Small Boxes for Carpenter's Rules

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We consider "carpenter's rules", i.e., polygonal chains where the edges are considered as "links" which can rotate freely around the vertices. So we can identify a carpenter's rule with the sequence $l_1, ..., l_k$ of lengths of its links.

1 One-dimensional containers

Hopcroft, Joseph, and Whitesides [HJW85] considered the problem of folding a carpenter's rule to a minimal length line segment. They showed that this problem is NP-hard and gave a factor 2 approximation algorithm.

They also observed that any carpenter's rule can be folded to length at most two, so an interval I of length 2 can be considered as a "universal one-dimensional container" into which any carpenter's rule can be folded. In fact, if any initial segment of a carpenter's rule γ has been folded into I and ends at some point $p \in I$, p has distance at least 1 to one of the endpoints of I and the next link of γ can be placed into that direction.

On the other hand, it was shown in [HJW85] that there is no universal interval of length less than 2 (i.e. less than $2 - \varepsilon$ for some $\varepsilon > 0$). In fact, observe the construction in Figure 1, where a carpenter's rule γ of link lengths $1, 1 - \varepsilon, 1, 1 - \varepsilon, ...$ is considered. In order to fold γ into an interval of length less than $2 - \varepsilon$ it is necessary to completely bend back γ at each vertex. But this causes γ to occupy an interval of length $1 + k\varepsilon$ after $2k + 1$ links have been folded. For k large enough the total length is greater than $2 - \varepsilon$.

Calinescu and Dumitrescu [CD05] then found an FPTAS for the problem of finding the minimal folding of a carpenter's rule.

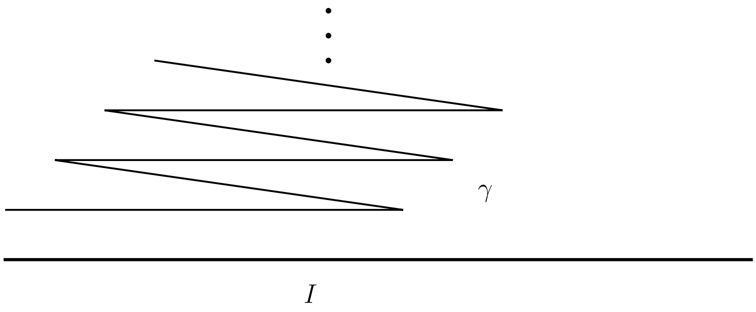

Figure 1: A carpenter's rule not fitting into any interval of length less than $2-\varepsilon$

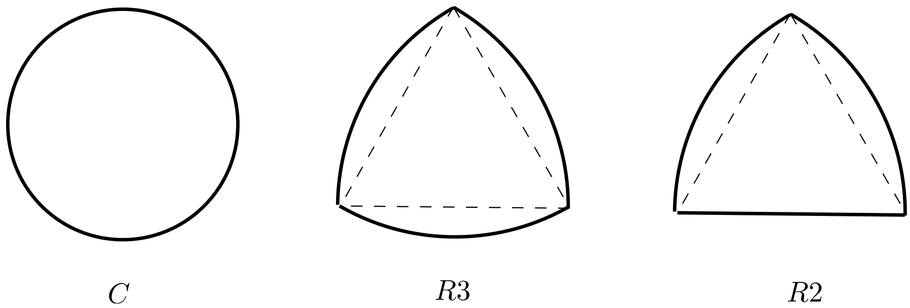

Figure 2: Universal boxes

2 Universal Boxes

Calinescu and Dumitrescu [CD05] then raised the question of a two-dimensional minimum area *universal box* (convex compact set) of width 1 that can contain any chain whose links have length at most 1.

Clearly, the circle C of diameter 1 is a universal box (of area $\pi/4 \approx 0.785$) since from any point on the boundary we can place any segment of length at most 1 so that its second endpoint lies on the boundary again. For the same reason, however, the *Releaux triangle* $R3$ of side length 1, see Figure 2b) is a universal box of area $(\pi - \sqrt(3))/2 \approx 0.704$.

