Some lifting theorems for bounded linear operators

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1 INTRODUCTION

Our aim in the present paper is to survey some results in connection with linear liftings of operators. That is, we are interested in the following problem.

> PROBLEM. Let Y and X be Banach spaces, $J \subset X$ a closed subspace and $T : Y \to X/J$ a bounded linear operator. Does there exist a bounded linear operator $L : Y \to X$ such that (q denoting the quotient map from X onto X/J) $qL = T$, in other words, such that the diagram
>
> $$\begin{tikzcd} Y \arrow[rr, "T"] \arrow[dr, "L"'] & & X/J \\ & X \arrow[ur, "q"'] & \end{tikzcd}$$
>
> commutes?

This problem is most often posed in the special case that $Y = X/J$ and $T = Id$. Under this assumption a lifting L is a right inverse for the quotient map.

It is clear that – with no additional assumptions made – this is not to be expected. For if L is a continuous linear right inverse for the quotient map, then the operator

$P = Lq$ is a continuous linear projection on X with $\ker P = J$. However, for example in the case that $X = \ell^\infty$ and $J = c_0$ no such projection exists [42].

In this paper we shall discuss the Ando-Choi-Effros theorem (see Section 2) giving sufficient conditions which ensure that a continuous linear lifting exists. Then we will apply it in various instances, and we reprove several classical theorems on linear extension operators (Section 3). In Section 4 we deal with Sobczyk's theorem on projections onto c_0 and obtain isomorphic representations of some spaces of analytic functions. This paper is – for the most part – of expository nature, most of its results are already known.

In order to state the Ando-Choi-Effros theorem, we need the notion of an M-ideal in a Banach space.

Definition 1.1 *A closed subspace J of a Banach space X is called an M-ideal if there is a linear projection P from X^* onto $J^\perp := \{x^* \in X^* \mid x^*(x) = 0 \ \forall x \in J\}$ satisfying*

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \qquad \forall x^* \in X^*.$$

This definition is due to Alfsen and Effros [2]. Here are some examples of M-ideals.

Example 1.2(a) Let S be a locally compact Hausdorff space. Then $J \subset C_0(S)$ is an M-ideal if and only if there is a closed subset D of S such that

$$J = J_D := \{x \in C_0(S) \mid x(s) = 0 \quad \text{for all } s \in D\}.$$

Clearly, $\mu \mapsto \chi_D \mu$ is the required projection onto $J_D^\perp$. For a detailed proof of the converse see [7, p. 40] or [20, Ex. I.1.4(a)]. Note that in particular c_0 is an M-ideal in ℓ^∞.

Example 1.2(b) Let A be the disk algebra, that is the complex Banach space of continuous functions on the closed unit disk which are analytic in the open unit disk. It will be convenient to consider A (via boundary values) as a subspace of $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Then J is a nontrivial M-ideal in A if and only if there is a closed subset $D \neq \emptyset$ of $\mathbb{T}$ with linear Lebesgue measure 0 such that (J_D as above)

$$J = J_D \cap A = \{x \in A \mid x(t) = 0 \quad \text{for all } t \in D\}.$$

This follows from Example 1.2(a) and Propositions 3.3 and 3.4 below which are applicable by the F. and M. Riesz theorem. More generally, the M-ideals of a function algebra coincide with the annihilators of p-sets, see [21] or [20, Th. V.4.2].

Example 1.2(c) Let K be a compact convex set in a Hausdorff locally convex topological vector space. As usual, $A(K)$ denotes the space of real-valued affine continuous functions on K. Let us recall the definition of a split face of K [1, p. 133]. A face F of K is called a split face if there is another face F' such that every $k \in K \backslash (F \cup F')$ has a *unique* representation

$$k = \lambda k_1 + (1-\lambda) k_2 \ \text{ with } k_1 \in F, \ k_2 \in F', \ 0 < \lambda < 1.$$

It is known that every closed face of a simplex is a split face [1, p. 144].

Then J is an M-ideal in $A(K)$ if and only if there exists a closed split face F of K such that

$$J = J_F \cap A(K) = \{x \in A(K) \mid x(k) = 0 \quad \text{for all } k \in F\}.$$

The proof of this fact can be given along the lines of (b), the crucial step being the measure theoretic characterisation of closed split faces (see [1, Th. II.6.12]) which replaces the use of the F. and M. Riesz theorem.

