \cap O_2(G)H$. This gives

$$\frac{|O_2(G)||P \cap H|}{|O_2(G) \cap P \cap H|} = |P \cap O_2(G)H|.$$

By definition $|O_2(G) : O_2(G) \cap H| = n_\emptyset$ and $|P \cap O_2(G)H| = \frac{|G|_2}{|G:O_2(G)H|_2} = \frac{|G|_2}{2^e}$ by assumption. This gives $|P \cap H| = \frac{|G|_2}{2^e n_\emptyset}$ and $|P : P \cap H| = 2^e n_\emptyset = |P \cap K|$, so P is a group to a subloop by 6.2. Finally $|X|_2 = |G : H|_2 = |G : O_2(G)H|_2 |O_2(G)H : H|_2 = 2^e |O_2(G) \cap K| = 2^e n_\emptyset$. □

As a consequence we get an analogue of Lagrange's Theorem:

Corollary 10.7 *Let $Y \leq X$ be a subloop. Then $|Y|$ is a divisor of $|X|$.*

Proof. By 10.6, we have $|Y|_2 \leq |X|_2$: To Y a subloop of size $|Y|_2$ exists, which is soluble by 6.9. To this subloop a 2-group $U \leq G$ exists with $|U \cap K| = |Y|_2$. As $|P \cap K| = |X|_2$ for any Sylow-2-subgroup of G, $|Y|_2$ is a divisor of $|X|_2$.

Suppose Y is nonsoluble, so $|Y|_{2'} \neq 1$. To Y some subgroup $U \leq G$ exists with $U = (U \cap H)(U \cap K)$, $U = \langle U \cap K \rangle$ and $|Y| = |U : U \cap H| = |U \cap K|$. We may use Theorem 5 on U. The map $\theta : U \to G : u \mapsto O_2(G)u$ gives a homomorphism from U into $\overline{G}$ and an injection from $U/(O_2(U) \cap O_2(G))$ into $G/O_2(G)$.
But elements of odd order from $U \cap H$ map to elements of odd order in $\overline{H}$.
This shows, that components of type Alt_5 in $U/O_2(U)$ cannot project into components of type Alt_6 in $\overline{G}$, so components of $U/O_2(U)$ project surjectively into components of $G/O_2(G)$. Recall, that $|U : U \cap H|_{2'}$ is the product of $\frac{|C_j : C_j \cap \overline{H}|}{2}$ for C_j the components of $U/O_2(U)$. Similarly $|G : H|_{2'} = \prod\limits_{i=1}^{e} \frac{q_i+1}{2}$. By the injection map, which preserves $\overline{H}$-containement, therefore $|Y|_{2'} = |U : U \cap H|_{2'}$ divides $|G : H|_{2'} = |X|_{2'}$. □

We will show, that the subloops of size $|X|_2$ have some nice properties, but need before some facts about $\mathrm{PGL}_2(q)$:

Lemma 10.8 *Let $Z \cong \mathrm{PGL}_2(q)$ with $q = 9$ or $q \geq 5$ a Fermat prime. Let B a Borel subgroup of G and $\mathcal{C}$ the class of involutions outside $\mathrm{PSL}_2(q)$.*

(i) B acts in two orbits on $\mathrm{Syl}_2(Z)$: one orbit of size q and one of size $\frac{|B|}{2}$.

(ii) If $P \in \mathrm{Syl}_2(Z)$, then either $P \cap B \in \mathrm{Syl}_2(B)$ or $|P \cap B| = 2$.

(iii) Let $D = \langle A \rangle$ for $A \subseteq \{1\} \cup \mathcal{C}$ with $D = O_2(D)$ and $D = (D \cap B)A$. Then a $Q \in \mathrm{Syl}_2(Z)$ exists with $D \leq Q$ and $Q \cap B \in \mathrm{Syl}_2(B)$.

