CHAPTER II

Geometric properties of M-ideals

II.1 M-ideals and best approximation

In this section we shall establish several approximation theoretic properties of M-ideals. Also, some preparatory work for Section II.3 will be done where we shall study the difference between M-ideals and M-summands in some detail.

Let us first recall some notions from approximation theory. A subspace J of a Banach space X is called *proximinal* if

$$\forall\, x \in X \quad \exists\, y \in J \quad \|x-y\| = d(x,J)\,.$$

The set of all such y is called the set of best approximants and denoted by $P_J(x)$. Thus J is proximinal if and only if $P_J(x) \neq \emptyset$ for all $x \in X$. The set-valued map P_J is called the *metric projection*. J is called a *Chebyshev* subspace if $P_J(x)$ is a singleton for each $x \in X$. Finally, the *metric complement* J^θ is defined as

$$J^\theta = \{x \in X \mid \|x\| = d(x,J)\} = \{x \in X \mid 0 \in P_J(x)\}.$$

To begin with, let us study these concepts for an M-summand J in X. P denotes the M-projection with range J.
First of all, J is proximinal, and $P_J(x)$ is a ball with radius $d(x,J) = \|x - Px\|$ and centre Px. To see this note that for any $y \in J$

$$\|y-x\| = \max\{\|y-Px\|, \|x-Px\|\}.$$

As a consequence, $\operatorname{lin} P_J(x) = J$ for each $x \in X \backslash J$. So J is highly non-Chebyshev. Another consequence is the existence of a continuous linear selection for P_J, namely P.

(A *selection* f of a set-valued map F is a function such that $f(x) \in F(x)$ for all x.) Furthermore,

$$\{x \in X \mid \|Px\| < \|x\|\}$$

is an open subset of J^θ.

We wish to examine the validity of these results for M-ideals J. Naturally, one has to face certain limitations. For example, the M-ideal c_0 in ℓ^∞ is easily seen to be proximinal (this is not accidental, cf. Proposition 1.1 below), but

$$P_{c_0}(\mathbf{1}) \quad = \quad \{(s_n) \in c_0 \mid |s_n - 1| \le 1 \ \text{ for all } \ n\}$$

is not a ball (though still "big"), there is no continuous linear selection for P_{c_0} (since there is no continuous linear operator whatsoever from ℓ^∞ onto c_0), and the metric complement $c_0^\theta = \{(s_n) \in \ell^\infty \mid \|(s_n)\| = \limsup |s_n|\}$ has empty interior.

Proposition 1.1 *M-ideals are proximinal.*

PROOF: Let $J \subset X$ be an M-ideal, and let $x \in X$ with $d = d(x, J) > 0$. We shall inductively construct a sequence (y_n) in J with

$$\|y_{n+1} - y_n\| \le \left(\frac{3}{4}\right)^n \tag{1}$$

$$\|y_n - x\| \le d + \left(\frac{3}{4}\right)^{n-1} \tag{2}$$

for $n \ge 1$. Once this is achieved, (1) yields that (y_n) is a Cauchy sequence, while (2) gives $\|\lim y_n - x\| \le d$, whence $P_J(x) \neq \emptyset$. To show how the induction proceeds let $\varepsilon > 0$. First, choose $y \in J$ satisfying $\|y - x\| \le d + \varepsilon$. Then consider the balls $B(x, d + \varepsilon/2)$ and $B(y, \varepsilon/2)$. Since the distance of the centres does not exceed the sum of the radii, they have a point in common, and both balls meet J. Now J has the 2-ball property (Theorem I.2.2), so there is

$$z \in J \cap B(x, d + 3\varepsilon/4) \cap B(y, 3\varepsilon/4).$$

Applying this procedure with the sequence $(\varepsilon_n) = ((\frac{3}{4})^{n-1})$ we obtain the desired y_n. $\Box$

Next, we are going to discuss the largeness of the set of best approximants. We shall meet the following notion again in Section II.3.

Definition 1.2 *A closed convex bounded subset B of a Banach space Y is called a pseudoball of radius r if its diameter is $2r > 0$ and if for each finite collection $y_1, \ldots, y_n$ of points with $\|y_i\| < r$ there is $y \in B$ such that*

$$y + y_i \in B \ \text{ for } i = 1, \ldots, n.$$

If $\rho = \sup\{s \ge 0 \mid B \ \text{ contains a ball of radius } \ s\}$, then the grade of B is defined to be

$$g(B) = 1 - \frac{\rho}{r}.$$

Singletons are considered as pseudoballs with radius 0 and grade 0.

Equivalently B is a pseudoball of radius $r > 0$ if and only if

$$\bigcap_i \ (y_i + B) \neq \emptyset$$

for each finite family $y_1, \dots, y_n$ satisfying $\|y_i\| < r$.
Note that B is a closed ball of radius $r > 0$ if and only if

$$\bigcap_{\|y\|<r} (y + B) \neq \emptyset,$$

in which case the intersection consists of the centre of the ball. In a pseudoball there is only a "centre" for any finite set of directions. Also, note that $g(B) = 0$ if and only if B is a ball. Thus, the larger the grade of B is the more proper a pseudoball is B. The idea of properness for M-ideals will be studied in Section II.3.

To see a simple example, let $0 \leq s \leq 1$. It is easy to check that

$$B_s = \ \{(s_n) \in c_0 \ \mid \ |s_n - s| \leq 1 \ \text{ for all } \ n \in \mathbb{N}\}$$

is a pseudoball in c_0 with radius 1 and $g(B_s) = s$.

The following result explains our interest in pseudoballs here.

Proposition 1.3

(a) *Let J be an M-ideal in X, and let $x \in X$. Then $P_J(x)$ is a pseudoball in J with radius $d(x, J)$.*

(b) *If $P_J(x)$ is a pseudoball of radius $d(x, J)$ for all $x \in X$, then J is an M-ideal.*

Proof: (a) Obviously, $P_J(x)$ is a closed convex set with diameter $\leq 2 \cdot d(x, J)$. There is no loss in generality in assuming $d(x, J) = 1$. So, let $y_1, \dots, y_n \in \operatorname{int} B_J$ be given. Then we have

$$P_J(x + y_i) = J \cap B(x + y_i, 1) \neq \emptyset \tag{1}$$

since J is proximinal (Proposition 1.1), and

$$\operatorname{int} \bigcap_i B(x + y_i, 1) \neq \emptyset \tag{2'}$$

since $\|y_i\| < 1$. By Theorem I.2.2(v) one concludes

$$\bigcap_i \ (y_i + P_J(x)) = J \ \cap \ \bigcap_i B(x + y_i, 1) \neq \emptyset.$$

Hence $P_J(x)$ is a pseudoball of radius $d(x, J)$.

(b) We shall verify the 3-ball property (Th. I.2.2(iv)). Let x, y_i, ε be given as in I.2.2(iv). Let $r < d(x, J) < r + \varepsilon$. Then there is $y \in J$ such that

$$-ry_i + y \in P_J(x), \quad i = 1, 2, 3.$$

Hence
$$\|x + y_i - y\| \leq d(x, J) + (1 - r)\|y_i\| \leq 1 + \varepsilon. \qquad \square$$

We shall prove in Theorem 3.10 that every pseudoball arises in this way. This and the preceding proof show that B is a pseudoball if there is a "centre" for any 3 directions.