Also, one easily observes that a truncated version $R2$ of the Releaux triangle where one circular arc is replaced by a straight segment, still has this property and is a universal box of area $\pi/3 - \sqrt{3}/4 \approx 0.614$. $R2$ was presented in [CD05] and is the smallest universal box known so far. In [CD05] also a lower bound of $3/8 = 0.375$ for the area of any universal box is shown by presenting a chain of length 3 that cannot be placed into any smaller box.

3 k-universal Boxes

Let us call a convex set a *k-universal box* if its diameter is 1 and any carpenter's rule with k segments can be folded into it. Again, we can ask for the smallest area A_k of a k-universal box for distinct values of k.

Remarks:

a) $A_1 = A_2 = 0$

b) (see [CD05]) $A_3 \geq 3/8$

c) No polygon is 3-universal.

To see remark c) observe that the diameter of a polygon is attained only between two vertices. Therefore, for any given polygon the chain $1, x, 1$ where x is not a distance bewteen two vertices cannot be folded into it.

3.1 A 6-universal Box

The question comes to mind whether the box $R1$, see Figure 3, where two circular arcs of the Releaux triangle are replaced by straight segments is still universal or k universal for some k. $R1$ has area $\pi/6 \approx 0.523$.

In fact, we will show:

a) $R1$ **is 6-universal, so** $A_6 \leq \pi/6 \approx 0.523$**.**

b) $R1$ **is not 7-universal.**

In order to prove these propositions we separate the points of $R1$ into four types (see Figure 3): Type 1 are the two endpoints A, B of the circular arc, type 2 are the points in the interior of the circular arc, type 3 is the common point C of the two straight segments of the boundary, and type 4 are all remaining points.

Proposition a) follows from

Claim: Any chain γ of length 3 can be placed inside $R1$ starting with a point of type 1.

In fact, if the claim holds then we can place any chain of length 6 by placing the second half of the chain from point A and also the first half in reverse order which gives altother a placement of the whole chain.

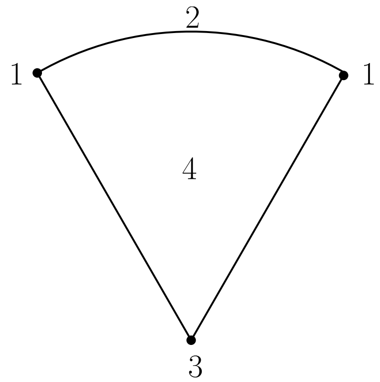

Figure 3: Types of points in $R1$

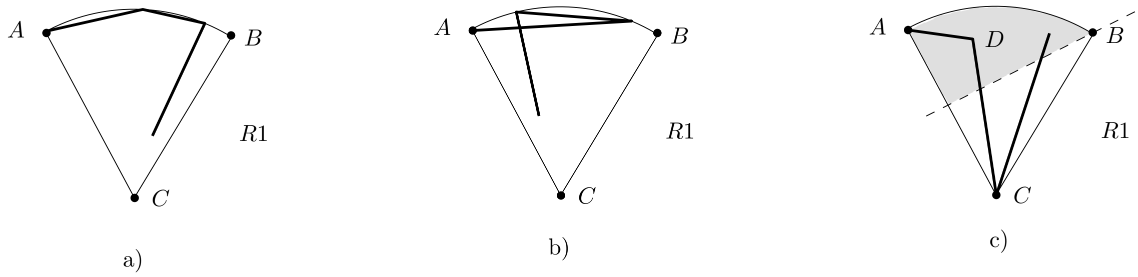

Figure 4: Placing chains of length 3 starting at point A

To prove the claim (wlog for point A), let the consecutive segments of γ have lengths a,b, and c, respectively. We consider three cases:

1.
If $a + b < 1$ (see Figure 4 a)) then we can place the first segment from A to a point of type 2, the second from there to another point of type 2, and the third one from there to some point of type 3 or 4.

2.
If $a > b$ (see Figure 4 b)) we can go from A to a point of type 2, from there to another point of type 2, and from there to type 3 or 4.

