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M-Ideals in Banach Spaces and Banach Algebras

Preface

The present notes centre around the notion of an M-ideal in a Banach space, introduced by E. M. Alfsen and E. G. Effros in their fundamental article "Structure in real Banach spaces" from 1972. The key idea of their paper was to study a Banach space by means of a collection of distinguished subspaces, namely its M-ideals. (For the definition of an M-ideal see Definition 1.1 of Chapter I.) Their approach was designed to encompass structure theories for C^*-algebras, ordered Banach spaces, L^1-preduals and spaces of affine functions on compact convex sets involving ideals of various sorts. But Alfsen and Effros defined the concepts of their M-structure theory solely in terms of the norm of the Banach space, deliberately neglecting any algebraic or order theoretic structure. Of course, they thus provided both a unified treatment of previous ideal theories by means of purely geometric notions and a wider range of applicability. Around the same time, the idea of an M-ideal appeared in T. Ando's work, although in a different context.
The existence of an M-ideal Y in a Banach space X indicates that the norm of X vaguely resembles a maximum norm (hence the letter M). The fact that Y is an M-ideal in X has a strong impact on both Y and X since there are a number of important properties shared by M-ideals, but not by arbitrary subspaces. This makes M-ideals an important tool in Banach space theory and allied disciplines such as approximation theory. In recent years this impact has been investigated quite closely, and in this book we have aimed at presenting those results of M-structure theory which are of interest in the general theory of Banach spaces, along with numerous examples of M-ideals for which they apply.
Our material is organised into six chapters as follows. Chapter I contains the basic definitions, examples and results. In particular we prove the fundamental theorem of Alfsen and Effros which characterises M-ideals by an intersection property of balls. In Chapter II we deal with some of the stunning properties of M-ideals, for example their proximinality. We also show that under mild restrictions M-ideals have to be complemented subspaces, a theorem due to Ando, Choi and Effros. The last section of Chapter II is devoted to an application of M-ideal methods to the classification of L^1-preduals. In Chapter III we investigate Banach spaces X which are M-ideals in their biduals. This geometric assumption has a number of consequences for the isomorphic structure of X. For instance, a Banach space has Pełczyński's properties (u) and (V) once it is an M-

ideal in its bidual; in particular there is the following dichotomy for those spaces X: a subspace of a quotient of X is either reflexive or else contains a complemented copy of c_0. Chapter IV sets out to study the dual situation of Banach spaces which are L-summands in their biduals. The results of this chapter have some possibly unexpected applications in harmonic analysis which we present in Section IV.4. Banach algebras are the subject matter of Chapter V. Here the connections between the notions of an M-ideal and an algebraic ideal are discussed in detail. The most far-reaching results can be proved for what we call "inner" M-ideals of unital Banach algebras. These can be characterised by having a certain kind of approximation of the identity. Luckily the M-ideals which are not inner seem to be the exception rather than the rule. The final Chapter VI presents descriptions of the M-ideals in various spaces of bounded linear operators. In particular we address the problem of which Banach spaces X have the property that the space of compact operators on X is an M-ideal in the space of bounded linear operators, a problem which has aroused a lot of interest since the appearance of the Alfsen-Effros paper. We give two characterisations of those spaces X, one of them following from our work in Chapter V, the other being due to N. Kalton.

Each chapter is accompanied by a "Notes and Remarks" section where we try to give precise references and due credits for the results presented in the main body of the text. There we also discuss additional material which is related to the topics of the chapter in question, but could not be included with complete proofs because of lack of space.

Only a few prerequisites are indepensable for reading this book. Needless to say, the cornerstones of linear functional analysis such as the Hahn-Banach, Krein-Milman, Krein-Smulian and open mapping theorems are used throughout these notes, often without explicitly mentioning them. We also assume the reader to be familiar with the basics of Banach algebra theory including the Gelfand-Naimark theorem representing a commutative unital C^*-algebra in the form $C(K)$, and with various special topics such as the representation of the extreme functionals on a $C(K)$-space as multiples of Dirac measures or the principle of local reflexivity (an explicit statement of which can be found in Theorem V.1.4). Other concepts that we need but are not so well-known will be recalled as required. For our notation we refer to the list of symbols.

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A final piece of advice to those readers who would like to relax from M-ideals, M-summands and M-structure theory: we think it's a good idea to resort to [313].

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