Proposition 1.4 *Let $B \subset Y$ be a pseudoball of radius r. Then*
$$\operatorname{int} B(0, r) \ \subset \ \frac{1}{2}(B - B) \ \subset \ B(0, r).$$

PROOF: Let $y \in Y$, $\|y\| < r$. Choose
$$z \in (-y + B) \cap (y + B).$$

Then
$$y = \frac{1}{2}((y + z) - (-y + z)) \in \frac{1}{2}(B - B).$$
The other inclusion is clear. $\square$

Combining Propositions 1.3 and 1.4 we arrive at the following stunning approximation properties of M-ideals.

Corollary 1.5 *If J is an M-ideal in X and $x \in X \backslash J$, then every element in J can be represented as a linear combination of two points from $P_J(x)$. More precisely,*
$$\operatorname{int} B_J(0, d(x, J)) \quad \subset \quad \frac{1}{2}(P_J(x) - P_J(x)) \quad \subset \quad B_J(0, d(x, J)).$$
In particular, $P_J(x)$ is compact if and only if J is finite dimensional.

Thus M-ideals are far from being Chebyshev subspaces. As a result, no strictly convex space can contain a nontrivial M-ideal (otherwise the corresponding metric projection would have to be single-valued).

Before we continue investigating the metric projection in more detail we want to give a characterisation of pseudoballs in terms of the bidual space.

Theorem 1.6 *For a closed convex bounded subset $B \subset Y$ and $r \geq 0$, equivalence between* (i) *and* (ii) *holds:*

(i) *B is a pseudoball of radius r.*

(ii) *The weak* closure of B in Y^{**} is a ball with radius r (and centre y_B^{**}, say).*

In this case
$$r \cdot g(B) = d(y_B^{**}, Y). \qquad (*)$$

PROOF: We may assume $r = 1$.
(i) $\Rightarrow$ (ii): Let $K = \overline{B}^{w*}$. It is easy to check that for any $y^* \in Y^*$ the set $y^*(K) = \overline{y^*(B)}$ is a pseudoball in $\mathbb{K}$ with radius $\|y^*\|$, and it is easy to prove, too, that pseudoballs in $\mathbb{K}$ are actually balls. For $y^* \in Y^*$ let $\ell(y^*)$ denote the centre of the ball $y^*(K)$.

CLAIM: ℓ is a continuous linear functional on Y^*.

Obviously ℓ is homogeneous. To prove additivity choose, given $\varepsilon > 0$ and $y_1^*, y_2^* \in Y^*$, $y_1, y_2, y_3 \in Y$ such that

$$\begin{array}{llllll} \|y_1\| & < & 1, & \mathbb{R} \ni & \langle y_1^*, y_1\rangle & \geq (1-\varepsilon^2/2) \quad \|y_1^*\| \\ \|y_2\| & < & 1, & \mathbb{R} \ni & \langle y_2^*, y_2\rangle & \geq (1-\varepsilon^2/2) \quad \|y_2^*\| \\ \|y_3\| & < & 1, & \mathbb{R} \ni & \langle y_1^*+y_2^*, y_3\rangle & \geq (1-\varepsilon^2/2) \quad \|y_1^*+y_2^*\|. \end{array}$$

We infer

$$|\langle y_1^*, y\rangle - \ell(y_1^*)| \quad \leq \quad \varepsilon\|y_1^*\|$$

and in the same way

$$\begin{aligned} |\langle y_2^*, y\rangle - \ell(y_2^*)| \quad &\leq \quad \varepsilon\|y_2^*\| \\ |\langle y_1^*+y_2^*, y\rangle - \ell(y_1^*+y_2^*)| \quad &\leq \quad \varepsilon\|y_1^*+y_2^*\|, \end{aligned}$$

from which the additivity of ℓ follows. The continuity of ℓ is clear.
Let us write y_B^{**} instead of ℓ. It is left to show that

$$K = B_{Y^{**}}(y_B^{**}, 1).$$

This is an immediate consequence of the Hahn-Banach theorem, since

$$y^*(K) \quad = \quad B_{\mathbb{K}}(\langle y_B^{**}, y^*\rangle,\ \|y^*\|) \quad = \quad y^*(B_{Y^{**}}(y_B^*, 1))$$

for all $y^* \in Y^*$.

(ii) $\Rightarrow$ (i): A moment's reflection shows that we have to prove

$$\operatorname{int}(B_Y \times \cdots \times B_Y) \subset B \times \cdots \times B - \Delta_B$$

where

$$\Delta_B = \{(y, \ldots, y) \mid y \in B\} \subset Y^n$$

and $n \in \mathbb{N}$ is arbitrary. By a separation theorem due to Tukey [604] (see also [178, p. 461] or [336, Cor. 22.5]) this will follow from

$$B_Y \times \cdots \times B_Y \quad \subset \quad \overline{B \times \cdots \times B - \Delta_B}. \tag{1}$$

To prove this inclusion we first note

$$\overline{B}^{w*} \times \cdots \times \overline{B}^{w*} - \Delta_{\overline{B}^{w*}} \quad \subset \quad \overline{B \times \cdots \times B - \Delta_B}^{w*} \tag{2}$$

thanks to the weak* continuity of $(y_1^{**}, \ldots, y_n^{**}, y^{**}) \mapsto (y_1^{**} - y^{**}, \ldots, y_n^{**} - y^{**})$. Now $\overline{B}^{w*}$ is a ball of radius 1 so that the left hand side of (2) contains $B_{Y^{**}} \times \cdots \times B_{Y^{**}}$ and a fortiori $B_Y \times \cdots \times B_Y$.
The desired inclusion easily follows from this and the Hahn-Banach theorem. It remains to prove formula $(*)$. We first observe

$$B_{Y^{**}}(y_B^{**}, 1) \cap Y = B.$$

This and the triangle inequality give

$$B(y, 1 - \|y_B^{**} - y\|) \subset B$$

and hence

$$g(B) \leq \|y_B^{**} - y\|$$

for any $y \in Y$. This shows "$\leq$" in $(*)$.

