NIGEL KALTON'S WORK IN ISOMETRICAL BANACH SPACE THEORY

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1. INTRODUCTION

Nigel Kalton was one of the greatest mathematicians of the last 40 years, although he did his best to conceal this fact. An outsider wouldn't recognise the mathematical giant that he was in this modest person who was always friendly and good-humoured and who was more than willing to share his ideas with everyone.

Nigel published more than 260 papers (including several books) not only in Banach and quasi-Banach space theory, but also in so diverse fields such as game theory, continued fractions, harmonic analysis, operator semigroups and convex geometry. Every single of these papers contains a deep contribution by Nigel, often taking care of the most difficult case that his coauthors would have to leave open without his help. He had a wide interest in mathematics, and his problem solving abilities were legendary. For instance, once after a colloquium talk on continued fractions he got hooked on the subject and redeveloped the theory for himself over one weekend, eventually solving the problem exposed in the talk.

I met Nigel for the first time at the conference on Banach spaces in Mons in 1987. It so happened that on the day after the conference we were both waiting for the same train to Paris, but not for the same coach: he told me that he always rides the first class, adding, "I'm snobbish." Of course he couldn't be more wrong in his self-assessment! Some years later he solved a big problem in M-ideal theory (see Section 2), a problem, where we, a group of fresh Ph.D.s in Berlin, couldn't get anywhere. He emailed me a file with his solution, and this was the beginning of our collaboration, in which more often than not I felt like a pedestrian next to a racing-car. In June 2010, after a talk at the conference in Valencia with a somewhat set-theoretical flavour, I reminded him of the quote from Star Trek, "It's mathematics, but not as we know it." I knew that this would strike his sense of humour; I did not know that this would be the last time I saw him.

In the next few sections I will try to survey some of Nigel's contributions to Banach space theory. I will restrict myself to problems of an isometric nature, but even this narrower area is still so rich that omissions and misconceptions will be inevitable. Certainly, the only way to do justice to Nigel's genius would be to not only paraphrase the main results, but to expound all the ideas contained in his papers. I have to leave this to an abler mathematician.

In the following, papers by Nigel will be cited in the form [K1] and other papers in the form [25]. The bibliography will first list Nigel's cited papers chronologically, and then the other papers alphabetically.

2. M-IDEALS

An M-*ideal* V of a Banach space E is a closed subspace such that the dual admits an ℓ_1-direct decomposition $E^* = V^\perp \oplus_1 W$ for some closed subspace $W \subset E^*$. (In other words, $V^\perp$ is an L-summand of E^*.) The notion of an M-ideal was introduced by E.M. Alfsen and E.G. Effros [1]; for a detailed study one may consult [10]. Of course, c_0 is an M-ideal in ℓ_∞, and Dixmier proved back in 1951 that $K(H)$, the space of compact operators on a Hilbert space H, is an M-ideal in $L(H)$, the space of bounded operators. By the end of the 1980s more examples of Banach spaces X for which $K(X)$ is an M-ideal in $L(X)$ were known; basically, these examples were ℓ_p-sums ($p > 1$) or c_0-sums of finite-dimensional spaces and certain of their subspaces and quotients. On the other hand, several necessary conditions were known: X has to be an M-embedded space (i.e., X is an M-ideal when canonically embedded into X^{**}), which implies for example that X^* has the RNP, and X must have the metric compact approximation property; this was proved by P. Harmand and Å. Lima [9]. For subspaces of $X \subset \ell_p$ the converse was proved by C.-M. Cho and W.B. Johnson [4]: the metric compact approximation property is sufficient for $K(X)$ to be an M-ideal in $L(X)$. (Later Nigel obtained this in complete generality, see Corollary 2.2.)

Let us recall what this approximation property means. A Banach space X has the *metric compact approximation property* if there is a net of compact operators $K_i\colon X \to X$ of norm ≤ 1 that converges pointwise to the identity: $K_i x \to x$ for all $x \in X$. Actually, an even stronger approximation property holds if $K(X)$ is an M-ideal in $L(X)$: one can achieve that $\limsup \|\mathrm{Id} - 2K_i\| \leq 1$ and both $K_i \to \mathrm{Id}$ and $K_i^* \to \mathrm{Id}$ pointwise (*shrinking unconditional metric compact approximation property*). Here the *shrinking* bit derives from the fact that also $K_i^* x^* \to x^*$ for all x^*, like for the projections associated with a shrinking basis, and the *unconditionality* is hidden in the norm condition $\limsup \|\mathrm{Id} - 2K_i\| \leq 1$; see the beginning of Section 4.

Although the M-ideal problem for $K(X)$ was intensively studied in the 1980s, there were no conditions on X known that were necessary and sufficient for $K(X)$ to be an M-ideal in $L(X)$; the best result by then was given by W. Werner [36]: $K(X)$ is an M-ideal in $L(X)$ if and only if X has the metric compact approximation property by means of a net (K_i) satisfying

$$\limsup \|SK_i + T(\mathrm{Id} - K_i)\| \leq \max\{\|S\|, \|T\|\} \qquad \forall S, T \in L(X).$$

This was a big achievement, but still the condition is so complicated that one cannot check easily that it is fulfilled for $X = \ell_2$.

This was the moment when Nigel got interested in the problem, probably after some eventually successful bugging by Gilles Godefroy. (It should be added that in the 1980s people working on M-ideals of compact operators often used techniques from Nigel's much quoted paper [K1].[1]) In his paper

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[1]It got quoted 47 times according to the Mathematical Reviews database; but Nigel once told me that it was accepted for publication only at the third attempt.

[K6] he offered an entirely new approach based on what he called property (M) and property (M^*). Here are the definitions. A Banach space X has *property* (M) if whenever (x_i) is a bounded weakly null net and $x, y \in X$ satisfy $\|x\| = \|y\|$, then

$$\limsup \|x_i + x\| = \limsup \|x_i + y\|, \tag{1}$$

and X has *property* (M^*) if whenever (x_i^*) is a bounded weak* null net and $x^*, y^* \in X^*$ satisfy $\|x^*\| = \|y^*\|$, then

$$\limsup \|x_i^* + x^*\| = \limsup \|x_i^* + y^*\|. \tag{2}$$

These notions can be recast in the language of types on Banach spaces; see Section 6 below.

