Complex reflection groups

A pseudo-reflection $s \in GL_r(\mathbb{C})$ *is an element such that* $\mathrm{Ker}(s - Id)$ *is an hyperplane.*

A finite complex reflection group *is a finite subgroup of* $GL_r(\mathbb{C})$ *generated by pseudo-reflections.*

The irreducible finite complex reflection groups have been classified by Shepard and Todd (1954). They consist of:

- $G(de, e, r)$: the monomial matrices with coefficients in μ_{de} and product of non-zero coefficients in μ_d (where μ_i is the group of i-th roots of unity in $\mathbb{C}$). We have $A_r = G(1, 1, r+1), B_r = G(2, 1, r), D_r = G(2, 2, r), I_2(e) = G(e, e, 2)$.
- exceptional groups denoted $G_4, \ldots, G_{37}$. We have $H_3 = G_{23}, F_4 = G_{28}, H_4 = G_{30}, E_6 = G_{35}, E_7 = G_{36}, E_8 = G_{37}$.

Geometric definition

Let $V = \mathbb{C}^r$*; a finite subgroup* $W \subset GL(V)$ *is a complex reflection group if and only if the variety* V/W *is smooth (or equivalently* $V/W \simeq V$*).*

The quotient of an affine variety $X = \mathrm{Spec}\, A$ by a finite group W exists in general, and we have $X/W = \mathrm{Spec}\, A^W$; but it is usually singular.
In our case $V = \mathrm{Spec}\, S$ where S =polynomial functions on $V = SV^*$, the symmetric algebra of the dual V^* of V. If we choose a basis $x_1, \ldots, x_n$ of V^* we have $S = \mathbb{C}[x_1, \ldots, x_n]$.
By a famous theorem of Hilbert and Noether, the polynomial invariants S^W are finitely generated; if $f_1, \ldots, f_r$ generate S^W

- Since W acts degree by degree we may take the f_i homogeneous.
- S^W is of transcendence degree n, thus we may assume $f_1, \ldots, f_n$ are algebraically independent.

Actually S^W is Cohen-Macaulay, *i.e.* it is a free module over $\mathbb{C}[f_1, \ldots, f_n]$.

Fundamental invariants

Let $f_1, \ldots, f_n$ be algebraically independent invariants, of degrees $d_1 \leq d_2 \ldots \leq d_n$.

Theorem (Shephard-Todd, Chevalley, Springer)

We have $|W| \leq d_1 \ldots d_n$, and if $d_1 \ldots d_n$ has been chosen minimal, the following are equivalent

1. $|W| = d_1 \ldots d_n$
2. *W is a complex reflection group.*
3. $S^W = \mathbb{C}[f_1, \ldots, f_n]$.

If these conditions hold and $\mathrm{Ref}(W)$ *is the set of pseudo-reflections of W, then we have* $|\mathrm{Ref}(W)| = \sum_{i=1}^{n}(d_i - 1)$.

The f_i are called the *fundamental invariants* of W; they are not unique but their degrees d_i are; they are called the *reflection degrees* of W.

Proof

We prove everything else assuming $(ii) \Rightarrow (iii)$.
The idea is the compare the growth of the graded algebras S^W and $\mathbb{C}[f_1, \ldots, f_n]$.
If $A = \oplus_i A_i$ is a graded algebra,

- we define $\mathrm{GradedDim}\, A = \sum_{i=0}^{\infty} t^i \dim A_i$
- for $w \in \mathrm{End}\, A$ we define $\mathrm{GradedTrace}(w \mid A) = \sum_i t^i \,\mathrm{Trace}(w \mid A_i)$.

$$\mathrm{GradedDim}(\mathbb{C}[f_1, \ldots, f_n]) = \prod_{i=1}^{n} \frac{1}{1 - t^{d_i}}$$

using that

- $\mathrm{GradedDim}(\mathbb{C}[f_j]) = \sum_i (t^{d_j})^i$ and
- $\mathbb{C}[f_1, \ldots, f_n] = \mathbb{C}[f_1] \otimes \ldots \otimes \mathbb{C}[f_n]$.

Proof (continued)

For $w \in GL(V)$ we have $\mathrm{GradedTrace}(w \mid S) = \frac{1}{\det(1-wt)|V^*}$

We use that

- if $\dim V = 1$ we have $\mathrm{GradedTrace}(w \mid S) = \sum_i (wt)^i = \frac{1}{(1-wt)}$
- then we may assume that w is triangular in the basis $x_1, \ldots, x_n$.