Conversely, if a ball $B(y,s)$ is contained in B, then

$$B_{Y^{**}}(y,s) \subset B_{Y^{**}}(y_B^{**}, 1).$$

This and the triangle inequality give

$$\|y_B^{**} - y\| \leq 1 - s$$

and hence $d(y_B^{**}, Y) \leq g(B)$. □

Corollary 1.7 *Let J be an M-ideal in X, and let P be the associated M-projection from X^{**} onto $J^{\perp\perp}$. Consider the pseudoball $B = P_J(x)$ for some $x \in X$. Then $Px = y_B^{**}$, the centre of $\overline{B}^{w*}$. More precisely: the canonical isometry i_J^{**} from J^{**} onto $J^{\perp\perp}$ maps y_B^{**} onto Px. Moreover (if $d(x,J) = 1$)*

$$g(P_J(x)) = d(Px, J) = \sup\{|\langle y^*, x\rangle| \mid y^* \in B_{J^*}\}.$$

PROOF: We consider J^* via unique Hahn-Banach extensions as a subspace of X^* (Remark I.1.13). We have for $y^* \in J^*$

$$y^*(B_{J^{**}}(y_B^{**}, 1)) = B_{\mathbb{K}}(\langle y_B^{**}, y^*\rangle, \|y^*\|).$$

On the other hand we know from Theorem 1.6 (w.l.o.g. $d(x,J) = 1$)

$$\begin{aligned} B_{\mathbb{K}}(\langle y_B^{**}, y^*\rangle, \|y^*\|) &= \overline{y^*(B)} \\ &\subset \overline{y^*(B_X(x,1))} \\ &= B_{\mathbb{K}}(\langle y^*, x\rangle, \|y^*\|). \end{aligned}$$

Hence $\langle y_B^{**}, y^*\rangle = \langle x, y^*\rangle$ for all $y^* \in J^*$, which gives $y_B^{**} = Px$. □

We now return to best approximation and investigate the problem of whether there is a continuous (possibly linear) selection for the metric projection. We recall the definition of the Hausdorff metric d_H on the set of closed subsets of X:

$$d_H(A,B) = \sup(\{d(a,B) \mid a \in A\} \cup \{d(b,A) \mid b \in B\})$$

Proposition 1.8 *The metric projection associated to an M-ideal J in X is Lipschitz continuous with respect to the Hausdorff metric. More precisely*

$$d_H(P_J(x_1),\ P_J(x_2)) \leq 2 \cdot \|x_1 - x_2\| \qquad \text{for all } x_1, x_2 \in X.$$

PROOF: The assertion is clearly true if either of x_1 or x_2 is in J. For the general case we have to show: given $x_1, x_2 \in X$, $\varepsilon > 0$, and $y_2 \in P_J(x_2)$, there is $y_1 \in P_J(x_1) = J \cap B(x_1, d(x_1, J))$ such that

$$\|y_1 - y_2\| \quad \leq \quad 2 \cdot \|x_1 - x_2\| + \varepsilon.$$

By the strict 2-ball property (Theorem I.2.2(v)) this will be assured by

$$\|x_1 - y_2\| < 2 \cdot \|x_1 - x_2\| + \varepsilon + d(x_1, J) \tag{$*$}$$

(since int $(B(a_1, r_1) \cap B(a_2, r_2)) \neq \emptyset$ if $\|a_1 - a_2\| < r_1 + r_2$ and $r_1, r_2 > 0$). Finally, $(*)$ is a consequence of

$$\begin{aligned} \|x_1 - y_2\| &\leq \|x_1 - x_2\| + \|x_2 - y_2\| \\ &= \|x_1 - x_2\| + d(x_2, J) \\ &\leq \|x_1 - x_2\| + d(x_2 - x_1, J) + d(x_1, J) \\ &< 2 \cdot \|x_1 - x_2\| + d(x_1, J) + \varepsilon. \end{aligned}$$

□

REMARKS: (a) The constant 2 is optimal here. In fact, let $X = \ell^\infty$, $J = c_0$, $x_1 = (0, 0, 0, \ldots)$, $x_2 = (1, 1, 1, \ldots)$, $y_2 = (2, 0, 0, \ldots)$. Then

$$d_H(P_J(x_1),\ P_J(x_2)) \quad \geq \quad d(y_2, P_J(x_1)) \quad = \quad \|y_2 - x_1\| = 2.$$

(b) A similar argument shows

$$d_H(x_1 - P_J(x_1), x_2 - P_J(x_2)) \leq d_H(x_1 - x_2, J).$$

Theorem 1.9 *Let J be an M-ideal in X.*

(a) *There exists a continuous homogeneous mapping $\pi : X \to J$ with $\pi(x) \in P_J(x)$ for all $x \in X$ (i.e. a continuous selection for P_J) which is quasiadditive, meaning $\pi(x + y) = \pi(x) + y$ for $x \in X$, $y \in J$.*

(b) *There exists a continuous homogeneous mapping $f : X/J \to X$ with $f(x+J) \in x + J$ and $\|f(x + J)\| = \|x + J\|$ for all $x \in X$ (i.e. a continuous norm preserving lifting for the quotient map).*

PROOF: Consider the set-valued homogeneous mapping Σ on X/J defined by

$$\Sigma\,(x + J) \quad = \quad \{x\} - P_J(x).$$

It is well-defined, and by Proposition 1.8 Σ is continuous for the Hausdorff metric, a fortiori it is lower semicontinuous (that is to say for an open set U, the set $\{x + J \mid \Sigma(x + J) \cap U \neq \emptyset\}$ is open in X/J). Furthermore the $\Sigma(x + J)$ are closed, convex and bounded. Of course, the same is true for the restriction Σ_0 of Σ to the unit sphere in X/J so that Michael's selection theorem [442, Theorem 3.2$''$] applies to yield a continuous selection

σ_0 for Σ_0. The technique described in [442, p. 376] permits the extension of σ_0 to a homogeneous selection σ for Σ on X/J.
Now, (a) is achieved by letting $\pi(x) = x - \sigma(x+J)$, and (b) by $f = \sigma$. □

Note that (a) and (b) above are in complete duality: given π as in (a), $f(x+J) = x - \pi(x)$ fulfills the requirements of (b), and given f as in (b), $\pi(x) = x - f(x+J)$ fulfills (a).

The question of whether the mappings π and f above can be chosen to be Lipschitzian (or even linear) arises immediately. Let us first address the question of Lipschitz projections. This problem remains open. It is pointed out in [510] and [655] that a Lipschitz set-valued map need not have a Lipschitz selection (see the Notes and Remarks); on the other hand, there does exist a 2-Lipschitz projection from $\ell^\infty_{\mathbb{R}}$ onto c_0 [414]. Here we give a lower estimate for the Lipschitz constants of such projections in terms of the grade of certain pseudoballs.

Proposition 1.10 *Suppose J is an M-ideal in X. Let*

$$g^*(J,X) := \sup\{g(P_J(x)) \mid d(x,J) = 1\}.$$

If there exists a Lipschitz projection π from X onto J, then its Lipschitz constant L is at least $2 \cdot g^*(J,X)$.

The number $g^*(J,X)$ defined above will be put into its proper milieu in the Notes and Remarks section. It will be remarked there that $g^*(J,X) = 1$ in a number of cases, e.g. for the familiar M-ideals J_D in $C(K)$ if D is not clopen, for $J = K(H)$ in $X = L(H)$ or, more generally, in the case $X = J^{**}$. On the other hand, clearly $g^*(J,X) = 0$ if J is an M-summand.

Proof: Let $x \in X$, $d(x,J) = 1$, and consider $B = P_J(x) = B_X(x,1) \cap J$. Then

$$B \;=\; \pi(B) \;\subset\; \pi(B_X(x,1)) \;\subset\; B_J(\pi(x), L).$$

Now we take $\sigma(J^{**}, J^*)$-closures and use Proposition 1.3, Theorem 1.6, and Corollary 1.7 to obtain

$$B_{J^{**}}(Px, 1) \;\subset\; B_{J^{**}}(\pi(x), L)$$

where P is the M-projection from X^{**} onto $J^{\perp\perp} \cong J^{**}$. Hence

$$2Px - y \;\in\; B_{J^{**}}(\pi(x), L) \quad \text{for all } \; y \in B$$

so that by Theorem 1.6 and Corollary 1.7

$$\begin{aligned} L \;&\geq\; 2 \cdot \|Px - (y + \pi(x))/2\| \\ &\geq\; 2 \cdot d(Px, J) \\ &=\; 2 \cdot g(B). \end{aligned}$$

□

The grade of the pseudoballs $P_J(x)$ is the appropriate tool for investigating the (possibly empty) interior of the metric complement J^θ.

