# Daugavet's proof of Daugavet's theorem

Dirk Werner

The theorem in question, described by Daugavet as "almost obvious, but at the same time unexpected", is this.

**Theorem.** *If* $A$: $C[a,b] \to C[a,b]$ *is a compact linear operator, then*

$$\|\mathrm{Id} + A\| = 1 + \|A\|. \tag{1}$$

Here is an account of Daugavet's argument from [1], using his notation.

One first observes that it is enough to consider a finite-rank operator $A$ since these operators are dense in the space of compact operators on $C[a,b]$. Such an operator has the form

$$Ax = \sum_{k=1}^{n} \varphi_k(x) z_k \tag{2}$$

with $z_k \in C[a,b]$ and continuous linear functionals $\varphi_k \in C[a,b]^*$ that can be represented by Riemann-Stieltjes integrals

$$\varphi_k(x) = \int_a^b x(t)\, d\sigma_k(t),$$

where $\sigma_k$ is a function of bounded variation. Denote

$$\max_k \|z_k\| = M. \tag{3}$$

Now let $\varepsilon > 0$. Pick $x_0 \in C[a,b]$ such that $\|x_0\| = 1$ and $\|Ax_0\| > \|A\| - \varepsilon/2$. Put $y_0 = Ax_0$ and let $\Delta \subset [a,b]$ be a subinterval on which $|y_0(t)| > \|A\| - \varepsilon/2$. Replacing $x_0$ with $-x_0$ if necessary we can even assume that $y_0(t) > \|A\| - \varepsilon/2$ on $\Delta$. Further pick a subinterval $I = [t_0-\delta, t_0+\delta] \subset \Delta$ such that for $k = 1, \dots, n$

$$\mathrm{Var}(\sigma_k|_I) \le \frac{\varepsilon}{4nM}. \tag{4}$$

Indeed, if $\Delta$ is written as a union of $m$ non-overlapping closed intervals $I_1, \dots, I_m$, then one of the $I_l$ will work provided $m \ge (8nM/\varepsilon) \max_k \mathrm{Var}(\sigma_k|_\Delta)$.

Now let $x_1 \in C[a,b]$ be the function that coincides with $x_0$ off $I$, $x_1(t_0) = 1$, and $x_1$ is linear on $[t_0 - \delta, t_0]$ and on $[t_0, t_0 + \delta]$; put $y_1 = Ax_1$. Obviously $\|x_1\| = 1$, and it follows from (2), (3) and (4) that

$$\|y_1 - y_0\| \le \frac{\varepsilon}{2}. \tag{5}$$

Indeed, by (2)

$$y_1 - y_0 = Ax_1 - Ax_0 = \sum_{k=1}^{n} (\varphi_k(x_1) - \varphi_k(x_0)) z_k$$

and, since $\|x_1 - x_0\| \le 2$,

$$|\varphi_k(x_1) - \varphi_k(x_0)| = \left| \int_{t_0-\delta}^{t_0+\delta} (x_1(t) - x_0(t))\, d\sigma_k(t) \right| \le 2 \operatorname{Var}(\sigma_k|_I) \le \frac{\varepsilon}{2nM}$$

by (4), which implies (5) by (3).

One now has

$$\|\mathrm{Id} + A\| \ge \|x_1 + Ax_1\| \ge x_1(t_0) + y_1(t_0) = 1 + y_0(t_0) - [y_0(t_0) - y_1(t_0)].$$

But $y_0(t_0) \ge \|A\| - \varepsilon/2$ and $y_0(t_0) - y_1(t_0) \le \|y_0 - y_1\| \le \varepsilon/2$ by (5). Hence

$$\|\mathrm{Id} + A\| \ge 1 + \|A\| - \varepsilon,$$

and the theorem is proved.

[1] I. K. Daugavet. *On a property of completely continuous operators in the space C*. Uspekhi Mat. Nauk **18.5** (1963), 157–158 (Russian).