Homework on Buildings *(Berlin Summer School, 2009)*

1. Let Γ be a thick generalized n-gon and let $G = \mathrm{Aut}(\Gamma)$. For each vertex u, let Γ_u denote the set of vertices adjacent to u and let $G_u^{[1]}$ denote the pointwise stablizer of Γ_u in the stabilizer G_u. Let $\alpha = (x_0, x_1, \ldots, x_n)$ be a path of length n and let G_α be the pointwise stabilizer of α in G.

   (i) Show that
   $$G_{x_0}^{[1]} \cap G_\alpha \subset G_{x_n}^{[1]}.$$

   (ii) Show that
   $$G_{x_0}^{[1]} \cap G_{x_1}^{[1]} \cap G_\alpha = 1.$$

2. Let (K, V, q) be an anisotropic quadratic space and suppose that $\dim_K V = 2$. Show that there exists a field E such that E/K is a quadratic extension and a K-linear isomorphism ψ from E to V such that
   $$q(\psi(u)) = q(\psi(1)) \cdot N(u)$$
   for all $u \in E$, where N is either the norm of the extension E/K if E/K is separable or $N(u) = u^2$ if E/K is inseparable.

3. Let (K, V, q) be an anisotropic quadratic space and suppose that $|K| < \infty$.

   (i) Show that if $\dim_K V = 2$, then q is surjective.

   (ii) Show that $\dim_K V \leq 2$.

4. Let K be a skew field of characteristic two, let σ be an involution of K, let
   $$K^\sigma = \{a \in K \mid a^\sigma = a\}$$
   and let
   $$K_\sigma = \{a + a^\sigma \mid a \in K\}.$$
   Then K^σ and K_σ are additive subgroups of K with $K_\sigma \subset K^\sigma$. Let $U = K^\sigma/K_\sigma$ and let $(u + K_\sigma)t = tut^\sigma + K_\sigma$ for all $u \in K^\sigma$ and all $t \in K$. Show that this map makes U into a right vector space over K.

5. Let
   $$(K, K_0, \sigma, V, q)$$
   be an anisotropic pseudo-quadratic space and suppose that $|K| < \infty$. Show that $\dim_K L = 1$.

6. Let E/K be a separable quadratic extension with norm N and let $\alpha \in K \backslash N(E)$. Let Q denote the quaternion algebra $(E/K, \alpha)$, let N be the reduced norm of Q and let σ be the standard involution of Q.

   (i) Show that $N(u) = u \cdot u^\sigma = u^\sigma \cdot u$ for all $u \in Q$.

   (ii) Show that $u^{-1} = u^\sigma / N(u)$ for each non-zero $u \in Q$.

   (iii) Let $T(u) = u + u^\sigma$ for all $u \in Q$ and let
   $$f(u, v) = T(u^\sigma \cdot v)$$
   for all $u, v \in Q$. Show that f is a bilinear form on Q as a vector space over K and that eE is the subspace $E^\perp$ with respect to this form.

   (iv) Find K_σ and K^σ. Are they equal?

7. Let $(X, (U_x)_{x \in X})$ be a Moufang set.

   (i) Show that for each ordered pair (x, y) of distinct elements x, y in X and for each $g \in U_x$, there exist unique elements $a, b \in U_y$ such that the product $\mu_{x,y}(g) := agb$ interchanges x and y.

   (ii) Let x, y, g, a, b be as in (i). (Note that a and b are non-trivial.) Show that
   $$\mu_{y,x}(a) = \mu_{y,x}(b) = \mu_{x,y}(g).$$
   [Hint: Rewrite the expression agb to obtain a product of three terms with a in the middle.]

8. Let $(m_{ij})_{i,j \in S}$ be a Coxeter matrix, let $\{s_i \mid i \in S\}$ be a set of symbols, one for each element of S, let
   $$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$
   be the corresponding Coxeter group and let Σ be the corresponding S-colored chamber system. Thus Σ is the graph with vertex set W whose i-colored edges (for each $i \in S$) are all pairs of the form $\{x, xs_i\}$ for some $x \in W$. The group W acts on Σ by left multiplication.

   (i) Show that the (setwise) stabilizer in W of an edge of Σ is a group of order two.

   (iii) Let $u, v \in W$ and let $\gamma = (x_0, x_1, \ldots, x_t)$ be a path of minimal length such that $x_0 = u$ and $x_t = v$. Show that γ cannot pass though two edges having the same stabilizer in W.

9. Let Γ be a bipartite graph such that every vertex has at least three neighbors and let $\Delta = \Delta_\Gamma$ be the corresponding chamber system of rank two. Show that if Γ is a generalized n-gon for some $n \geq 2$ or a tree, then Δ is a thick building. Is Δ irreducible?