As $P = \frac{1}{|W|}\sum_{w\in W} w$ is the projector on S^W, we have

$$\mathrm{GradedTrace}(P) = \mathrm{GradedDim}(S^W) = \frac{1}{|W|}\sum_{w\in W}\frac{1}{\det(1-wt)}.$$

Note that the two series we converge for $0 \leq t < 1$. We develop them around 1.

Proof (continued)

We find

$$\prod_{i=1}^{n}\frac{1}{1-t^{d_i}} \sim \frac{1}{d_1\ldots d_n}\frac{1}{(1-t)^n} + \frac{\sum_{i=1}^n (d_i - 1)}{2d_1\ldots d_n}\frac{1}{(1-t)^{n-1}} + \text{higher terms.}$$

and

$$\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det(1-wt)} \sim \frac{1}{|W|}\frac{1}{(1-t)^n} + \frac{|\operatorname{Ref}(W)|}{2|W|}\frac{1}{(1-t)^{1-n}} + \text{higher terms,}$$

the second term since

- for a reflection of eigenvalue -1 we have $\frac{1}{1-(-1)t} \to 1/2$ as $t \to 1$,
- and for a non real eigenvalue ζ we have $\frac{1}{1-\zeta t} + \frac{1}{1-\zeta^{-1}t} \to 1$ as $t \to 1$.

End of proof

Comparing we get $|W| \le d_1 \dots d_n$, and if equality then $|\operatorname{Ref}(W)| \ge \sum_i (d_i - 1)$; both are equalities if $\mathbb{C}[f_1, \dots, f_n] = S^W$.

Let $W' = \langle \operatorname{Ref}(W) \rangle$. If we assume $(ii) \Rightarrow (iii)$ we have $S^{W'} = \mathbb{C}[f'_1, \dots, f'_n]$ thus

$$\mathbb{C}[f'_1, \dots, f'_n] = S^{W'} \supset S^W \supset \mathbb{C}[f_1, \dots, f_n].$$

We now use

Lemma

If $\mathbb{C}[f'_1, \dots, f'_n] \supset \mathbb{C}[f_1, \dots, f_n]$ are polynomial algebras where $d_i = \deg f_i$, $d'_i = \deg f'_i$ are increasing, then $d'_i \le d_i$.

Thus $|\operatorname{Ref}(W)'| = \sum_i (d'_i - 1) \le \sum_i (d_i - 1) \le |\operatorname{Ref}(W)|$ so we have equality everywhere so $d_i = d'_i$ and $|W'| = \prod d'_i = \prod d_i \ge |W|$ so $W' = W$.

Coinvariant algebra

Let I be the ideal of S generated by the elements of S^W of strictly positive degree. The *coinvariant algebra* is $S_W := S/I$.
As I is graded, S_W is also. As W is finite, I admits a W-stable complement H, thus isomorphic to S_W as a W-module.

Theorem (Chevalley)

If W is generated by $\operatorname{Ref}(W)$, the natural map $S_W \otimes S^W \to S$ given by the isomorphism $S_W \simeq H$ and multiplication is a graded W-invariant isomorphism; further S_W is isomorphic to $\mathbb{C}[W]$ as a W-module.

The first part implies the second as a consequence of Galois theory, considering the extension $\operatorname{Frac}(S)$ of $\operatorname{Frac}(S^W)$.
It follows from the theorem that S^W is a graded sub-algebra of S such that as an S^W-module S admits a finite homogeneous basis (take a basis of H). Chevalley deduces that S^W itself is a polynomial algebra.

Reflection cosets

We now consider $\phi \in N_{\mathrm{GL}(V)}(W)$ and the coset $W\phi = \phi W$, called a *reflection coset*.
We are interested in eigenspaces of elements of $W\phi$.
For ζ a root of unity we denote by $V_\zeta(w\phi)$ the ζ-eigenspace of $w\phi$.
We assume ϕ semi-simple (this is automatic if W is irreducible on V).

ϕ acts on S^W and

- *We may choose the f_i eigenvectors of ϕ.*
- *If ε_i are the corresponding eigenvalues, the pairs (d_i, ε_i) are unique,*
- *and $\phi \in W \Leftrightarrow \forall i, \varepsilon_i = 1$.*

For the last assertion, if $\varepsilon_i = 1$ for all i then the f_i are algebraically independent invariants of $\langle W, \phi\rangle$ so $|\langle W, \phi\rangle| \le d_1 \dots d_n = |W|$.