3.
The remaining case is that $a + b \geq 1$ and $a \leq b$ (see Figure 4 c)). We place the first two segments between A and C, so that their common endpoint D lies to the right of the line segment $\overline{AC}$. Since it also lies to the left of the bisector of $\overline{AC}$ and inside the circle of radius 1 around C. Consequently, it lies inside the shaded area in Figure 4c). Therefore, D and the whole first two segments lie in $R1$. The third segment can be placed from point C to some point of type 1 or 4.

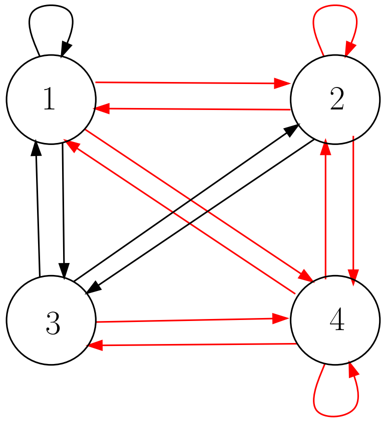

Figure 5: Possible type transitions by placing segments

In order to prove proposition b) let us call a link in a chain "black" if it has length 1 and "red" if its length is less than 1. As easily can be verified, the diagram in Figure 5 shows all possible transitions when placing a link.

More precisely, a black (red) arrow from i to j means that if a black (red) link is placed with one endpoint on a point of type i then the other endpoint could be on a point of type j. Any black-red chain that has no corresponding path in the diagram cannot be folded into $R1$. As easily can be verified there is no path colored "black-red-black-red-black-red-black" in the diagram, therefore, e.g., the chain with lengths 1, 1/2, 1, 1/2, 1, 1/2, 1 cannot be folded into $R1$.

3.2 A 4-universal Box

For chains consisting of fewer than 6 segments we consider the box $R1/2$ which results from $R1$ by replacing half of the circular arc by a straight segment, see Figure 6. The length of this segment is $\alpha = \sqrt{2-\sqrt{3}} \approx 0.518$. The area of $R1/2$ is $\pi/12 + 1/4 \approx 0.512$. Let A, B, C denote the vertices and κ the circular segment as shown in Figure 6. We will show:

- **a) R1/2 is 4-universal, so $A_4 \leq \pi/12 + 1/4 \approx 0.512$.**
- **b) R1/2 is not 5-universal.**

To prove the 4-universality we first observe that it is sufficient to show that sequences of lengths $1, a, b, 1$ with $a, b \in (0, 1]$ can be folded into $R1/2$. In fact, it must be possible to fold any such sequence, and, on the other hand, an arbitrary sequence c, a, b, d can be folded, if the sequence $1, a, b, 1$ can be.