It is easy to see that (M^*) implies (M); for the converse see the remarks following Theorem 2.3.

Nigel's theorem is as follows.

Theorem 2.1. *The following assertions about a Banach space X are equivalent:*

  (i) $K(X)$ *is an M-ideal in $L(X)$.*
 (ii) X *has property (M), does not contain a copy of ℓ_1 and has the unconditional metric compact approximation property, i.e., there is a net of compact operators satisfying $K_i x \to x$ for all x and $\limsup \|\mathrm{Id} - 2K_i\| \le 1$.*
(iii) X *has property (M^*) and has the unconditional metric compact approximation property.*

Actually, in his paper Nigel only deals with the case of separable spaces and the sequential versions of (M) and (M^*), but the extension to the general case doesn't offer any difficulties; it can be found in [10, page 299] for example. One should remark that the sequential property (M) is trivially satisfied in Schur spaces (where by definition weakly convergent sequences are norm convergent), but these are excluded by the requirement that ℓ_1 does not embed into X in Theorem 2.1.

Let me point out that it is trivial to verify that the ℓ_p-spaces for $1 < p < \infty$ and in particular Hilbert spaces satisfy Nigel's conditions; to see that ℓ_p has (M) just note that for ℓ_p we have

$$\limsup \|x + x_i\|^p = \|x\|^p + \limsup \|x_i\|^p \tag{3}$$

so that (1) becomes obvious.

Using property (M) Nigel has been able to solve a number of open problems in the theory of M-ideals. For example, he proved a general version of the theorem of Cho and Johnson mentioned above:

Corollary 2.2. *If $K(X)$ is an M-ideal in $L(X)$ and $E \subset X$ has the compact metric approximation property, then $K(E)$ is an M-ideal in $L(E)$ as well.*

He also showed that Orlicz sequence spaces and more generally modular sequence spaces can be renormed to have property (M); if in addition such a space X has a separable dual, then X can be renormed so that $K(X)$ becomes an M-ideal in $L(X)$. In the other direction, a separable non-atomic

order continuous Banach lattice can be renormed to have property (M) if and only if it is lattice isomorphic to L_2.

Another interesting corollary is that spaces X with property (M) contain subspaces isomorphic to ℓ_p; more precisely, there exists $1 \le p < \infty$ so that ℓ_p embeds almost isometrically into X (cf. Section 3 for this concept) or c_0 embeds almost isometrically into X. The proof relies on a deep theorem due to J.-L. Krivine [21]; a particular consequence is the theorem, originally obtained by J. Lindenstrauss and L. Tzafriri, that an ifinite-dimensional subspace of an Orlicz sequence space h_M contains a copy of some ℓ_p or of c_0.

Concerning $L_p = L_p[0,1]$ it was known by 1990 that $K(L_p)$ is not an M-ideal in $L(L_p)$ for $p \ne 2$, and conversely that an M-ideal in $L(L_p)$ is necessarily a two-sided ideal for $1 < p < \infty$; indeed this is so since L_p and its dual are uniformly convex [5]. But in our joint paper [K10] it was shown that, for $p \ne 1, 2, \infty$, there are no nontrivial M-ideals in $L(L_p)$ whatsoever (nontrivial meaning different from $\{0\}$ and the whole space), and if $1 < p, q < \infty$ then $K(\ell_p(\ell_q^n))$ is the only nontrivial M-ideal in $L(\ell_p(\ell_q^n))$. In both these results we had to assume complex scalars; the proofs use arguments involving hermitian operators.

In the paper [K14], with coauthors G. Androulakis and C.D. Cazacu, Nigel takes his construction of spaces with property (M) still further in that he considers Fenchel-Orlicz spaces [33]. The definition of these spaces is similar to that of Orlicz sequence spaces, but they are built on a Young function on $\mathbb{R}^n$ rather than $\mathbb{R}$ and consist of vector-valued sequences. Now Nigel and his coauthors proved that Fenchel-Orlicz spaces can be renormed to have propert (M) and that many interesting Banach spaces have a represention as a Fenchel-Orlicz space. This is in particular so for the "twisted sums" Z_p, $1 < p < \infty$, from [K3]. These are "extreme" counterexamples to the three-space problem for ℓ_p: Z_p is not isomorphic to ℓ_p, yet contains a subspace Y_p isomorphic to ℓ_p such that Z_p/Y_p is isomorphic to ℓ_p as well. In the language of homological algebra, Z_p is a nontrivial twisted sum of ℓ_p with itself, i.e., there is a short exact sequence $0 \to \ell_p \to Z_p \to \ell_p \to 0$ that does not split. Nigel has contributed a lot to twisted sums, but this is another story.

In a subsequent publication [K9] Nigel pursued an idea mentioned at the end of [K6], namely to decide whether the unconditionality assumption, i.e., $\limsup \|\mathrm{Id} - 2K_i\| \le 1$, in Theorem 2.1 is actually needed. It turns out that this is not so.

Theorem 2.3. *For a separable Banach space X, $K(X)$ is an M-ideal in $L(X)$ if and only if X has property (M), does not contain a copy of ℓ_1 and has the metric compact approximation property.*

The proof of the if-part consists of two steps: first to show that X has property (M^*) as well, which is much more difficult than the implication $(M^*) \Rightarrow (M)$, and then to construct, using property (M^*), from a compact approximation of the identity satisfying $\limsup \|K_n\| \le 1$ another compact approximation of the identity satisfying $\limsup \|\mathrm{Id} - 2L_n\| \le 1$. This is done by a skipped blocking decomposition argument. Meanwhile other and simpler arguments for the second step have been given by Å. Lima [23], E. Oja [28] and O. Nygaard and M. Põldvere [26].