Note that $\mathrm{GradedTrace}(\phi \mid S^W) = \prod_i \frac{1}{1-\varepsilon_i t^{d_i}}$.

Theorem of Solomon

Theorem (Solomon)

$(S \otimes \Lambda V)^W \simeq k[f_1, \dots, f_n] \otimes \Lambda[df_1, \dots, df_n]$ where $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ where $dx_1, \dots, dx_n$ denotes a basis of $\Lambda^1 V \simeq V$.

This translates into an identity of bi-graded dimensions, and of bi-graded traces of ϕ:

$$|W|^{-1} \sum_{w \in W} \frac{\det(1 - yw\phi)}{\det(1 - xw\phi)} = \frac{\prod_i (1 - \varepsilon_i y x^{d_i - 1})}{\prod_i (1 - \varepsilon_i x^{d_i})}$$

Expanding both series around $x = \zeta^{-1}$, and setting $T = \frac{y - \zeta^{-1}}{x - \zeta^{-1}}$, we get

Theorem of Pianzola-Weiss

Theorem (Pianzola-Weiss)

$$\sum_{w\in W} T^{\dim(V_\zeta(w\phi))} = \prod_{\{i|\zeta^{d_i}=\varepsilon_i\}} (T+d_i-1) \prod_{\{i|\zeta^{d_i}\neq\varepsilon_i\}} d_i$$

This has the following consequences

- $\max\{\dim V_\zeta(w\phi) \mid w\phi \in W\phi\} = |a(\zeta)|$ where $a(\zeta)=\{i \mid \zeta^{d_i}=\varepsilon_i\}$.
- The exponent of W is $\operatorname{lcm} d_i$.
- If W is irreducible on V, then $|ZW| = \gcd d_i$.

Theorem of Springer-Lehrer

Let U be a ζ-eigenspace of dimension $|a(\zeta)|$ of some element of $W\phi$. Then

Theorem (Springer-Lehrer)

- *Any other $V_\zeta(w\phi)$ of dimension $|a(\zeta)|$ is W-conjugate to U.*
- *$N_W(U)/C_W(U)$ is a complex reflection group on U with degrees $\{d_i \mid i \in a(\zeta)\}$.*
- *The hyperplanes of $N_W(U)/C_W(U)$ are the traces on U of the hyperplanes of W which do not contain U.*

Sketch of proof

$\pi : v \mapsto (f_1(v),\ldots,f_n(v))$ is the orbit map $V \mapsto V/W$. The fibers are the W-orbits. Now $v \in V_\zeta(w\phi)$ iff $\phi(v)$ and ζv are in the same orbit, *i.e.* if for all i we have $f_i(\zeta v) = f_i(\phi v)$ or equivalently $\zeta^{d_i} f_i(v) = \varepsilon_i f_i(v)$, *i.e.* either $f_i(v)=0$ or $i \in a(\zeta)$.

Sketch of proof

We have thus

$$H := \bigcup_{w\in W} V_\zeta(w\phi) = \bigcap_{i\notin a(\zeta)} f_i^{-1}(0).$$

By general properties in algebraic geometry, the dimension of the irreducible components of such an intersection is at least $|a(\zeta)|$, thus the irreducible components have to be the $V_\zeta(w\phi)$ of maximal dimension.

Consider the map $\pi : H \to \mathbb{C}^{|a(\zeta)|}$ given by $v \mapsto \{f_i(v) \mid i \in a(\zeta)\}$.
Then π is surjective since it is the restriction of the orbit map to the inverse image of $\mathbb{C}^{|a(\zeta)|}$. Thus it is closed.
The image of any $|a(\zeta)|$-dimensional eigenspace U is irreducible closed of dimension $|a(\zeta)|$ so it is equal to $\mathbb{C}^{|a(\zeta)|}$.

Sketch of proof (end)

This implies that

- The $\{f_i \mid i \in a(\zeta)\}$ restricted to U are algebraically independent.
- The maximal $V_\zeta(w\phi)$ have the same image by the orbit map so they are conjugate;
- thus $|W/N_W(U)|$ is the number of maximal $V_\zeta(w\phi)$.
- By Pianzola-Weiss this number is $\prod_{i\notin a(\zeta)} d_i$

Finally the $\{f_i \mid i \in a(\zeta)\}$ restricted to U are algebraically independant invariants of the group $N_W(U)/C_W(U)$ acting on U, and $|N_W(U)/C_W(U)| = \prod_{i\in a(\zeta)} d_i$, so by the Shepard-Todd-Chevalley this group is a c.r.g. on U.