Example 1.2(d) In a C^*-algebra the M-ideals coincide with the closed two-sided ideals.

This was first proved in [2] and [38], see also [20, Th. V.4.4].

For future reference we state an equivalent characterisation of M-ideals by means of an intersection property of balls, the so-called 3-ball property.

Theorem 1.3 *A closed subspace J of a Banach space X is an M-ideal if and only if for all $x \in B_X$ (the closed unit ball of X), all $y_1, y_2, y_3 \in B_J$ and all $\varepsilon > 0$ there is some $y \in J$ such that*

$$\|x + y_i - y\| \le 1 + \varepsilon \qquad (i = 1, 2, 3).$$

PROOF: [25, Th. 6.17] or [20, Th. I.2.2]. □

A detailed exposition of M-ideal theory will appear in [20].

2 AROUND THE ANDO-CHOI-EFFROS THEOREM

As already mentioned, the Ando-Choi-Effros theorem presents additional sufficient conditions for a continuous linear lifting to exist. As usual, AP stands for ‘approximation property’, BAP for ‘bounded approximation property’, and MAP for ‘metric approximation property’.

Theorem 2.1 (Ando-Choi-Effros)
Suppose J is an M-ideal in the Banach space X, Y is a separable Banach space, and $T \in L(Y, X/J)$ with $\|T\| = 1$. Assume further

(a) *Y has BAP*

or

(b) *J is an L^1-predual.*

Then there is a continuous linear lifting L for T. More precisely, we obtain a lifting with

$$\|L\| \le \lambda \qquad \text{if } Y \text{ has } \lambda\text{-AP},$$

resp.

$$\|L\| = 1 \qquad \text{under assumption (b)}.$$

PROOF: See the original papers [5] and [12]. Ando’s proof is elaborated in [20, II.2.1]. □

In the next sections some consequences of this theorem will be presented. But first of all let us discuss to what extent the additional assumptions in Theorem 2.1 are really needed. As for the separability one just has to activate the example of the noncomplemented M-ideal c_0 in ℓ^∞. (In view of the examples in [13] or [26, Prop. 3.5]

Theorem 2.1 does not even extend to weakly compactly generated spaces Y.) Instead of the separability of Y and (a) or (b) one may of course use the weaker assumptions

(a′) T factors through a separable space with BAP

resp.

(b′) J is an L^1-predual, and T has separable range.

As for the approximation property we first note a proposition.

Proposition 2.2 *For every separable Banach space Y there are a separable Banach space X enjoying MAP and an M-ideal J in X such that $Y \cong X/J$.*

Applying Proposition 2.2 to a space Y without BAP we infer that there can be no continuous linear lifting for the quotient map, because otherwise Y would be isomorphic to a complemented subspace of X and hence would have BAP.

Proof of Proposition 2.2: We let (E_n) be an increasing sequence of finite dimensional subspaces of Y such that $\bigcup E_n$ is dense and define

$$\begin{aligned} X &= \{(x_n) \mid x_n \in E_n, \ \ \lim x_n \text{ exists}\}, \\ J &= \{(x_n) \mid x_n \in E_n, \ \ \lim x_n = 0\}. \end{aligned}$$

(These spaces are, of course, equipped with the sup norm.)

It is quickly verified that Y is isometric to X/J. Moreover, J is an M-ideal. To prove this we use the 3-ball property (Theorem 1.3). In fact, given normed vectors $\xi = (x_n) \in X$, $\eta_i = (y_n^i) \in J$ $(i = 1, 2, 3)$ and $\varepsilon > 0$, choose N such that

$$\|y_n^i\| \le \varepsilon \qquad \text{for } \ n > N \ \text{ and } \ i = 1, 2, 3.$$

If $\eta = (y_n)$ with $y_n = x_n$ $(n \le N)$, $y_n = 0$ $(n > N)$, then

$$\|\xi + \eta_i - \eta\| \le 1 + \varepsilon.$$

It remains to notice that X has MAP since the contractive finite rank projections $P_m : (x_n) \mapsto (x_1, \dots, x_m, x_m, x_m, \dots)$ converge strongly to the identity. □

Also, it is not enough to assume that $J^\perp$ is norm one complemented by *some* projection as is shown by the example of a subspace J of ℓ^1 such that $\ell^1/J = L^1[0,1]$. However, in this setting we can prove the following.