This theorem was proved while I was a visitor at the University of Missouri in 1993. Let me commit myself to some personal recollections at this stage. The day I arrived, Nigel asked me what I was working on. One of the questions had to do with Banach spaces X for which $K(X \oplus_p X)$ is an M-ideal in $L(X \oplus_p X)$, like $X = \ell_p$. The conjecture was that such a space should be similar to ℓ_p, more precisely such an X should embed almost isometrically into an ℓ_p-sum of finite-dimesional spaces. In fact, in the paper [34] I had previously formulated the bold conjecture that all spaces for which $K(X)$ is an M-ideal in $L(X)$ are stable in the sense of Krivine and Maurey [22] and that one should be able to deduce from that that $K(X \oplus_p X)$ is an M-ideal in $L(X \oplus_p X)$ for some p. The first half was disproved by Nigel in [K6] whereas he did prove the second part to be correct. Almost on the spot he suggested an idea how to tackle the problem. Of course, it took me some time to digest it, and after a week or so I understood what he had in mind. I then suggested to use an ultrapower argument at some stage of the proof, upon which Nigel said, "Oh, I missed that point!" – only to come up with a much better idea that eventually solved the problem. I also pointed out a relation to work by Bill Johnson and Morry Zippin, and Nigel asked, "Can they do it without the approximation property?", which was the case, and he added, "Then we can do without the approximation property too!" I'll describe the outcome in the next section.

3. Almost isometric embeddings

Let us start with some vocabulary. We say that a (separable) Banach space X has property (m_p) if, whenever $x_n \to 0$ weakly,

$$\limsup \|x + x_n\|^p = \|x\|^p + \limsup \|x_n\|^p \qquad \forall x \in X$$

if $p < \infty$, resp.

$$\limsup \|x + x_n\| = \max\{\|x\|, \limsup \|x_n\|\} \qquad \forall x \in X$$

for $p = \infty$. It is clear that ℓ_p has (m_p) for $p < \infty$ (cf. (3) above) and that c_0 has (m_∞). If $K(X \oplus_p X)$ is an M-ideal, then X has (m_p), as proved by Nigel [K6].

The Johnson-Zippin space C_p is an ℓ_p-sum of a sequence of finite-dimensional spaces $E_1, E_2, \ldots$ that are dense in all finite-dimensional spaces with respect to the Banach-Mazur distance. A Banach space X embeds almost isometrically into Y if for each $\varepsilon > 0$ there is a subspace $X_\varepsilon \subset Y$ such that $d(X, X_\varepsilon) \leq 1 + \varepsilon$, $d(X, X_\varepsilon)$ denoting the Banach-Mazur distance. We use a similar definition for X to be almost isometric to a quotient of Y. Note that any two versions of C_p (built on different E_k) embed almost isometrically into each other.

In [K9], the following result is proved.

Theorem 3.1. *Suppose X is a separable Banach space not containing ℓ_1. Let $1 < p < \infty$. Then X has (m_p) if and only if X embeds almost isometrically into C_p. Likewise, X has (m_∞) if and only if X embeds almost isometrically into c_0.*

The proof uses again a skipped blocking decomposition technique.

Because of the duality of the property (m_p), one obtains the following corollary.

Corollary 3.2. *Let $1 < p < \infty$. If $X \subset C_p$, then X is almost isometric to a quotient of C_p, and if $X = C_p/Z$, then X is almost isometric to a subspace of C_p.*

This is an almost isometric refinement of a result due to Johnson and Zippin who proved the corresponding isomorphic result [15]. On the other hand, isomorphic versions of Theorem 3.1 using tree conditions were later obtained by Nigel for $p = \infty$ [K19] and E. Odell and Th. Schlumprecht for $p < \infty$ [27].

In the context of L_p-spaces more can be proved. First of all, if $X \subset L_p = L_p[0, 1]$ has property (M), then it has (m_r) for some r; if $1 < p \le 2$, then $p \le r \le 2$, and if $2 < p < \infty$, then $r = 2$ or $r = p$. We now have [K9]:

Theorem 3.3. *Suppose $1 < p < \infty$, $p \ne 2$, and let $X \subset L_p$ be infinite-dimensional. Then the following are equivalent:*
  (i) *B_X, the unit ball of X, is compact in L_1 (i.e., with respect to the topology inherited from L_1).*
  (ii) *X has property (m_p).*
  (iii) *X embeds almost isometrically into ℓ_p.*
If $p > 2$, then (i)–(iii) are equivalent to each of the following:
  (iv) *X is isomorphic to a subspace of ℓ_p.*
  (v) *X does not contain a copy of ℓ_2.*

Our proof was (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) for $p > 2$, where (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is trivial, as is (iii) $\Rightarrow$ (ii). Originally the equivalence of (iv) and (v) for $p > 2$ is due to Bill Johnson and Ted Odell [12], see also [11].

Concrete examples of subspaces of L_p with (m_p) are the Bergman spaces consisting of all analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ for which $\int_{\mathbb{D}} |f(x + iy)|^p \, dx \, dy < \infty$. Likewise, the "little" Bloch space has (m_∞). In [K22], Nigel proved that also the "little" Lipschitz space has (m_∞), thus showing that it is an M-ideal in its bidual, a problem left open in [3].

Results concerning isometric embeddings of subspaces of L_p into ℓ_p were proved by F. Delbaen, H. Jarchow and A. Pełczyński [6]; as is often the case in isometric considerations in L_p, one has to distinguish whether or not p is an even integer.

In a conversation in 1998, Nigel once suggested to prove a result similar to Theorem 3.1 for property (M). His idea was to show that such spaces embed into Fenchel-Orlicz spaces. I am sure that he knew an outline of the argument, but as far as I know the proof has never been written down.

There is also an isomorphic version of Theorem 3.1 for $p = \infty$, devised by Nigel and his coauthors G. Godefroy and G. Lancien [K17]. The main result of their paper is that the Banach space c_0 is determined by its metric, that is:

Theorem 3.4. *If a Banach space X is Lipschitz isomorphic to c_0, that is, if there is a bijective map $T: X \to c_0$ with T and T^{-1} Lipschitz, then X is linearly isomorphic to c_0, that is, T can be chosen linear.*

To prove this, they first show that X embeds isomorphically into c_0, for which an isomorphic version of property (m_∞) and of Theorem 3.1 is needed. To conclude the proof of Theorem 3.4 one has to appeal to known properties of subspaces of c_0. Theorem 3.4 is one of the most remarkable achievements in the nonlinear theory of Banach spaces.