Proof. Notice, that $|\mathrm{Syl}_2(B)| = q$. So for $T \in \mathrm{Syl}_2(B)$, the B-conjugation action on $N_Z(T)$ gives one orbit of length q. Recall, that a subgroup $T \in \mathrm{Syl}_2(B)$ is the normalizer of a Cartan-subgroup: A 2-point stabilizer in the 3-transitive action of Z on $q + 1$ points.
As B is a point stabilizer, B acts 2-transitively on the q other points. In particular B acts transitively on the $\frac{q(q-1)}{2}$ 2-point-stabilizers, which are not subgroups of B. Therefore B acts transitively on the remaining $\frac{q(q-1)}{2}$ Sylow-2-subgroups, so (i) holds.
(ii) is a consequence of (i), as $P \cap B$ is the orbit stabilizer.
For (iii) let $D \leq P$ with $P \in \mathrm{Syl}_2(G)$ and assume $|P \cap B| = 2$. Otherwise we may choose $Q = P$.
As P is dihedral, D is dihedral and $|D \cap B| \leq |P \cap B| = 2$. Notice that $P \cap B \neq Z(P)$, as else $|P \cap B| > 2$.
Let $P \cap B = \langle i \rangle$ and suppose $|D| > 4$. Then i is a $\mathrm{PSL}_2(q)$-involution, which has a conjugate $i^d \in D$ with $i^d \neq i, d \in D$. This contradicts $D \subseteq \langle i \rangle A = A \cup iA$, as $i^d \notin (A \cup iA)$. We conclude $|D| \leq 4$.
If $D = 1$, the statement is trivial.
If $|D| = 2$, let $c \in D \cap \mathcal{C}$. Then c fixes $B \cap P$, so $c \in N_Z(C_B(B \cap P))) =: Q \in \mathrm{Syl}_2(Z)$.
If $|D| = 4$, then $D \cap \mathcal{C} = \{c_1, c_2\}$ with $c_1c_2 \in B \cap P$. Again $D \leq N_Z(C_B(B \cap P))) =: Q \in \mathrm{Syl}_2(Z)$. □

Lemma 10.9 *Let $U \leq G$ with $U = O_2(U)$, $U = \langle U \cap K \rangle$ and $U = (U \cap H)(U \cap K)$.*
Then a Sylow-2-subgroup Q of G exists with $U \leq Q$ and $Q \cap O_2(G)H \in \mathrm{Syl}_2(O_2(G)H)$.

Proof. For fixed $i \in I$ let $Z := \pi_i(\overline{G})$, $D := \pi_i(\overline{U})$,$B := \pi_i(\overline{H})$ and $A := \pi_i(\overline{U} \cap \overline{K})$. Then Z, B, A, D satisfy the prerequisites of 10.8 (iii), as $\mathcal{C}$ is $\pi_i(\overline{K}) - \{1\}$: By the homomorphism property of π_i: $\pi_i(\overline{U}) = \pi_i(\overline{U} \cap \overline{H})\pi_i(\overline{U} \cap \overline{K})$. But $\pi_i(\overline{U} \cap \overline{H}) \leq \pi_i(\overline{U}) \cap \pi_i(\overline{H}) = D \cap B$.
By 10.8(iii) then $\pi_i(U) \leq Q_i$ for some $Q_i \in \mathrm{Syl}_2(\pi_i(\overline{G}))$ with $Q_i \cap \overline{H} \in \mathrm{Syl}_2(\pi_i(\overline{H}))$.
We get such a Q_i for all $i \in I$.
If we set Q as the preimage of $\prod_{i \in I} Q_i$, we have $\overline{U} \leq \overline{Q}$ with $Q \cap O_2(G)H \in \mathrm{Syl}_2(O_2(G)H)$ □

We get an analogue of Sylow's Theorem:

Corollary 10.10 *For any soluble subloop $Y \leq X$ some subloop $Z \leq X$ exists with:*

(i) $Y \leq Z$ and $|Z| = |X|_2$.

(ii) All subloops of X of size $|X|_2$ are conjugate under H, the group of inner automorphisms.