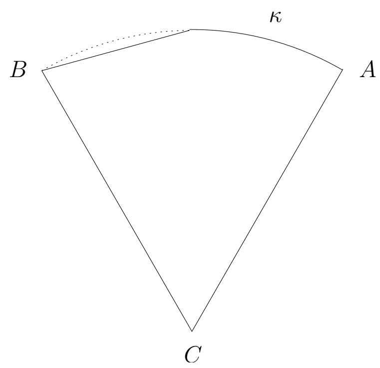

Figure 6: The 4-universal box $R1/2$

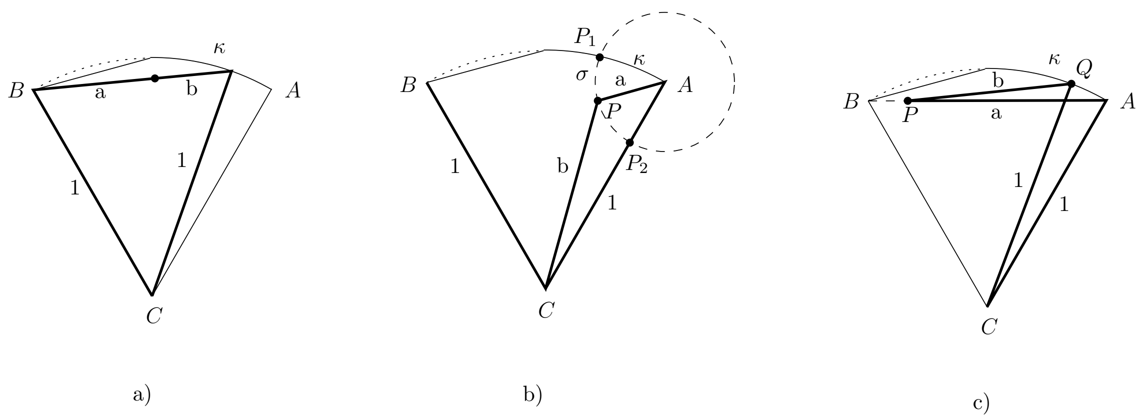

Figure 7: Folding 4-chains into $R1/2$

Then we observe that $R1/2$ is 3-universal. To see this we can argue as before that it suffices to show that any sequence $1, a, 1$ can be folded into $R1/2$. If $a \leq \alpha$ we can do this by going from C to A, from A to some point on κ and from there back to C. If $a > \alpha$, we can go from C to B to some point on κ and back to C.

Returning to length-4 sequences $1, a, b, 1$ we first observe that if $a + b \leq 1$ we can reduce folding the whole sequence to folding the length-3 sequence $1, a+b, 1$ which is possible by the 3-universality of $R1/2$, see Figure 7a). So we may assume that $a + b > 1$.

If one of the segments has length at most α, wlog $a \leq \alpha$, we put the first segment from C to A, see Figure 7b). The endpoint of the second segment then can be placed on the circular arc σ of radius a around A lying inside $R1/2$ whose endpoints are some point on $P_1 \in \kappa$ and a point P_2 on the line segment $\overline{AC}$. Since the distance of P_1 to C is 1 and the distance of P_2 to C is $1 - a < b$ there must be some point P on σ which has distance b to C.

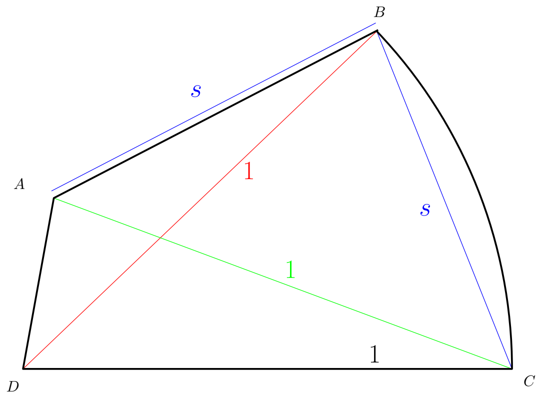

Figure 8: The 4-universal box $S2$

The chain then can be placed from C to A to P to C to A.

The remaining case are sequences $1, a, b, 1$ with $a, b > \alpha$, see Figure 7c). Without loss of generality let us assume that $a \geq b$. We place this sequence by going from C to A, and then from A to the point $P \in \overline{AB}$ that has distance a from A. Observe that the distance from P to the upper endpoint of κ is at most $\alpha < b$ and to A it is $a \geq b$. So there must be some point $Q \in \kappa$ having distance b to P and we can place the chain $C \to A \to P \to Q \to C$.

To see proposition b) consider the chain $1, 1/2, 1, 1/2, 1$. Observe first that the suffix $1/2, 1$ cannot be placed starting from points C. So the middle segment of length 1 must be placed between A and B. So either the sequence $1/2, 1$ would have to start from B or the sequence $1, 1/2$ would have to end in B (which is equivalent). This is impossible.

3.3 A better 4-universal box

Consider the box $S2$ in Figure 8. Observe that the conditions that $\overline{DC} = \overline{DB} = \overline{AC} = 1$ and $\overline{AB} = \overline{BC}$ do not identify the figure uniquely, there is still one degree of freedom. However, as is shown below each such box is 4-universal. $S2$ is the one with smallest area which is approximately 0.485.

a) $S2$ **is 4-universal, so** $A_4 < 0.486$**.**

b) $S2$ **is not 5-universal.**

Proof.