In their paper [K12], Nigel together with G. Godefroy and D. Li addressed the problem of extending results like Theorem 3.3 to the case of subspaces of L_1. There are two intrinsic difficulties that do not occur in the case $p > 1$. For one thing, the Haar basis is unconditional in L_p for $p > 1$, but not in L_1, and this was an essential ingredient in the proof of Theorem 3.3. Also, the L_1-topology on the unit ball of a subspace $X \subset L_p$ is certainly locally convex, whereas the L_r-spaces for $r < 1$ are not locally convex and hence the L_r-topology on the unit ball of a subspace $X \subset L_1$ need not be locally convex. (Here we enter the world of quasi-Banach spaces, one of Nigel's favourite areas.) So in order to be able to study when subspaces of L_1 embed into ℓ_1, one has to assume some unconditionality.

A separable Banach space has the *unconditional metric approximation property* (UMAP for short) if there is a sequence of finite rank operators such that $F_n x \to x$ for all x and $\|\mathrm{Id} - 2F_n\| \to 1$; the latter obviously implies that $\|F_n\| \to 1$ as well. We have already encountered a variant in Theorem 2.1; the definition of UMAP is due to Nigel and Pete Casazza [K5]. We shall have more to say on this in Section 4.

In the next theorem, one of the main results from [K12], τ_m denotes the topology of convergence in measure, i.e., the topology of the F-space L_0.

Theorem 3.5. *Let X be a subspace of L_1 with the approximation property. The following statements are equivalent.*
  (i) *X has the UMAP, and B_X is relatively compact for the topology τ_m.*
  (ii) *B_X is τ_m-compact and τ_m-locally convex.*
  (iii) *For any $\varepsilon > 0$, there exists a weak* closed subspace X_ε of ℓ_1 with Banach-Mazur distance $d(X, X_\varepsilon) \le 1 + \varepsilon$.*

By a result of Rosenthal, B_X is τ_m-relatively compact if and only if X fulfills a strong quantitative version of the Schur property, called the 1-strong Schur property.

The paper [K12] also contains a very interesting counterexample as to possible generalisations of the previous theorem.

Theorem 3.6. *There exists a subspace X of L_1 with the approximation property, whose unit ball is τ_m-compact but not τ_m-locally convex. In particular, X fails the UMAP.*

The construction of this space would not have been possible without Nigel's insight into the nature of L_p-spaces for $0 \le p < 1$—for most of us a no-go area—in particular the strange world of needle points and the failure of the Krein-Milman theorem there, cf. [30].

Another type of embedding result is contained in Nigel's work with Alex Koldobsky [K24]; it concerns subspaces of the quasi-Banach space L_p for $p < 1$. It is known that a Banach space X embeds into some L_p, $p < 1$, isomorphically if and only if X embeds into all L_r, $0 < r < 1$, isomorphically. (The embedding into L_r for $r \le p$ is clear since L_p embeds into those L_r

isometrically; the issue is the range of r between p and 1.) Nigel proved in [K4] that embedding into L_1 is equivalent to the embeddability of $\ell_1(X)$ into L_p. As for the corresponding isometric question, A. Koldobsky [20] showed that there is a finite-dimensional Banach space that embeds isometrically into $L_{1/2}$, but not into L_1. Using stable random variables, Nigel and Alex Koldobsky obtain a vast generalisation.

Theorem 3.7. *Let $0 < p < 1$; then there is an infinite-dimensional Banach space E_p that embeds isometrically into L_p, but not into any L_r for $p < r \leq 1$.*

The space E_p has a representation as $\ell_1 \oplus_{N_p} \mathbb{R}$ by means of some very cleverly chosen absolute norm N_p on $\mathbb{R}^2$, and the embedding of E_p into L_p is realised by means of a sequence of independent random variables having a 1-stable (i.e., Cauchy) distribution. In addition to this example E_p, a second example is constructed that is isomorphic to a Hilbert space, viz. $F_p = \ell_2 \oplus_{\tilde{N}_p} \ell_2$.

4. Unconditionality

We have already mentioned the notion of unconditional metric approximation property (UMAP) on page 7. The UMAP was introduced in the paper [K5] by Nigel and Pete Casazza. Let us explain what "unconditional" refers to here. Following Nigel and Pete one can obtain from an approximating sequence (F_n) with $\|\mathrm{Id} - 2F_n\| \to 1$, for a given $\varepsilon > 0$, another approximating sequence (F'_n) such that for $A_n = F'_n - F'_{n-1}$ and all N

$$\left\| \sum_{n=1}^{N} \varepsilon_n A_n \right\| \leq 1 + \varepsilon$$

whenever $\varepsilon_n = \pm 1$; this should be compared to the estimate

$$\left\| \sum_{n=1}^{N} \varepsilon_n e_n^*(x) e_n \right\| \leq (1 + \varepsilon)\|x\|$$

for $(1 + \varepsilon)$-unconditional bases. Hence the epithet "unconditional." Replacing finite rank operator by compact operators in the approximating sequence, one arrives at the notion of unconditional metric compact approximation property (UMCAP). Apart form studying these properties, [K5] also contains the proof of the following stunning result concerning the ordinary metric approximation property (MAP).

Theorem 4.1. *If a separable Banach space has the MAP, then it even has the commuting MAP, meaning that there are commuting finite rank operators with $\|T_n\| \leq 1$ and $T_n x \to x$ for all x.*

For reflexive spaces, the same type of conclusion is proved for the UMAP in [K5] as well, but in a paper by Nigel and G. Godefroy [K13], the result was proved in full generality.