Proposition 1.11 *Let J be an M-ideal in X, and let*

$$D = \{x \in X \mid \|x\| = d(x, J) = 1\}.$$

(a) *If $B_J(y_0, 2r) \subset P_J(x_0)$ for some $x_0 \in D$, then $B_X(x_0 - y_0, r) \subset J^\theta$.*

(b) *If $B_X(x_0, r) \subset J^\theta$ for some $x_0 \in D$, then $P_J(x_0)$ contains a ball of radius $2r - \varepsilon$ for whatever $\varepsilon > 0$.*

Therefore, J^θ has empty interior if and only if

$$g_*(J, X) := \inf\{g(P_J(x)) \mid d(x, J) = 1\} = 1.$$

PROOF: (a) We may assume $y_0 = 0$. Define

$$|x| = \sup\{|\langle x, y^*\rangle| \ \mid \ y^* \in B_{J^*}\}$$

(cf. Remark I.1.13). This is a seminorm, and from Lemma I.1.5 we get

$$\|x\| = \max\{|x|,\ d(x, J)\}. \tag{1}$$

Now let $z \in X$, $\|z - x_0\| < r$. We wish to show $|z| < d(z, J)$ (so that $z \in J^\theta$), then (a) will follow from the closedness of J^θ.
First note $|z - x_0| < r$ and $d(z - x_0, J) < r$ by (1). Secondly,

$$|x_0| \le 1 - 2r. \tag{2}$$

Proof of (2): Given $\varepsilon > 0$ choose $y^* \in S_{J^*}$ such that

$$|x_0| \le \langle y^*, x_0\rangle + \varepsilon.$$

Next choose $y \in B_J$ such that

$$\langle y^*, y\rangle \le -1 + \varepsilon.$$

Note $2ry \in P_J(x_0)$, i.e. $\|x_0 - 2ry\| = 1$. The estimate

$$\begin{aligned} |x_0| &\le \langle y^*, x_0\rangle + \varepsilon \\ &= \langle y^*, x_0 - 2ry\rangle + 2r\langle y^*, y\rangle + \varepsilon \\ &\le 1 + 2r(-1 + \varepsilon) + \varepsilon \end{aligned}$$

now proves (2).
Altogether one obtains

$$|z| \ < \ |x_0| + r \ \le \ 1 - r \ = \ d(x_0, J) - r \ < \ d(z, J)$$

as desired.

(b) Let P be the M-projection from X^{**} onto $J^{\perp\perp} \cong J^{**}$. In view of Theorem 1.6 and Corollary 1.7 we have to show $d(Px_0, J) \le 1 - 2r$. This is equivalent to

$$|\langle y^*, x_0\rangle| \ \le \ 1 - 2r \qquad \text{for all } \ y^* \in B_{J^*}.$$

To prove this for some given $y^* \in S_{J^*}$, fix $\varepsilon > 0$ and choose $y \in \operatorname{int} B_J$ such that $\langle y^*, y\rangle$ is real and so close to 1 that

$$|\langle y^*, y\rangle \pm \alpha| \leq 1 \text{ only if } |\alpha| \leq \varepsilon.$$

In particular $\langle y^*, y\rangle \geq 1 - \varepsilon$.
Since $P_J(x_0)$ is a pseudoball (Proposition 1.3), we may find $\overline{y} \in P_J(x_0)$ with $\overline{y} \pm y \in P_J(x_0)$. Let us first observe

$$|\langle y^*, \overline{y}\rangle - \langle y^*, x_0\rangle| \;\leq\; \varepsilon. \tag{3}$$

In fact,

$$\langle y^*, \overline{y}\rangle \pm \langle y^*, y\rangle \;\in\; y^*(P_J(x_0)) \;\subset\; B_{\mathbb{K}}(\langle y^*, x_0\rangle, 1)$$

(cf. Theorem 1.6) so that (3) follows. On the other hand, for $z = (1-r)x_0 + r\overline{y}$ we have

$$y^*(z) \;=\; y^*(z + ry) - ry^*(y) \;\leq\; (1-r) - r(1-\varepsilon) \tag{4}$$

where we used $\|x_0 - (z + ry)\| \;\leq\; r$ (since $\overline{y} + y \in P_J(x_0)$), consequently

$$\|z + ry\| \;=\; d(z + ry, J) \;=\; d((1-r)x_0, J) \;=\; 1 - r.$$

(3) and (4) together give our claim. $\square$

With the help of Remark 3.8(d) below we conclude from Proposition 1.11 for example that J^θ has empty interior if J is an M-ideal in J^{**}. Elementary calculations show that $J_D^\theta \;\subset\; C(K)$ has empty interior if and only if D has empty interior.

II.2 Linear projections onto M-ideals

In this section we shall investigate whether there are *linear* mappings π and f in the setting of Theorem 1.9 above. Since there are noncomplemented M-ideals (e.g. c_0 in ℓ^∞; an entirely elementary argument for this fact is contained in [633]) we cannot generally expect such a linear projection or lifting to exist. We next present a sufficient condition. It involves Grothendieck's approximation property which we now recall.

Let X be a Banach space. Then X has the *approximation property (AP)* if $Id \in \overline{F(X)}$, the closure being taken with respect to the topology τ of uniform convergence on compact sets. We say that X has the λ-*approximation property* (λ-*AP*) where $\lambda \geq 1$ if $Id \in \overline{B_{F(X)}(0, \lambda)}^{\tau}$. Usually the 1-AP is called *metric approximation property (MAP)*. Finally X has the *bounded approximation property (BAP)* if it has the λ-AP for some λ.
If one requires only approximation by compact operators instead of finite rank operators, X is said to have the *compact approximation property (CAP)* resp. λ-*compact approximation property* (λ-*CAP)* etc.

We remark that the closure with respect to τ in the definition of the λ-AP and λ-CAP may be replaced by the closure in the strong operator topology or even the weak operator topology (note that the latter two topologies yield the same dual space [178, p. 477]).

For a detailed exposition of the approximation properties see [158], [280], [421], [422]. We mention that the classical $L^p(\mu)$- and $C(K)$-spaces enjoy the MAP. However, each of the spaces ℓ^p ($1 \le p < \infty$, $p \ne 2$) and c_0 has a closed subspace failing the MCAP (even the CAP) [422, p. 107 and passim], [354].
The main result of this section, due to Ando and, independently, Choi and Effros, is as follows.