Proposition 2.3 *Let $J \subset X$ be a closed subspace, and suppose that there is a contractive linear projection from X^* onto $J^\perp$. Then every operator $T \in L(Y, X/J)$ which is a limit of finite rank operators can be lifted to an operator $L \in L(Y, X)$, which is a limit of finite rank operators, too. Also, given $\varepsilon > 0$, one may require that $\|L\| \le (1+\varepsilon)\|T\|$.*

Proof: We let $\overline{F}(.,.)$ denote the norm closure of the finite rank operators. We claim that

$$\overline{F}(Y, X/J) \cong \overline{F}(Y, X)/\overline{F}(Y, J) \tag{$*$}$$

which immediately gives the assertion of Proposition 2.3.

Now $\overline{F}(Y, X) \cong Y^* \widehat{\otimes}_\varepsilon X$, the completed injective tensor product of Y^* and X; thus $(*)$ is equivalent to the claim that $Id \otimes q$ is a quotient map on $Y^* \widehat{\otimes}_\varepsilon X$, where of course

$q : X \to X/J$ denotes the canonical mapping. This in turn is equivalent to $(Id \otimes q)^*$ being an isometric embedding of $(Y^*\widehat{\otimes}_\varepsilon X/J)^*$ into $(Y^*\widehat{\otimes}_\varepsilon X)^*$. But these duals can be represented by spaces of integral operators:

$$\begin{aligned}(Y^*\widehat{\otimes}_\varepsilon X/J)^* &\cong I(Y^*,(X/J)^*),\\ (Y^*\widehat{\otimes}_\varepsilon X)^* &\cong I(Y^*,X^*).\end{aligned}$$

Thus, yet another reformulation of $(*)$ reads: The canonical map of $I(Y^*,(X/J)^*)$ into $I(Y^*,X^*)$ is an isometry. But this follows from the supposed existence of a contractive projection P from X^* onto $J^\perp \cong (X/J)^*$. Indeed, if $S \in I(Y^*,(X/J)^*)$,

$$\begin{aligned}\|S \mid I(Y^*,X^*)\| &\le \|S \mid I(Y^*,J^\perp)\|\\ &= \|PS \mid I(Y^*,J^\perp)\|\\ &\le \|P\| \cdot \|S \mid I(Y^*,X^*)\|\\ &= \|S \mid I(Y^*,X^*)\|.\end{aligned}$$

□

Let us observe that Proposition 2.3 does not hold for merely compact operators. Suppose it did. Let E be a Banach space without the approximation property, and let Y be a Banach space such that there is a compact operator $T : Y \to E$ which is not approximable by finite rank operators (cf. [30, p. 32]). Let X and J be as in Proposition 2.2 such that $X/J \cong E$. If T were liftable to a compact operator $L : Y \to X$, then L would be approximable since X has the MAP, and so would be $T = qL$. This is a contradiction.

It is noteworthy that there is always a nonlinear continuous projection onto an M-ideal and thus a nonlinear continuous lifting for the quotient map; this follows from Michael's selection theorem and the fact that the metric projection onto an M-ideal is continuous with respect to the Hausdorff metric. (For details, cf. [23], [17], [43] or [20, II.1.9].)

3 LINEAR EXTENSION OPERATORS

We now turn to the consequences of Theorem 2.1. It contains several well-known results on the existence of linear extension operators as a special case.

Corollary 3.1 (Borsuk-Dugundji)
Let K be a compact Hausdorff space and let $D \subset K$ be a closed metrizable subset. Then there is a linear extension operator $T : C(D) \to C(K)$ with $\|T\| = 1$, i.e. $(Tx)(t) = x(t)$ for $x \in C(D)$ and $t \in D$.

PROOF: Consider the M-ideal $J_D = \{x \in C(K) \mid x|_D = 0\}$ (Example 1.2(a)). Then $C(D) \cong C(K)/J_D$ meets the requirements of Theorem 2.1. (As a matter of fact, both (a) and (b) are fulfilled: $C(D)$ has MAP, and J_D is an L^1-predual.) □

Corollary 3.2 (Pełczyński)
Let A be the disk algebra, and suppose D is a subset of the unit circle with Lebesgue measure 0*. Then there is a contractive linear extension operator from $C(D)$ to A.*

PROOF: Consider the M-ideal $J = \{x \in A \mid x|_D = 0\}$ (Example 1.2(b)). By the Rudin-Carleson theorem [18, p. 58] we have $C(D) \cong A/J$, hence the result. $\square$

More general corollaries can be formulated along the same lines on the basis of the Glicksberg peak interpolation theorem [18, p. 58, Th. 12.5 and Th. 12.7] and Proposition 4.8 below, along with the discussion preceding it.