Theorem 4.2. *If a separable Banach space has the UMAP, then it even has the commuting UMAP.*

The key to the proof is to look at some norm-1 approximating sequence (F_n) and to study the limiting projection

$$Px^{**} = w^{*}\text{-}\lim_{n\to\infty} F_n^{**}x^{**}$$

in the bidual and to prove that its range is weak* closed. The paper [K13] also contains an embedding result that is similar in spirit to Theorem 3.1 and Theorem 3.3:

Theorem 4.3. *A Banach space X has the UMAP if and only if, for each $\varepsilon > 0$, X embeds isometrically as a $(1 + \varepsilon)$-complemented subspace into a space with a $(1+\varepsilon)$-unconditional Schauder basis.*

Another outgrowth of [K5] is the concept of a u-ideal that is studied in detail in the influential paper [K7] by Nigel, G. Godefroy and P. Saphar. Let $X \subset Y$ be Banach spaces. X is said to be a *u-summand* in Y if there is a projection P from Y onto X with $\|\mathrm{Id} - 2P\| = 1$; equivalently, one may decompose $Y = X \oplus X_s$ in such a way that $\|x + x_s\| = \|x - x_s\|$ whenever $x \in X$, $x_s \in X_s$. An easy example of a u-decomposition is the decomposition of $f \in C[-1, 1]$ into its even and odd part. An important special case is when $Y = X^{**}$; for example, the bidual of L_1 admits such a decomposition, which is even ℓ_1-direct: $(L_1)^{**} = L_1 \oplus_1 (L_1)_s$, the so-called Yosida-Hewitt decomposition. (In technical terms, L_1 is an example of an L-embedded space; see Chapter IV in [10].) Likewise, X is a *u-ideal* in Y if $X^\perp$, its annihilator, is a u-summand in Y^*. By definition, every M-ideal is a u-ideal, but also every order ideal in a Banach lattice is a u-ideal. When working with complex scalars, it is more appropriate to replace the condition $\|\mathrm{Id} - 2P\| = 1$ by $\|\mathrm{Id} - (1 + \lambda)P\| = 1$ for all scalars of modulus 1; correspondingly, one then speaks of *h-summands* and *h-ideals*. In these definitions, u stands for "unconditional" and h for "hermitian."

The extensive paper [K7] contains a wealth of information concerning u-ideals and h-ideals. I will mention only a few aspects. First of all, there are certain similarities to M-ideals, for example, the u-projection is uniquely determined, and a u-ideal that does not contain a copy of c_0 is a u-summand. On the other hand, if X is a u-ideal in its bidual, then the u-projection need not be the canonical projection from the decomposition $X^{***} = X^\perp \oplus X^*$, as for M-embedded spaces. (An example is $X = L^1$.) Let us say that X is a strict u-ideal (in its bidual) in this case.

One of the main results in [K7] characterises strict u-ideals by means of a quantitative version of Pełczyński's property (u). To explain this, some notation is needed. Let $x^{**} \in X^{**}$ be such that there is a sequence in X converging to x^{**} in the weak* topology of X^{**}. Define

$$\kappa_u(x^{**}) = \inf\left\{\sup_n\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|: \varepsilon_k = \pm 1,\ x_k \in X,\ x^{**} = w^{*}\text{-}\sum_{k=1}^\infty x_k\right\}$$

and let $\kappa_u(X)$ be the smallest constant such that $\kappa_u(x^{**}) \leq C\|x^{**}\|$ for all x^{**} in the sequential closure of X in (X^{**}, w^*). κ_h is defined in the same way, replacing $\varepsilon_k = \pm 1$ with $|\varepsilon_k| = 1$, $\varepsilon_k \in \mathbb{C}$. The Banach space X has property (u) if $\kappa_u(X) < \infty$.

Theorem 4.4. *Suppose X does not contain a copy of ℓ_1. Then X is a strict u-ideal in X^{**} if and only if $\kappa_u(X) = 1$.*

As for spaces of operators, Nigel and his collaborators obtain the following results.

Theorem 4.5. *Let X be a separable reflexive Banach space. Then X has the UMCAP if and only if $K(X)$ is a u-ideal in $L(X)$.*

Generally, the results for h-ideals are more satisfactory and precise, since one can apply the powerful machinery of hermitian operators; for example (the notion of complex UMCAP should be self-explanatory):

Theorem 4.6. *Let X be a complex Banach space with a separable dual. Then X has the complex UMCAP if and only if $K(X)$ is an h-ideal in $L(X)$ and X is an h-ideal in X^{**}.*

Theorem 4.7. *Let X be a separable complex Banach space with the MCAP. Then $K(X)$ is an h-ideal in its bidual $K(X)^{**}$ if and only if X is an M-ideal in X^{**} and has the complex UMCAP.*

In their paper [K30] (with S.R. Cowell) Nigel takes up the question of embedding into a space with an unconditional basis, as in Theorem 4.3, but without the assumption of the approximation property. Such a result was given by W.B. Johnson and B. Zheng [13], but now the aim is to find an (almost) isometric version. The key to this are the asymptotic unconditionality properties (au) and (au^*). A separable Banach space X has *property (au)* if

$$\lim(\|x + x_d\| - \|x - x_d\|) = 0$$

whenever (x_d) is a bounded weakly null net; in [K19] Nigel referred to this property as "X is of unconditional type." If X^* is separable, it is equivalent to use weakly null sequences instead, and one obtains a notion that has been known by the acronym WORTH in the literature. Likewise, X has *property (au^*)* (previously "X is of shrinking unconditional type") if

$$\lim(\|x^* + x_n^*\| - \|x^* - x_n^*\|) = 0$$

whenever (x_n^*) is a (necessarily bounded) weak* null sequence; due to the weak* metrisability of the unit ball there is no need to consider nets here. These notions are reminiscent of the properties (M), (M^*) and (m_p), and (au^*) has also been considered by Å. Lima [23] under the name (wM^*). In general (au^*) implies (au), and the converse is true under additional hypotheses. The main result of [K30] says the following.