Theorem 2.1 *Suppose J is an M-ideal in the Banach space X, Y is a separable Banach space, and $T \in L(Y, X/J)$ with $\|T\| = 1$. Assume further*

(a) *Y has the BAP*

or

(b) *J is an L^1-predual.*

Then there is a continuous linear lifting L for T, i.e. $L \in L(Y, X)$ such that $qL = T$ where $q : X \to X/J$ denotes the quotient map. More precisely, we obtain a lifting with

$$\|L\| \le \lambda \qquad \text{if } Y \text{ has the } \lambda\text{-AP},$$

resp.

$$\|L\| = 1 \qquad \text{under assumption (b)}.$$

Before giving the proof we would like to make some remarks.

Remarks 2.2 (a) Instead of the separability of Y and (a) or (b) one may of course use the weaker assumptions

(a) *T factors through a separable space with the BAP*

resp.

(b′) *J is an L^1-predual, and T has a separable range.*

(b) If Theorem 2.1 is applicable to $Y = X/J$ and L is a lifting for Id_Y, then Lq is a continuous linear projection whose kernel is J so that J is a complemented subspace of X.

(c) One might wonder to what extent the additional assumptions in Theorem 2.1 are really needed. As for the separability one just has to activate the standard example of the noncomplemented M-ideal c_0 in ℓ^∞. (In view of the examples in [137] or [417, Prop. 3.5] Theorem 2.1 does not even extend to weakly compactly generated spaces Y.) Also, it is not enough to assume that $J^\perp$ is norm one complemented by some projection as is shown by the example of a subspace J of ℓ^1 such that $\ell^1/J = L^1[0,1]$. As for the approximation property we first note an easy proposition.

Proposition 2.3 *For every separable Banach space Y there are a separable Banach space X enjoying the MAP and an M-ideal J in X such that $Y \cong X/J$.*

Applying Proposition 2.3 to a space Y without the BAP we infer that there can be no continuous linear lifting for the quotient map, because otherwise Y would be isomorphic to a complemented subspace of X and hence would have the BAP.

PROOF OF PROPOSITION 2.3: We let (E_n) be an increasing sequence of finite dimensional subspaces of Y such that $\bigcup E_n$ is dense and define

$$\begin{aligned} X &= \{(x_n) \mid x_n \in E_n, \quad \lim x_n \text{ exists}\}, \\ J &= \{(x_n) \mid x_n \in E_n, \quad \lim x_n = 0\}. \end{aligned}$$

(These spaces are, of course, equipped with the sup norm.)
It is quickly verified that Y is isometric to X/J. Moreover, J is an M-ideal. To prove this we use the 3-ball property (Th. I.2.2(iv)). In fact, given normed vectors $\xi = (x_n) \in X$, $\eta_i = (y_n^i) \in J$ $(i = 1,2,3)$ and $\varepsilon > 0$, choose N such that

$$\|y_n^i\| \le \varepsilon \qquad \text{for } n > N \text{ and } i = 1,2,3.$$

If $\eta = (y_n)$ with $y_n = x_n$ $(n \le N)$, $y_n = 0$ $(n > N)$, then

$$\|\xi + \eta_i - \eta\| \le 1 + \varepsilon.$$

It remains to notice that X has the MAP since the contractive finite rank projections $P_m : (x_n) \mapsto (x_1, \dots, x_m, x_m, x_m, \dots)$ converge strongly to the identity. □

We now proceed to the proof of Theorem 2.1. The essential part of the argument is purely finite dimensional, and we shall present it in the following lemma.

Lemma 2.4 *Let X be a Banach space, $J \subset X$ an M-ideal and $q : X \to X/J$ the quotient map. Suppose further that E is a finite dimensional space and $F \subset E$ is a subspace. Let $T \in L(E, X/J)$ with $\|T\| = 1$. We assume*

(a) *there exists a contractive projection π from E onto F*

or

(b) *J is an L^1-predual.*

Then, given a contractive linear lifting $L_F : F \to X$ for $T|_F$ and $\varepsilon > 0$, there exists a contractive linear lifting $L_E : E \to X$ for T such that $\|L_E|_F - L_F\| \le \varepsilon$.

For the time being, let us take this lemma for granted and give the

PROOF OF THEOREM 2.1:

We will first assume (b). Let (E_n) be an increasing sequence of finite dimensional subspaces of Y such that its union is dense. Let $E_0 = \{0\}$ and $L_0 = 0$. Using Lemma 2.4 one may inductively define a sequence $L_n : E_n \to X$ of contractions such that

$$qL_n = T|_{E_n} \qquad \text{and} \qquad \|L_n|_{E_{n-1}} - L_{n-1}\| \le 2^{-n}$$

for all $n \in \mathbb{N}$. For $y \in \bigcup E_n$ the sequence $(L_n y)$ is eventually defined and Cauchy. Hence

$$Ly := \lim_{n\to\infty} L_n y \tag{$*$}$$

exists for these y, and $(*)$ defines a linear operator which can be extended to a contraction (still called L) from Y to X. It is clear that $qL = T$, i.e. L is a lifting.

The proof using assumption (a) is more delicate. We let $c(X)$, $c(X/J)$, ... be the sup-normed spaces of convergent sequences of vectors in X, X/J, ... Then T induces a contraction $c(T) : c(Y) \to c(X/J)$ defined by

$$c(T)((y_n)_n) = (Ty_n)_n.$$

Further note that $c(q) : c(X) \to c(X/J)$, defined in the same vein, is a quotient map with kernel $c(J)$ so that

$$c(X/J) \cong c(X)/c(J).$$

Finally we remark that $c(J)$ is an M-ideal in $c(X)$ as can easily be verified either by means of the 3-ball property (Theorem I.2.2) or by directly checking the definition. (This is a special case of Proposition VI.3.1.)

Now that we know that Y is a separable space with the λ-AP there is a sequence of finite rank operators (S_n) converging strongly to Id_Y with $\|S_n\| \le \lambda$. Next we define an auxiliary subspace $H \subset c(Y)$ as the closed linear span of the sequences

$$(S_1 y, \dots, S_{m-1} y, S_m y, S_m y, \dots)$$

where $m \in \mathbb{N}$, $y \in Y$. Note $(S_n y)_n \in H$ for all $y \in Y$. For $m \in \mathbb{N}$ and $(y_n)_n \in H$ we define

$$\pi_m((y_n)_n) = (y_1, \dots, y_{m-1}, y_m, y_m, y_m, \dots).$$

Obviously, the π_m form an increasing sequence of contractive finite rank projections on H converging strongly to Id_H. If we let $E_m = \operatorname{ran}(\pi_m)$ and use the same technique as in the first part (based on Lemma 2.4(a) applied to the M-ideal $c(J)$ in $c(X)$) we obtain a contractive linear lifting

$$\Lambda : H \to c(X)$$

for $c(T)|_H$, i.e.

$$c(q)\Lambda = c(T)|_H.$$

To get the desired lifting for T we define

$$Ly = \text{limit } \Lambda((S_n y)_n).$$

Then

$$\|L\| \le \|\Lambda\| \cdot \sup_n \|S_n\| \le \lambda$$

and

$$\begin{aligned} q(Ly) &= \text{limit } c(q)(\Lambda((S_n y)_n)) \\ &= \text{limit } (c(q)\Lambda)((S_n y)_n) \\ &= \lim_n T(S_n y) \\ &= Ty. \end{aligned}$$