We first show that $S2$ is 3-universal. As we saw before it suffices to show that any chain of lengths $1, a, 1$ can be placed inside $S2$. If $a \leq s$ this can be

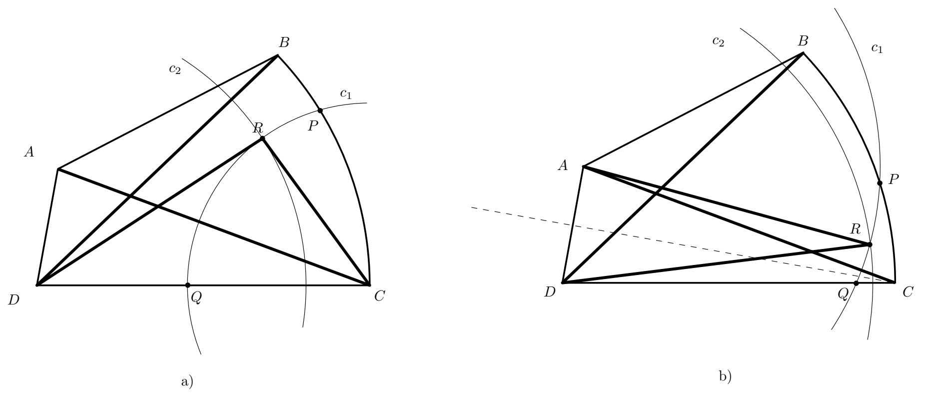

Figure 9: Placing a chain $1, a, b, 1$ inside $S2$ a) if $a \leq s$ b) if $a > s$

done by placing the middle link on the circular arc $\overset{\frown}{BC}$ and the outer links between $\overset{\frown}{BC}$ and D. If $a > s$, then there is some point P on $\overset{\frown}{BC}$ which has distance a from A and we can go from A to P to D.

For the 4-universality it suffices to show that chains of lengths $1, a, b, 1$ can be placed inside $S2$. Again we can assume that $a + b > 1$ since otherwise, because of the 3-universality of $S2$ we can place the chain by keeping the two middle links stretched. We will also assume wlog that $a \leq b$.

If $a \leq s$ (see Figure 9 a)) the circle c_1 around C of radius a intersects the arc $\overset{\frown}{BC}$ at some point P which has distance $1(\geq b)$ from D. On the other hand c_1 intersects the line segment $\overline{DC}$ in a point Q that has, because of $a+b > 1$, distance $< b$ from D. Consequently the circle c_2 of radius b around D must intersect c_1 in some point R between P and Q, i.e., inside $S2$. The chain can then be placed with its vertices at, say, A, C, R, D, B.

If $b \geq a > s$ (see Figure 9 b)) the circle c_1 of radius a around A intersects $\overset{\frown}{BC}$ in some point P and $\overline{DC}$ in some point Q. Since the bisector between A and D passes through C, Q is closer to D than to A, i.e., the distance of D to Q is $< a \leq b$. Therefore, the circle c_2 of radius b around D must intersect c_1 in some point R between P and Q, i.e., inside $S2$. The chain can then be placed with its vertices at, say, C, A, R, D, B.

To see that $S2$ is not 5-universal, consider any chain $1, a, 1, b, 1$ with $s < a, b < 1$. The segments of length 1 can only be placed between two points from A, D or on $\overset{\frown}{BC}$, let us call them *width points.* Therefore all vertices of the chain must be placed on width points. The only width points of distance a with $s < a < 1$ can be A and some interior point of $\overset{\frown}{BC}$. Therefore, the only way to place the prefix $1, a, 1$ is between C and D. Since from neither one a width point lies at distance b, the chain cannot be placed.

4 Open Problems

- Find a 5-universal box smaller than $R1$.
- Find a 3-universal box smaller than $S2$ trying to get closer to the lower bound. For 3-universality there are small improvements of $S2$ possible (maybe for 4-universality, as well).
- Find better lower bounds for 4-universal, 5-universal, 6-universal and universal boxes. The bound 3/8 is based on one single 3-chain.
- Find an algorithm to find a minimum area box for a given chain $l_1, ..., l_k$. It seems it is possible by solving exponentially many quadratic programs. Is the 2-d problem also NP-hard?

References

[CD05] G. Calinescu and A. Dumitrescu. The carpenter's ruler folding problem. In J. Goodman, J. Pach, and E. Welzl, editors, *Combinatorial and Computational Geometry*, Mathematical Sciences Research Institute Publications, pages 155–166. Cambridge University Press, 2005.

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