Theorem 4.8. *A separable Banach space X has property (au^*) if and only if X embeds almost isometrically into a space Y with a shrinking 1-unconditional basis; if X is reflexive, Y can be taken to be reflexive as well.*

I will briefly mention other contributions by Nigel on the topic of unconditional bases, Schauder decompositions and expansions. In a series of papers with P. Casazza or F. Albiac and C. Leránoz, Nigel investigated uniqueness of unconditional bases in Banach or quasi-Banach spaces ([K11], [K15], [K16], [K20], [K23]). For example, in [K16] it is proved that although the Tsirelson space T admits a unique unconditional basis, this is not so

for $c_0(T)$. In the paper [K25] with A. Defant the question of existence of an unconditional basis in the space $P(^mE)$ of m-homogeneous on a Banach space E is considered. Sean Dineen had conjectured that $P(^mE)$ has an unconditional basis if and only if E is finite-dimensional. This conjecture is vindicated in [K25]; the proof uses local Banach space theory and some greedy basis theory.

See also Theorem 5.8 below for unconditional expansions in L_1.

5. Operators on L_1

Nigel looked at operators on function spaces like L_1 or indeed L_p for $p \leq 1$ in a vast number of papers; I will report on a small sample of his results.

In [K2] he devised a very useful representation theorem for operators on L_p, $0 \leq p \leq 1$, by means of random measures. For $p = 1$ the result is as follows.

Theorem 5.1. *If T: $L_1[0,1] \to L_1[0,1]$ is a bounded linear operator, then there are measures μ_x on $[0,1]$, with $x \mapsto \mu_x \in M[0,1]$ weak* measurable, such that*

$$(Tf)(x) = \int_0^1 f(s)\, d\mu_x(s) \quad a.e.$$

Moreover,

$$\|T\| = \sup_{\lambda(B)>0} \frac{1}{\lambda(B)} \int_0^1 |\mu_x(B)|\, dx.$$

Decomposing the measures μ_x into their atomic and continuous parts μ_x^a and μ_x^c allows to define the corresponding operators

$$(T^a f)(x) = \int_0^1 f(s)\, d\mu_x^a(s), \quad (T^c f)(x) = \int_0^1 f(s)\, d\mu_x^c(s).$$

This permits Nigel to derive the following variant of a result due to P. Enflo and T.W. Starbird [7].

Theorem 5.2. *If $T^a \neq 0$, there is a Borel set B of positive measure such that $T|_{L_1(B)}$ is an into-isomorphism whose range is complemented.*

Let me remark that it is still an open problem whether a complemented infinite-dimensional subspace of L_1 is isomorphic to L_1 or ℓ_1. However, Enflo and Starbird have proved that L_p is primary for $p \geq 1$. This means that whenever L_p is isomorphic to a direct sum $X \oplus Y$, then X or Y is isomorphic to L_p. Using his representation theorem, Nigel extends this result to $p < 1$ and obtains a new proof for $p = 1$. Thus:

Theorem 5.3. *L_p is primary for $0 < p < \infty$.*

The representation theorem 5.1 is also used in the paper [K8] of Nigel and Beata Randrianantoanina, where they show that a surjective isometry on a real rearrangement invariant space X on $[0,1]$ different from L_2 has the form $(Tf)(s) = a(s)f(\sigma(s))$; if X is not isomorphic to L_p for any $1 \leq p \leq \infty$, then in addition $|a| = 1$ a.e. and σ is measure preserving.

In [K18], Nigel and his coauthors G. Godefroy and D. Li take up the line of reasoning based on Theorem 5.1 to obtain results of a more isometric flavour, for example a quantitative version of a result due to D. Alspach [2].

Theorem 5.4. *There is a function φ with $\lim_{\alpha \to 0^+} \varphi(\alpha) = 1$ such that if $T \colon L_1 \to L_1$ satisfies*

$$\|f\| \le \|Tf\| \le \alpha\|f\| \qquad \forall f \in L_1,$$

then there is an isometry $J \colon L_1 \to L_1$ such that $\|T - J\| \le \varphi(\alpha)$.

In other words, there is always an into isometry close to a given into near-isometry.

The paper also contains an example, of a similar nature as that of Theorem 3.6, to show that this result does not extend to (near-) isometries from subspaces of L_1 to L_1.

Specifically, Nigel and his coauthors investigate certain subspaces of L_1 that they call *small subspaces* and operators that they call *strong Enflo operators*. The latter means that in the representation of T by the measures μ_x one has $\mu_x(\{x\}) \ne 0$ on a set of positive measure. A subspace X of L_1 is called small if the mapping $f \mapsto f|_A$ from X to $L_1(A)$ is not surjective for any $A \subset [0,1]$ of positive measure. In other words, the inclusion $B_{L_1(A)} \subset kB_X$ is false whenever $\lambda(A) > 0$ and $k > 0$, where we consider $L_1(A) \subset L_1[0,1]$ in a natural way. The following short-hand notation is now handy. Say

$$M \underset{\sim}{\subseteq} N$$

for two subset of L_1 if there is a positive constant k such that $M \subset kN$, i.e., M is absorbed by N. Hence X is not small if $B_{L_1(A)} \underset{\sim}{\subseteq} B_X$.

It is worthwhile to equip L_1 with the topology τ_m of convergence in measure, defined above Theorem 3.5. For a subspace $X \subset L_1$, let C_X be the closure of B_X in L_1 with respect to τ_m. The subspace X is called *nicely placed* if $B_X = C_X$, that is, if its unit ball is closed for the topology τ_m. By a theorem due to A.V. Buhvalov and G.J. Lozanovskiĭ [10, page 183], X is nicely placed if and only if the Yosida-Hewitt projection associated to the decomposition

$$(L_1)^{**} = L_1 \oplus (L_1)_s$$

maps $X^{\perp\perp}$ onto X. Hence, $X \subset L_1$ is nicely placed if and only if it is L-embedded, i.e., $X^{**} = X \oplus_1 X_s$. Nicely placed subspaces were introduced in [8] and studied intensively in many papers, see Chapter IV in [10] for a survey.

Returning to the paper [K18], let us note the following result.

Theorem 5.5. *A subspace X of L_1 is small if and only if no strong Enflo operator satisfies $T(B_{L_1}) \underset{\sim}{\subseteq} C_X$. Consequently, a nicely placed subspace is small if and only if no operator from L_1 to X is a strong Enflo operator.*

Since strong Enflo operators have a nonzero atomic part, they fix a copy of L_1 by Theorem 5.2; therefore a nicely placed subspace of L_1 not containing L_1 is small.