This concludes the proof of Theorem 2.1. □

Proof of Lemma 2.4: We shall work with the Banach space $L(E,X)$. There is a canonical isometry (since $\dim E<\infty$)

$$L(E,X)^{**}\cong L(E,X^{**}) \tag{$*$}$$

which we shall employ in the sequel. ($(*)$ is a special case of Grothendieck's duality theory of tensor products [280], but a direct argument is available in [152].) Let us consider the following subspaces of $L(E,X)$:

$$\begin{aligned} W &= \{S\in L(E,X)\mid \operatorname{ran}(S)\subset J\} \quad (\cong L(E,J)) \\ V &= \{S\in W\mid \ker(S)\supset F\} \end{aligned}$$

Using $(*)$ one checks

$$\begin{aligned} W^{\perp\perp} &= \{S\in L(E,X^{**})\mid \operatorname{ran}(S)\subset J^{\perp\perp}\} \quad (\cong L(E,J^{\perp\perp})) \\ V^{\perp\perp} &= \{S\in W^{\perp\perp}\mid \ker(S)\supset F\}. \end{aligned}$$

At this point we observe that W is an M-ideal in $L(E,X)$: in fact, if P denotes the M-projection from X^{**} onto $J^{\perp\perp}$, then $S\mapsto PS$ is the M-projection from $L(E,X^{**})$ onto $W^{\perp\perp}$. (We refer to Chapter VI for more general results.)
Now let $L_1\in L(E,X)$ be any extension of L_F such that $qL_1=T$ (this is possible since $\dim E<\infty$). Abbreviating $B_{L(E,X)}$ by B we claim

$$L_1\in\overline{B+V}. \tag{$**$}$$

To prove this, we think of L_1 as an element of $L(E,X^{**})$ and will show

$$L_1\in\overline{B+V}^{w*}. \tag{$***$}$$

In fact, let P be as above. We will first show how assumption (a) entails $(***)$. We decompose L_1 as

$$L_1=((Id-P)L_1+PL_1\pi)+PL_1(Id-\pi)$$

and note

- $PL_1(Id-\pi)\in V^{\perp\perp}$
- $\|PL_1\pi\|\le\|P\|\cdot\|L_F\|\cdot\|\pi\|\le 1$
- $\|(Id-P)L_1\|=\|T\|=1$

(since $\operatorname{ran}(Id-P)\cong(X/J)^{**}$ and the diagram

$$\begin{tikzcd}
 & X/J \arrow[r, "i_{X/J}"] & (X/J)^{**} \\
E \arrow[ur, "T"] \arrow[dr, "L_1"'] & & \\
 & X \arrow[uu, "q"'] \arrow[r, "i_X"] & X^{**} \arrow[uu, "Id-P"']
\end{tikzcd}$$

commutes) so that

- $\|(Id-P)L_1+PL_1\pi\| = \max\{\|(Id-P)L_1\|,\ \|PL_1\pi\|\} \le 1$

(cf. Lemma VI.1.1(c)).
It follows that

$$L_1 \in B_{L(E,X^{**})} + V^{\perp\perp} = \overline{B}^{w*} + \overline{V}^{w*} = \overline{B+V}^{w*}.$$

This proves $(***)$ under assumption (a). If (b) is assumed we know that $J^{\perp\perp}$ is isometric to an L^∞-space and hence enjoys the Hahn-Banach extension property (cf. e.g. [385, p. 86ff.] for that matter). In particular, there is a contractive extension $\Lambda : E \to J^{\perp\perp}$ of $PL_F : F \to J^{\perp\perp}$. We now decompose L_1 as

$$L_1 = ((Id-P)L_1 + \Lambda) + (PL_1 - \Lambda)$$

and deduce $L_1 \in \overline{B+V}^{w*}$ as above.
Given $(**)$ we conclude the proof of the lemma as follows. By $(**)$ there are $S_1 \in B$, $S_2 \in V$ such that

$$\|L_1 - (S_1+S_2)\| \le \varepsilon/2.$$

Let $L_2 = L_1 - S_2$. Then L_2 is a lifting for T extending L_F which is nearly a contraction ($\|L_2\| \le 1+\varepsilon/2$). We wish to disturb L_2 so as to obtain a lifting which nearly extends L_F but *is* a contraction. The decisive tool to achieve this is an intersection property of M-ideals. To wit:

$$\begin{aligned} L_2 &\in (L_1+V)\cap(1+\varepsilon/2)B \\ &\subset \overline{B+V}\cap(1+\varepsilon/2)B \qquad\qquad \text{by } (**) \\ &\subset \overline{B+W}\cap(1+\varepsilon/2)B \\ &\subset B \ + \ \varepsilon(B\cap W) \end{aligned}$$

where we used Lemma 2.5 below in the last line. Thus there is a contraction L_E with

$$\|L_E - L_2\| \le \varepsilon \quad \text{and} \quad \operatorname{ran}(L_E - L_2) \subset J.$$

It follows

$$\|L_E|_F - L_F\| \le \varepsilon \quad \text{and} \quad qL_E = T,$$

as desired. ☐

It remains to prove:

Lemma 2.5 *For an M-ideal J in a Banach space X and $\varepsilon > 0$*

$$\overline{B_X + J}\cap(1+\varepsilon/2)B_X \ \subset \ B_X + \varepsilon B_J$$

holds.

PROOF: For $x \in \overline{B_X + J}$ certainly $d(x, J) \le 1$ holds. Since J is proximinal (1.1) we have $B(x,1) \cap J \ne \emptyset$, and, if $\|x\| \le 1 + \varepsilon/2$, $B(0,\varepsilon) \cap B(x,1)$ has nonempty interior. Therefore there is $y \in B(0,\varepsilon) \cap B(x,1) \cap J$ by the strict 2-ball property (Theorem I.2.2(v)), and $x = (x-y) + y \in B_X + \varepsilon B_J$, as requested. □

Theorem 2.1 contains several well-known results on the existence of linear extension operators as a special case. We give a sample.

Corollary 2.6 (Borsuk-Dugundji)
Let K be a compact Hausdorff space and let $D \subset K$ be a closed metrizable subset. Then there is a linear extension operator $T : C(D) \to C(K)$ with $\|T\| = 1$, i.e. $(Tx)(t) = x(t)$ for $x \in C(D)$ and $t \in D$.

PROOF: Consider the M-ideal $J_D = \{x \in C(K) \mid x_{|D} = 0\}$ (Example I.1.4(a)). Then $C(D) \cong C(K)/J_D$ meets the requirements of Theorem 2.1. (As a matter of fact, both (a) and (b) are fulfilled: $C(D)$ has the MAP, and J_D is an L^1-predual.) □

Corollary 2.7 (Pełczyński)
Let A be the disk algebra, and suppose D is a subset of the unit circle with Lebesgue measure 0. Then there is a contractive linear extension operator from $C(D)$ to A.

PROOF: Consider the M-ideal $J = \{x \in A \mid x_{|D} = 0\}$ (Example I.1.4(b)). By the Rudin-Carleson theorem [239, p. 58] we have $C(D) \cong A/J$, hence the result. □

More general corollaries can be formulated along the same lines on the basis of the Glicksberg peak interpolation theorem [239, p. 58, Th. 12.5 and Th. 12.7] and Theorem V.4.2 below.