Next we show how to obtain the Michael-Pełczyński-Ryll-Nardzewski theorem as a consequence of Theorem 2.1. For this we need to discuss M-ideals in subspaces of $C(K)$.

We recall from the fundamental Example 1.2(a) that the M-ideals in $C(K)$ coincide with the ideals J_D. The following proposition says that the J_D are in fact the ancestors of all M-ideals, since every Banach space is a subspace of some $C(K)$.

Proposition 3.3 *Let X be a closed subspace of $C(K)$, and let J be an M-ideal in X. Then there is a closed subset D of K such that $J = J_D \cap X$.*

PROOF: Let $D = \{k \in K \mid \delta_k|_X \in J^\perp\}$. Then D is a closed set, and $J \subset J_D \cap X$ by construction. If the inclusion were proper, we could separate a certain $x_0 \in J_D \cap X$ from J by a functional $p \in J^\perp$. We may even assume $p \in \operatorname{ex} B_{J^\perp}$ by the Krein-Milman theorem and thus $p \in \operatorname{ex} B_{X^*}$. (It is easily verified that for an M-ideal $\operatorname{ex} B_{J^\perp} \subset \operatorname{ex} B_{X^*}$.) Such a p is of the form $p(x) = \lambda \cdot x(k)$ for some $k \in K$, $|\lambda| = 1$. Since $p \in J^\perp$ we must have $k \in D$ and hence $x_0(k) = 0$. On the other hand $p(x_0) \neq 0$ since p is a separating functional: a contradiction. $\square$

We turn to the converse of this proposition, which, of course, will generally not hold. The following is obvious.

Proposition 3.4 *Let X be a closed subspace of $C(K)$. Then $J_D \cap X$ is an M-ideal in X if the L-projection $\mu \mapsto \chi_D \mu$ from $C(K)^*$ onto $J_D^\perp$ leaves $X^\perp$, the annihilator of X in $C(K)^*$, invariant.*

Let us formulate an important example where M-ideals are induced in a subspace. Suppose X is a closed subspace of $C(K)$ and suppose $D \subset K$ is closed. We let $X|_D$ be the space of all restrictions $\{x|_D \mid x \in X\}$. Following [31] one says that $(X|_D, X)$ has the *bounded extension property* if there exists a constant C such that, given $\xi \in X|_D$, $\varepsilon > 0$ and an open set $U \supset D$, there is some $x \in X$ such that

$$\begin{aligned} x|_D &= \xi, \\ \|x\| &\leq C \cdot \|\xi\|, \\ \|x(k)\| &\leq \varepsilon \qquad \text{for } k \notin U. \end{aligned}$$

(Note that $X|_D$ is closed under this assumption.) For example, if $X \subset C(K)$ is a subalgebra and $D \subset K$ is a subset of the form $f^{-1}(\{1\})$ for some $f \in B_X$ (a 'peak set'), then the pair $(X|_D, X)$ has the bounded extension property: Let $\xi \in X|_D$ and $g \in X$ such that $g|_D = \xi$. First of all we remark that replacing f by $(1+f)/2$ permits us to assume that in addition $|f(k)| < 1$ if and only if $k \notin D$. Then $g \cdot f^n$ meets the requirements of the above definition if n is large enough. (The argument easily extends to intersections of peak sets, the so-called p-sets.)

We now have:

Proposition 3.5 *If $(X|_D, X)$ has the bounded extension property, then $J_D \cap X$ is an M-ideal in X.*

PROOF: To begin with we observe that the constant C appearing in the definition of the bounded extension property may be chosen as close to 1 as we wish. To see this let $U \supset D$ be an open set and let $\varepsilon > 0$. Applying the bounded extension property with $U_1 = U$ and ε yields an extension x_1 of a given $\xi \in X|_D$ (w.l.o.g. $\|\xi\| = 1$) such that $\|x_1\| \le C$ and $|x_1| \le \varepsilon$ off U_1. Then we repeat this procedure with $U_2 = \{k \mid |x_1(k)| < 1+\varepsilon/2\} \cap U_1$ and obtain an extension x_2 such that $\|x_2\| \le C$ and $|x_2| \le \varepsilon$ off U_2. In the third step one applies the bounded extension property with $U_3 = \{k \mid |x_2(k)| < 1 + \varepsilon/2\} \cap U_2$ etc. This yields a sequence of extensions (x_n) with $\|x_n\| \le C$ and $|x_n| \le \varepsilon$ off U_n. Let $x = \frac{1}{N}\sum_{n=1}^{N} x_n$. Obviously we have $x|_D = \xi$ and $|x| \le \varepsilon$ off U. Finally, if $k \in U$, then by construction $|x_n(k)|$ is big (but $\le C$) for at most one n, and $|x_n(k)| \le 1+\varepsilon/2$ otherwise. It follows