Another feature of operators that are not strong Enflo operators is the equation

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

known as the *Daugavet equation*. By the above, this is fulfilled for all operators valued in a small nicely placed subspace. The Daugavet equation is one of the technical ingredients in the proof of the main result of [K18]:

Theorem 5.6. *Let X and Y be small subspaces of L_1 and suppose that there is an isomorphism $S\colon L_1/X \to L_1/Y$ with $\max\{\|S\|, \|S^{-1}\|\} < 1 + \delta < 1.25$. Then there is an invertible operator $U\colon L_1 \to L_1$ such that $\|U\|\,\|U^{-1}\| \leq (1+\delta)/(1-25\delta)$ and $d_{\mathscr{H}}(U(B_X), B_Y) \leq 71\delta/(1-25\delta)$, where $d_{\mathscr{H}}$ denotes the Hausdorff distance.*

If X and Y are additionally assumed to be nicely placed, the conclusion can be strengthened: If $\|S\|\,\|S^{-1}\| = 1 + \alpha < 2$, then U above can be chosen to map X onto Y and $\|U\|\,\|U^{-1}\| \leq (1+\alpha)/(1-\alpha)$. A particular consequence in this setting is: If L_1/X and L_1/Y are isometric, then so are X and Y.

Leaving the field of small subspaces let us turn to *rich subspaces*. These were introduced by A. Plichko and M. Popov [29]; later the definition was slightly modified [18] to accommodate the general setting of Banach spaces with the Daugavet property introduced in [17]. A Banach space X has the *Daugavet property* if the Daugavet equation

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

holds for all compact operators $T\colon X \to X$; examples include $L_1[0,1]$, $C[0,1]$, $L_\infty[0,1]$, the disc algebra, L_1/H^1 and many other function spaces, but also more pathological spaces in the spirit of Theorem 3.6 [19]. A subspace Y of X is called rich if whenever $Y \subset Z \subset X$, then Z has the Daugavet property. (There are other equivalent reformulations of this, see e.g. the survey [35].)

I did some work on these notions with Vova Kadets and other coauthors. Upon hearing about some of our results, Nigel immediately contributed the following theorem, published in our joint paper [K21], showing that rich subspaces are indeed the largest possible proper subspaces of L_1. Recall that C_X is the closure of B_X for the topology of convergence in measure.

Theorem 5.7. *A subspace $X \subset L_1$ is rich if and only if, for each 1-codimensional subspace $H \subset X$, $\frac{1}{2}B_{L_1} \subset C_H$. On the other hand, if $rB_{L_1} \subset C_X$ for some $r > \frac{1}{2}$, then $X = L_1$.*

One of the consequences of Nigel's representation theorem (Theorem 5.1) is not only the primariness of L_1, but more generally that whenever L_1 is isomorpic to an unconditional Schauder decomposition $X_1 \oplus X_2 \oplus \cdots$, then one of the X_k is isomorphic to L_1. In other words, this result ponders on possible or impossible representations of $\mathrm{Id} = \sum T_n$ as a pointwise unconditionally convergent series. In our joint paper [K26] (with V. Kadets) we investigate this question further. This paper introduces a class $\mathscr{C}$ of operators related to the narrow operators of Plichko and Popov [29] and to the not sign preserving operators of H.P. Rosenthal [31], but I will skip the exact definition and will only mention that compact operators belong to this class. Anyway, here is the result.

Theorem 5.8. *Let X be a Banach space and $T, T_n\colon L_1 \to X$ be bounded operators such that $Tf = \sum_n T_n f$ unconditionally for each $f \in L_1$. If each T_n is in $\mathscr{C}$, then T is in $\mathscr{C}$.*

Nigel's student R. Shvydkoy had obtained a similar result for the narrow operators in the case $X = L_1$ in his Ph.D. thesis.

Theorem 5.8 can be considered as a generalisation of A. Pełczyński's classical theorem that L_1 does not embed into a space with an unconditional basis. It also contains an unpublished (in fact unwritten, but occasionally quoted) result due to H.P. Rosenthal [32] as a special case: L_1 does not even sign-embed into a space with an unconditional basis.

6. Extensions of operators into $C(K)$-spaces

It is a classical fact that whenever $E \subset X$ are Banach spaces and T_0: $E \to L_\infty[0,1]$ is a bounded linear operator, then there exists an extension $T \colon X \to L_\infty[0,1]$ of the same norm; however, there need not exist a bounded linear extension whatsoever if $L_\infty[0,1]$ is replaced by $C[0,1]$. By constrast, Joram Lindenstrauss's memoir [24] presents a detailed study of those range spaces for which compact operators are extendible; it turns out, that, for all pairs $E \subset X$, every compact operator $T_0 \colon E \to F$ admits, for every $\varepsilon > 0$, an extension $T \colon X \to F$ of norm $\|T\| \leq (1 + \varepsilon)\|T_0\|$ if and only if F^* is isometric to a space $L_1(\Omega, \Sigma, \mu)$. In particular this is true for $F = C[0,1]$.

More recently, the problem of extension of bounded operators into $C(K)$ was reexamined by Bill Johnson and Morry Zippin [16] who obtained $(3 + \varepsilon)$-extensions for the pair (E, ℓ_1) provided E is weak* (meaning $\sigma(\ell_1, c_0)$-) closed.

In a series of papers that are enormous in wealth of ideas, technical mastery and also in size ([K19], [K27], [K28], [K29]) Nigel has contributed to this circle of ideas. I will now describe just a few of his findings. Let us say, following Nigel, that the pair (E, X), with E a subspace of the separable space X, has the λ-C-*extension property* if, given a bounded linear operator $T_0 \colon E \to C(K)$ into some separable $C(K)$-space, there is an extension $T \colon X \to C(K)$ of norm $\|T\| \leq \lambda \|T_0\|$. (The same problem can be studied for Lipschitz maps where the Lipschitz constant plays the role of the norm. This setting is investigated in detail in [K27], [K28].) In this language the Johnson-Zippin result says that (E, ℓ_1) has the $(3+\varepsilon)$-C-extension property for every $\varepsilon > 0$ if $E \subset \ell_1$ is weak* closed.