Corollary 2.8 (Michael and Pełczyński, Ryll-Nardzewski)
Suppose $X \subset C(K)$ and $D \subset K$ is closed such that the pair $(X_{|D}, X)$ has the bounded extension property. If $X_{|D}$ is separable and has the MAP, then there is a contractive linear extension operator from $X_{|D}$ into X.

PROOF: This follows from Corollary I.1.20. □

A slightly different type of corollary is the following.

Corollary 2.9 (Sobczyk)
If X is a separable Banach space and $Y \subset X$ is a closed subspace isometric to c_0, then there is a continuous linear projection π from X onto Y with $\|\pi\| \le 2$.

PROOF: $Y^{\perp\perp}$, which is canonically isometric to Y^{**}, is isometric to ℓ^∞, and Y (more precisely $i_X(Y)$) is an M-ideal in $Y^{\perp\perp}$, since c_0 is an M-ideal in ℓ^∞. Since $Y^{\perp\perp}$ is isometric to ℓ^∞ there is a contractive projection P from X^{**} onto $Y^{\perp\perp}$. Hence $Y^\perp$ is the kernel of a contractive projection on X^*, viz. $Q = i_X^* P^* i_{X^*}$. (To see this check $\operatorname{ran}(Id - Q) \subset Y^\perp \subset \ker Q$ which shows $Q(Id - Q) = 0$ and $\ker Q = Y^\perp$.) We now renorm X^* so that Q becomes an L-projection:

$$|x^*| := \|Qx^*\| + \|x^* - Qx^*\|$$

i.e.

$$(X^*, |\,.\,|) = \operatorname{ran}(Q) \oplus_1 Y^\perp.$$

Unfortunately $|\,.\,|$ need not be a dual norm, therefore we cannot conclude directly that Y is an M-ideal in some renorming of X. However, we have with respect to the dual norm

$$(X^{**}, |\,.\,|) = \ker(Q^*) \oplus_\infty Y^{\perp\perp}$$

so that $Y^{\perp\perp}$ is an M-summand in $(X^{**}, |\,.\,|)$. Note that $|\,.\,|$ and $\|\,.\,\|$ coincide on $Y^{\perp\perp}$; it follows that Y is an M-ideal in $(X^{**}, |\,.\,|)$ (Prop. I.1.17(b)), a fortiori Y is an M-ideal in the intermediate space $(X, |\,.\,|)$. By Theorem 2.1 there is a contractive (with respect to $|\,.\,|$) linear lifting L of the identity map $Id_{X/Y}$ with respect to quotient map $q : X \to X/Y$ (Y is an L^1-predual!), consequently $\pi = Id - Lq$ is a linear projection onto Y. It remains to estimate the norm:

$$\|\pi(x)\| = |\pi(x)| \le |\pi| \cdot |x| \le 2 \cdot \|x\|$$

since $\|x^*\| \le |x^*|$ for all $x^* \in X^*$ whence $|x^{**}| \le \|x^{**}\|$ for all $x^{**} \in X^{**}$. □

We hasten to add that the above argument is probably the most complicated proof of Sobczyk's theorem that has appeared in the literature; a very simple one can be found in [421, Th. 2.f.5]. Reading our proof from the bottom to the top will, however, yield an interesting result in Chapter III (Theorem III.3.11).

We finish this section with a proposition which is contained in the proof of Corollary 2.9 and worth stating explicitly.

Proposition 2.10 *Let X be a Banach space and $Y \subset X$ a subspace isometric to c_0. Then there is an equivalent norm on X which agrees with the original norm on Y so that Y becomes an M-ideal.*

II.3 Proper M-ideals

This section in devoted to the study of the distinction between M-ideals and M-summands.

Definition 3.1 *An M-ideal which is not an M-summand is called a* proper M-ideal. *A Banach space Y* can be a proper M-ideal *if it is isometric to a proper M-ideal in a suitable superspace X.*

Trivially, a reflexive space Y cannot be a proper M-ideal, since in the decomposition

$$X^* = Y^\perp \oplus_1 Y^*$$

the complementary L-summand Y^* is reflexive, hence weak* closed. On the other hand, if Y is an M-ideal in Y^{**} and if Y is not reflexive, then it is a proper M-ideal. (If Y were an M-summand, then Y would be a weak* closed subspace of Y^{**} by Theorem I.1.9. Goldstine's theorem then yields $Y = Y^{**}$.)

The approximation theoretic results of Section II.1 imply the following useful criterion.

Proposition 3.2 *For a closed subspace $J \subset X$, the following assertions are equivalent:*
(i) *J is an M-summand.*
(ii) *$P_J(x)$ is a ball of radius $d(x, J)$ for all $x \in X$.*

PROOF: (i) $\Rightarrow$ (ii) is elementary (compare p. 49).
(ii) $\Rightarrow$ (i): First of all J is an M-ideal by Proposition 1.3. Let P denote the M-projection from X^{**} onto $J^{\perp\perp}$. By Theorem 1.6 and Corollary 1.7 we obtain $Px \in J$ for all $x \in X$ from (ii). This means that P maps X onto J. $\square$

Proposition 3.2 has an interesting consequence. The 3-ball property of Theorem I.2.2(iv) implies:

- J is an M-ideal in X iff J is an M-ideal in $\operatorname{lin}(J \cup \{x\})$ for all $x \in X$.

Proposition 3.2 shows the same equivalence for M-summands! Thus:

Corollary 3.3 *Y can be a proper M-ideal if and only if there is a Banach space X containing Y as a proper M-ideal and* $\dim X/Y = 1$.

Another consequence is a characterisation of M-summands by means of an intersection property.

Proposition 3.4 *For a subspace J in X, the following assertions are equivalent:*
(i) *J is an M-summand.*
(ii) *For all families $(B(x_i, r_i))_{i \in I}$ of closed balls satisfying*

$$B(x_i, r_i) \cap J \neq \emptyset \quad \textit{for all } i \in I \tag{1}$$

and

$$\bigcap_i B(x_i, r_i) \neq \emptyset \tag{2}$$

the conclusion

$$\bigcap_i B(x_i, r_i) \cap J \neq \emptyset$$

obtains.

PROOF: (i) $\Rightarrow$ (ii) is Lemma I.2.1.
(ii) $\Rightarrow$ (i): We show that $P_J(x)$ is a ball of radius 1 if $d(x, J) = 1$. This is the case if and only if

$$\bigcap_{\|y\| < 1} (P_J(x) + y) \neq \emptyset. \tag{$*$}$$

This set equals

$$\bigcap_{\substack{\|y\| < 1 \\ \varepsilon > 0}} B_X(x + y, 1 + \varepsilon) \cap J,$$

and $(*)$ follows since the above collection of balls satisfies (1) and (2). $\square$

We next describe a class of Banach spaces which cannot be proper M-ideals.

Proposition 3.5 *Suppose Y is norm one complemented in Y^{**}. Then Y cannot be a proper M-ideal.*

Proof: Suppose Y is an M-ideal in X. Let P denote the M-projection from X^{**} onto $Y^{\perp\perp}$ and Q a contractive projection from $Y^{**} \cong Y^{\perp\perp}$ onto Y. Then $QP|_X$ is a contractive projection from X onto Y, hence Y is an M-summand (Corollary I.1.3). □

We remark that the assumption of Proposition 3.5 is equivalent to saying that Y is a 1-complemented subspace of a dual; this is well known.