$$|x(k)| \le \frac{1}{N}\Big(C + (N-1)(1+\varepsilon/2)\Big) \le 1 + \varepsilon$$

for sufficiently large N. (As a consequence of this one may remark that $X/(J_D \cap X) \cong X|_D$.)

Now we can prove that $J_D \cap X$ is an M-ideal employing the 3-ball property of Theorem 1.3. Thus, let y_1, y_2, y_3 in the unit ball of $J_D \cap X$, $x \in B_X$ and $\varepsilon > 0$. With the help of the $(1+\varepsilon)$-bounded extension property we may find $\widehat{x} \in X$ such that

$$\widehat{x}|_D = x|_D,$$

$$\|\widehat{x}\| \le (1+\varepsilon)\|x|_D\| \le 1 + \varepsilon,$$

$$|\widehat{x}(k)| \le \varepsilon \quad \text{if } \max_i |y_i(k)| \ge \varepsilon.$$

Let $y = x - \widehat{x}$ so that $y \in J_D \cap X$. Distinguishing whether or not $\max_i |y_i(k)| \ge \varepsilon$ one can immediately verify that

$$|(x + y_i - y)(k)| = |y_i(k) + \widehat{x}(k)| \le 1 + 2\varepsilon$$

for all $k \in K$, i.e.

$$\|x + y_i - y\| \le 1 + 2\varepsilon \qquad (i = 1, 2, 3).$$

□

Corollary 3.6 (Michael and Pełczyński, Ryll-Nardzewski)
Suppose $X \subset C(K)$ and $D \subset K$ is closed such that the pair $(X|_D, X)$ has the bounded extension property. If $X|_D$ is separable and has the MAP, then there is a contractive linear extension operator from $X|_D$ into X.

PROOF: This follows from Proposition 3.5 and Theorem 2.1. □

Next we present an application of Theorem 2.1 to potential theory. Let $U \subset \mathbb{R}^n$ be open and bounded. We put

$$H(U) = \{f \in C(\overline{U}) \mid f \text{ is harmonic in } U\}.$$

The classical Dirichlet problem requires to find, given $\varphi \in C(\partial U)$, some $f \in H(U)$ such that $f|_{\partial U} = \varphi$. This is generally impossible. However, there is always a generalised solution, called the Perron-Wiener-Brelot solution. Those points $x_0 \in \partial U$ such that for all $\varphi \in C(\partial U)$ and corresponding Perron-Wiener-Brelot solutions f the relation $\lim_{x\to x_0} f(x) = \varphi(x_0)$ is valid are called regular boundary points. The set of all regular boundary points is denoted by $\partial_r U$, and it is classical that $\partial_r U = \partial U$ if the boundary of U is sufficiently smooth or if U is simply connected and $n = 2$.

Concerning the weak solvability of the Dirichlet problem we now have:

Proposition 3.7 *Let $U \subset \mathbb{R}^n$ be open and bounded, and let $E \subset \partial_r U$ be compact. Then there is a contractive linear operator $L : C(E) \to H(U)$ such that $(L\varphi)|_E = \varphi$ for all $\varphi \in C(E)$.*

PROOF: This result will turn out to be almost obvious after we have reformulated it in terms of convexity theory. The space $H(U)$ is an order unit space and can hence be represented as a space of affine continuous functions $A(K)$. The crux of the matter is that here K is a Choquet simplex; see [11] or [15]. Moreover, it is known from [6] that $\partial_r U$ can be identified with the Choquet boundary of $H(U)$, that is $\operatorname{ex} K$. If we regard E as a compact subset of $\operatorname{ex} K$, then a corollary to Edwards' separation theorem states that every continuous function on E has a norm preserving extension to an affine continuous function on K [1, p. 91]. Let us denote $F = \overline{\operatorname{co}}\, E$ so that F is a split face since K is a simplex. Consequently, by Example 1.2(c),

$$\{f \in A(K) \mid f|_E = 0\} = J_F \cap A(K) =: J$$

is an M-ideal in $A(K)$, and the quotient space $A(K)/J$ is isometric with $C(E)$ by the above. By Theorem 2.1 there is a linear contractive lifting L for the quotient map, and reidentifying $A(K)$ with $H(U)$ yields the desired solution operator from $C(E)$ to $H(U)$. □