Now, Nigel improves the $(3 + \varepsilon)$-bound to $(1 + \varepsilon)$, and he also obtains a converse: If (E, ℓ_1) has the λ-C-extension property for some $\lambda > 0$ and additionally ℓ_1/E has an unconditional FDD, then there is an automorphism $S \colon \ell_1 \to \ell_1$ such that $S(E)$ is weak* closed. To prove this, he makes a detailed study of spaces embedding into c_0 and presents the tree characterisation of such spaces mentioned on page 6. He also proves that if (E, ℓ_1) has the $(1+\varepsilon)$-C-extension property for all $\varepsilon > 0$ (the "almost isometric C-extension property"), then E is weak* closed.

In [K29] Nigel goes on to develop an approach to the C-extension property that is rooted in the theory of types, introduced by J.L. Krivine and B. Maurey [22]. Let X be a separable Banach space. A *type* generated by a bounded sequence (x_n) (or more generally a bounded net) is a function of the form

$$\sigma \colon X \to \mathbb{R}, \quad \sigma(x) = \lim_{n \to \infty} \|x + x_n\|$$

(provided all these limits exist); it is called a weakly null type if $x_n \to 0$ weakly. Property (M) can now be rephrased by saying that for a weakly null type σ, $\sigma(x)$ depends only on $\|x\|$, and X has (m_p) whenever each weakly

null type has the form $\sigma(x) = (\|x\|^p + \lim_n \|x_n\|^p)^{1/p}$. Property (au) is a symmetry condition, viz. $\sigma(x) = \sigma(-x)$. Now X is said to have *property* (L) if two weakly null types coincide once they coincide at 0:

$$\sigma_1(0) = \sigma_2(0) \quad \Rightarrow \quad \sigma_1(x) = \sigma_2(x) \text{ for all } x.$$

Property (L^*) is defined similarly using weak* null types in the dual. Clearly, ℓ_p has properties (L) and (L^*) for $1 < p < \infty$, and so do certain renormings of Orlicz sequence spaces or Fenchel-Orlicz spaces. These renormings are similar in spirit to the renormings that yield property (M), but not identical. Indeed, if X has both properties (M) and (L) and fails to contain ℓ_1, then it has (m_p) for some $1 < p \leq \infty$.

With the help of types, Nigel is able to characterise the almost isometric C-extension property. I won't give the exact formulation of this result, but will note only one special case.

Theorem 6.1. *If X is a separable Banach space with (M^*) or (L^*) and if $E \subset X$, then (E, X) has the almost isometric C-extension property.*

In particular, this applies to (renormings of) the twisted sums Z_p. Nigel also provides a particular renorming of ℓ_2 to show that the existence of almost isometric extensions need not guarantee isometric extensions.

Moreover, Nigel gives the first examples of spaces with the following universal property: Whenever E embeds into a separable space X isometrically, then the pair (E, X) has the almost isometric C-extension property. Indeed, all weak* closed subspaces of ℓ_1 have this property as do the spaces of Theorem 3.6. The proof again uses the theory of types.

An intriguing study of the corresponding isomorphic property is contained in [K29]. Let us say that the pair (E, X) of separable spaces has the C-*extension property* if, given a bounded linear operator $T_0 \colon E \to C(K)$ into some separable $C(K)$-space, there is a bounded linear extension $T \colon X \to C(K)$; here it is enough to consider $K = [0, 1]$ by Milutin's theorem. A Banach space E has the *universal C-extension property* if (E, X) has the C-extension property whenever E embeds into X isometrically.

Nigel proves the following result:

Theorem 6.2. *A separable Banach space E has the universal C-extension property if and only if E is C-automorphic.*

The latter means that whenever $E_1 \subset C[0, 1]$ and $E_2 \subset C[0, 1]$ are isomorphic to E, then there is an automorphism $S \colon C[0, 1] \to C[0, 1]$ mapping E_1 to E_2. One can paraphrase this by saying that there is essentially only one way to embed E into $C[0, 1]$.

It is a classical result due to J. Lindenstrauss and A. Pełczyński [25] that c_0 is C-automorphic. Theorem 6.2 enables Nigel to show that $c_0(X)$ is C-automorphic as well if X is (for example $X = \ell_1$), but ℓ_p is not C-automorphic for $1 < p < \infty$. Indeed, for a certain superreflexive $Z \supset \ell_p$ with an unconditional basis, the pair (ℓ_p, Z) fails the C-extension property. If, however, $Z \supset \ell_p$ is a UMD-space with an unconditional basis, then (ℓ_p, Z) satisfies the C-extension property; in Nigel's own words, "the appearance of the UMD-condition is quite mysterious."

A technical device to prove these theorems are homogeneous mappings $\Phi\colon X^* \to Z^*$ that are weak* continuous on bounded sets such that $\Phi(x^*)$ extends x^* with a bound $\|\Phi(x^*)\| \leq \lambda\|x^*\|$, for all $x^* \in X^*$. Such mappings were introduced by M. Zippin [37] and are called Zippin selectors by Nigel. Based on this notion, the following technical key result on $X = \ell_p$ or, more generally, an ℓ_p-sum of finite-dimensional spaces, $1 < p < \infty$, is proved, where again types play an essential role:

Theorem 6.3. *Let X be as above. For a separable superspace $Z \supset X$, the pair (X, Z) has the C-extension property if and only if Z can be renormed so as to contain X isometrically and such that*

$$\lim_{n\to\infty} \|z + x_n\| \geq \lim_{n\to\infty} (\|z\|^p + \|x_n\|^p)^{1/p}$$

for all $z \in Z$ and all weakly null sequences $(x_n) \subset X$, provided both limits exist.

Many of Nigel's extension theorems have counterparts for Lipschitz functions; for a detailed study see [K27] and [K28].

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