Proof. For (i) observe, that to Y a subgroup U exists with $U = \langle U \cap K \rangle$, $U = (U \cap H)(U \cap K)$ and the subloop to U is exactly Y. By 10.9, there exists some $Q \in \mathrm{Syl}_2(G)$ with $U \leq Q$ and $\overline{Q} \cap \overline{H} \in \mathrm{Syl}_2(\overline{H})$. By 10.6, Q is a group to a subloop.
For (ii) we use the fact, that for $P \in \mathrm{Syl}_2(G)$: $\overline{P} = \prod_{i=1}^{e} \pi_i(\overline{P})$ and apply 10.8(ii) on $B := \pi_i(\overline{H})$ and $Z := \pi_i(\overline{G})$. □

The structure of groups to Bol loops of exponent 2 is now quite well understood. Next to do: Bruck loops of 2-power-exponent, as indicated in [Asch] and [AKP].

References

[Asch] M.Aschbacher, On Bol loops of exponent 2, *J.Algebra* **288** (2005) 99-136

[AKP] M.Aschbacher,M.Kinyon,J.D.Philips, Finite Bruck loops *Transactions of the AMS* **358**,No.7, 3061-3075

[Baer] R.Baer, Nets and groups, *Trans. Amer. Math. Soc* **47** (1939) 110-141

[BS] B.Baumeister, A.Stein *Self-invariant* 1-*Factorizations of Complete Graphs and Finite Bol Loops of Exponent* 2, to appear

[Bruck] R.H.Bruck, *A survey of Binary Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 20. Reihe: Gruppentheorie, Springer Verlag, Berlin etc 1958

[Burn] R.P.Burn, Finite Bol loops, *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 377-385

[Car] R.Carter, *Simple Groups of Lie Type*, John Wiley and Sons, London, 1972.

[Car2] R.Carter, *Finite Groups of Lie Type: conjugacy classes and complex characters* Wiley-Interscience, 1985

[Coo] B.N.Cooperstein *The maximal subgroups of* $G_2(2^n)$. J. Algebra **70**, 23-36, (1981)

[CLSS] A.M.Cohen, M.W.Liebeck, J.Saxl, G.M.Seitz *The local maximal subgroups of exceptional groups of Lie type, finite and algebraic* Proc. Lond. Math. Soc., III. Ser. 64, No.1, 21-48 (1992)

[ATLAS] J.H.Conway, R.T.Curtis, S.P.Norton, R.A.Parker ,R.A.Wilson *An ATLAS of finite groups* Oxford University Press, 1985

[Glaub1] G.Glauberman, *On Loops of odd order I*, J.Algebra **1** (1964), 374-396

[Glaub2] G.Glauberman, *On Loops of odd order II*, J.Algebra **8** (1968), 393-414

[GLS3] D.Gorenstein, R.Lyons, R.Solomon *The Classification of Finite Simple Groups*, number 3

[HN] S. Heiss, G.Nagy, Personal Communication, 1998.

[Hei] S. Heiss, *Self-invariant* 1*-factorizations of complete graps*, submitted for publication.

[K3D4] P.Kleidman *The maximal subgroups of the Steinberg triality groups* $^3D_4(q)$ *and of their automorphism groups* J.Algebra 115(1988), 182-199

[KL] P.Kleidman, M.Liebeck *The Subgroup Structure of the Finite Classical Groups* LMS Lecture Note Series 129, Cambridge University Press

[Lieb] M.W.Liebeck, The classification of finite Moufang loops, *Math. Proc. Cambridge Philos. Soc* **102** (1987),33-47

[LSS] M.W.Liebeck,J.Saxl,G.M.Seitz, *Subgroups of maximal rank in finite exceptional groups of Lie type* Proc. Lond. Math. Soc., III. Ser. 65, No.2, 297-325 (1992)

[Malle] G.Malle, *The maximal subgroups of* $^2F_4(q^2)$, J.Algebra **139** (1991) 52-69

[Nag1] G.Nagy, *A class of simple proper Bol loops*, *Preprint*

[Nag2] G.Nagy *Finite simple left Bol loops*

`http://www.math.u-szeged.hu/~nagyg/pub/simple_bol_loops.html`

[Rob] D.A.Robinson, Bol loops, *Trans. Amer. Math. Soc* **123** (1966), 341-354.

[Ung] A.A.Ungar, Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovectors Spaces Kluwer Academic Publishers, Doldrechts-Boston-London 2001