One can also recover results of Andersen [3] using Example 1.2(d). For the original proofs of the above linear extension theorems see [14], [32], [33], [31] and [34]; for Proposition 3.7 there is a direct potential theoretic approach in [10]. A particularly nice proof of Corollary 3.1 in the case of metrizable K can be found in [22, p. 103].

4 SOBCZYK'S THEOREM AS A THEOREM ON M-IDEALS

A slightly different type of corollary is Sobczyk's theorem. In order to prove this classical theorem by M-ideal methods, we first provide a renorming result.

Proposition 4.1 *Let X be a Banach space and $Y \subset X$ a subspace isometric to c_0. Then there is an equivalent norm on X which agrees with the original norm on Y so that Y becomes an M-ideal.*

PROOF: $Y^{\perp\perp}$, which is canonically isometric to Y^{**}, is isometric to ℓ^∞, and Y (more precisely $i_X(Y)$) is an M-ideal in $Y^{\perp\perp}$, since c_0 is an M-ideal in ℓ^∞. Since $Y^{\perp\perp}$ is isometric to ℓ^∞ there is a contractive projection P from X^{**} onto $Y^{\perp\perp}$. Hence $Y^\perp$ is the kernel of a contractive projection on X^*, viz. $Q = i_X^* P^* i_{X^*}$. (To see this check $\mathrm{ran}(Id - Q) \subset Y^\perp \subset \ker Q$ which shows $Q(Id - Q) = 0$ and $\ker Q = Y^\perp$.) We now renorm X^* so that Q becomes an L-projection:

$$|x^*| := \|Qx^*\| + \|x^* - Qx^*\|$$

i.e.

$$(X^*, |\,.\,|) = \mathrm{ran}(Q) \oplus_1 Y^\perp.$$

Unfortunately $|\,.\,|$ need not be a dual norm, therefore we cannot conclude directly that Y is an M-ideal in some renorming of X. However, we have with respect to the dual norm

$$(X^{**}, |\,.\,|) = \ker(Q^*) \oplus_\infty Y^{\perp\perp}$$

so that $Y^{\perp\perp}$ is an M-summand in $(X^{**}, |\,.\,|)$. Note that $|\,.\,|$ and $\|\,.\,\|$ coincide on $Y^{\perp\perp}$; it follows that Y is an M-ideal in $(X^{**}, |\,.\,|)$, a fortiori Y is an M-ideal in the intermediate space $(X, |\,.\,|)$. $\square$

Corollary 4.2 (Sobczyk)
If X is a separable Banach space and $Y \subset X$ is a closed subspace isometric to c_0, then there is a continuous linear projection π from X onto Y with $\|\pi\| \leq 2$.

PROOF: Let $(X, |\,.\,|)$ be the renorming devised by Proposition 4.1. By Theorem 2.1 there is a contractive (with respect to $|\,.\,|$) linear lifting L of the quotient map $q : X \to X/Y$ (Y is an L^1-predual!), consequently $\pi = Id - Lq$ is a linear projection onto Y. It remains to estimate the norm:

$$\|\pi(x)\| = |\pi(x)| \leq |\pi| \cdot |x| \leq 2 \cdot \|x\|$$

since $\|x^*\| \leq |x^*|$ for all $x^* \in X^*$ whence $|x^{**}| \leq \|x^{**}\|$ for all $x^{**} \in X^{**}$. $\square$

We hasten to add that the above argument is probably the most complicated proof of Sobczyk's theorem that has appeared in the literature; a very simple one can be found in [30, Th. 2.f.5]. The most important feature of our proof is that one can read it from the bottom to the top. This yields the following result on Banach spaces which are M-ideals in their biduals; we shall call such a Banach space M-embedded. (For the definition and basic properties of $\mathcal{L}^p$-spaces we refer to [27], [28], and [29].)

Theorem 4.3 *Let X be a separable $\mathcal{L}^\infty$-space which is an M-ideal in its bidual. Then X is isomorphic to c_0.*

PROOF: We will show that X is a complemented subspace of any separable superspace Y containing X. Zippin's famous characterisation of separably injective spaces [44] then gives the claim.

As every separable Banach space embed into $C[0,1]$ it is sufficient to prove that every subspace of $C[0,1]$ isometric to X is the range of a continuous linear projection. Since X is an $\mathcal{L}^\infty$-space, its bidual is injective. Note $X^{**} \cong X^{\perp\perp}$ and X is an M-ideal in $X^{\perp\perp}$. Now $X^{\perp\perp}$ is injective and is therefore complemented in $C[0,1]^{**}$. Let P denote a projection from $C[0,1]^{**}$ onto $X^{\perp\perp}$; hence

$$C[0,1]^{**} \simeq X^{\perp\perp} \oplus \ker P.$$

We now renorm $C[0,1]^{**}$ (and thus its subspace $C[0,1]$) to the effect that

$$(C[0,1]^{**}, |\,.\,|) \cong X^{\perp\perp} \oplus_\infty \ker P.$$

Observe that $\|\,.\,\|$ and $|\,.\,|$ coincide on $X^{\perp\perp}$. We conclude that, since $X^{\perp\perp}$ is an M-summand in $(C[0,1]^{**}, |\,.\,|)$, X (being an M-ideal in $X^{\perp\perp}$) is an M-ideal in $(C[0,1]^{**}, |\,.\,|)$; a fortiori X is an M-ideal in $(C[0,1], |\,.\,|)$.

We remark that $(C[0,1]/X)^{**}$ is isomorphic to $\ker P$, which has BAP (it is a complemented subspace of the space $C[0,1]^{**}$ which has MAP in its original norm). Therefore $C[0,1]/X$ is a separable space with BAP, hence, by Theorem 2.1, X is a complemented subspace of $C[0,1]$. $\square$

We do not know if the above theorem holds also in the nonseparable case. Its proof essentially used results which are limited to separable spaces; and it cannot be transferred to nonseparable spaces on a formal level since there are spaces not isomorphic to $c_0(\Gamma)$ such that every separable subspace embeds into c_0 isomorphically [24]. However, the example given in the latter paper is not weakly compactly generated whereas M-embedded spaces are [16].

Theorem 4.3 was also proved by G. Godefroy using a similar, yet somewhat more complicated argument [19]; the present proof has appeared in [39].

We wish to apply Theorem 4.3 in the following setting. We denote by $\mathbb{D}$ the open unit disk and let $\phi : [0,1] \to \mathbb{R}$ be a positive continuous decreasing function such that $\phi(0) = 1$ and $\phi(1) = 0$. We define

$$\begin{aligned} A_\infty(\phi) &= \{f : \mathbb{D} \to \mathbb{C} \text{ analytic} \mid \|f\|_\phi = \sup_{z\in\mathbb{D}} |f(z)| \cdot \phi(|z|) < \infty\}, \\ A_0(\phi) &= \{f \in A_\infty(\phi) \mid \lim_{|z|\to 1} |f(z)| \cdot \phi(|z|) = 0\}. \end{aligned}$$

These are Banach spaces under the norm $\|\,.\,\|_\phi$, and $A_\infty(\phi)$ is canonically isometric with $A_0(\phi)^{**}$ ([35], [9]). It is shown in [41] that $A_0(\phi)$ is an M-ideal in its bidual. On the other hand, for a number of weight functions ϕ it is known that $A_0(\phi)$ is an $\mathcal{L}^\infty$-space [36, Theorem 1]. Hence in this case $A_0(\phi)$ is even isomorphic to c_0; see also [37] for this result. The weight function $\phi(r) = 1 - r^2$ constitutes a case of special importance, since the corresponding space $A_0(\phi)$ is isometric to the well-known little Bloch space [4].

Actually, more general weighted spaces of analytic functions of one or several variables can be shown to be M-embedded, see [40, Section 3]. Therefore, we may add one more equivalence to [8, Prop. 3.4], where the authors prove that a certain exact sequence of Banach spaces of analytic functions is a so-called $\otimes$-sequence if and only if $A_0(\phi)$ is an $\mathcal{L}^\infty$-space. The previous discussion shows that '$A_0(\phi)$ is isomorphic to c_0' is another equivalent condition.

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