Corollary 3.6

(a) *If Y is isometric to a dual Banach space (or merely a 1-complemented subspace of a dual Banach space) then Y cannot be a proper M-ideal.*

(b) *A weak* closed M-ideal in a dual Banach space X^* is an M-summand which is the annihilator of an L-summand in X.*

Proof: (a) follows from Proposition 3.5, and (b) follows from (a), since the complementary M-summand is weak* closed by Theorem I.1.9. □

Remarks: (a) Proposition 3.5 and Corollary 3.6 are isometric results; the corresponding isomorphic versions are false in general. For example, consider $\ell^\infty = C(\beta\mathbb{N})$ and $t \in \beta\mathbb{N}\backslash\mathbb{N}$. Then the proper M-ideal $J_{\{t\}}$ is isomorphic to ℓ^∞ as is easily seen. (Compare, however, Corollary III.3.7(e).)

(b) Another argument for Corollary 3.6 appears in the Notes and Remarks section.

We are going to characterise Banach space which can be proper M-ideals intrinsically. It will be useful to introduce some notation. We first recall the notion of the characteristic $r(V, X^*)$ of a subspace V of a dual space X^*, which was introduced by Dixmier [163],

$$r(V, X^*) = \max\{r \geq 0 \mid rB_{X^*} \subset \overline{B_V}^{w*}\}.$$

Obviously, $0 \leq r(Y, X^*) \leq 1$. Also $r(V, V^{**}) = 1$ by Goldstine's theorem, and an application of the bipolar theorem shows

$$r(V, X^*) = \inf_{x \in S_X} \sup_{x^* \in B_V} |\langle x^*, x\rangle|.$$

We refer to [174, p. 42ff.] for detailed information.
We now define:

Definition 3.7

(a) *If J is a one-codimensional M-ideal in X then the* grade of J in X, *$g(J, X)$, is the number $r(J^*, X^*)$ in case J is proper and $g(J, X) = 0$ otherwise.*

(b) *For a Banach space Y we define the* grade of Y *by*

$$g(Y) = \sup g(Y, X)$$

where the sup is taken over all superspaces X of Y containing Y as a one-codimensional M-ideal. If there is such an X with $g(Y, X) = 1$ we call Y an extreme M-ideal.

Remarks 3.8 (a) Corollary 3.3 suggests studying one-codimensional M-ideals in order to find intrinsic characterisations.

(b) Note that $r(J^*, X^*)$ is well defined for proper one-codimensional M-ideals, since J^* is either weak* closed or weak* dense if $\dim(X/J) = 1$. (Recall from Remark I.1.13 that J^* is a subspace of X^* for M-ideals.) Also, $g(J, X) = 0$ only if J is an M-summand: $r(J^*, X^*) = 0$ implies that B_{J^*} and consequently J^* is $\sigma(X^*, X)$-closed since J^* has codimension one. (In general, there do exist weak* dense subspaces V with $r(V, X^*) = 0$, e.g. for $X = c_0$ [163], [174, p. 45]. As a matter of fact, this characterises non-quasireflexivity of X [150], [174, p. 99].) Thus we obtain the equivalence

$$Y \text{ can be a proper } M\text{-ideal} \quad \Longleftrightarrow \quad g(Y) > 0,$$

and the bigger $g(Y)$ is, the more proper an M-ideal can Y be.

(c) As an example consider the space X_r of convergent sequences equipped with the norm

$$\|(s_n)\|_r = \max\left\{\|(s_n)\|_\infty \,, \frac{1}{r}\lim |s_n|\right\}$$

where $0 < r \le 1$ and $\|\cdot\|_\infty$ is the usual sup-norm. An application of the 3-ball property (Theorem I.2.2(iv)) reveals that c_0 is an M-ideal in X_r (note that $\|\cdot\|_r = \|\cdot\|_\infty$ on c_0). Also, an easy computation shows $g(c_0, X_r) = r$. (Some readers might find the following representation of X_r as a G-space helpful:

$$X_r \cong \{x \in C(\{0\} \cup \alpha\mathbb{N}) \mid r \cdot x(0) = x(\infty)\}$$

where the latter space carries the sup-norm.)

(d) Suppose there exists $y^{**} \in Y^{**}$ such that Y is an M-ideal in $X = \operatorname{lin}(Y \cup \{y^{**}\})$. Goldstine's theorem implies $g(Y, X) = 1$. Moreover, Y is an extreme M-ideal. The converse is also true: the canonical operator $I_{X,Y^*} : X \to Y^{**}$ is isometric if $g(Y, X) = 1$, see the proof of Proposition 3.9. We note as a particular case

$$g(Y) = 1 \text{ if } Y \text{ is an } M\text{-ideal in } Y^{**}.$$

We now link the grade of an M-ideal Y with the grade of certain pseudoballs in Y.

Proposition 3.9 *Let Y be an M-ideal in X,* $\dim X/Y = 1$*. Then*

$$g(Y, X) = g(P_Y(x)) \qquad \forall x \in X \backslash Y.$$

PROOF: Since Y is an M-summand if and only if either of the two numbers equals 0 (Proposition 3.2 and Remark 3.8(b)) we may suppose that Y is a proper M-ideal, i.e. $g(Y, X) > 0$. A look at the definition of the characteristic reveals that the canonical operator (recall $Y^* \subset X^*$)

$$I_{X,Y^*} : X \to Y^{**}, \ \langle I_{X,Y^*}(x), y^* \rangle = \langle x, y^* \rangle$$

is an (into-) isomorphism with $\|I_{X,Y^*}^{-1}\| = g(Y,X)^{-1}$. On the other hand, we note that $g(P_Y(x))$ is actually independent of x (because codim $(Y) = 1$) and equals (cf. Cor. 1.7, including the notation used there), in the case $d(x,Y) = 1$,

$$g(P_Y(x)) = d(Px, Y) = \sup\{|\langle x, y^*\rangle| \mid y^* \in B_{Y^*}\}.$$

These two observations together prove Proposition 3.9. □

Theorem 3.10 *A Banach space Y can be a proper M-ideal if and only if Y contains a pseudoball B which is not a ball. In the latter case, there exists a Banach space X containing Y such that $\dim(X/Y) = 1$, Y is a proper M-ideal in X, and $g(Y,X) = g(B)$.*

PROOF: The "only if" part is a consequence of Proposition 1.3, Corollary 3.3 and Proposition 3.9. We now turn to the "if" part. Let $B \subset Y$ be a pseudoball which is not a ball. We consider the vector space $X = Y \oplus \mathbb{K}$. We will find a norm $|\,.\,|$ on X such that

a) the map $y \mapsto (y, 0)$ from Y into X is an isometry,

b) $P_{Y\oplus\{0\}}(x) = B \times \{0\}$ for some x,

c) $Y \cong Y \oplus \{0\}$ is an M-ideal in X.

The inequality $g(B) > 0$ then yields that $Y \oplus \{0\}$ is a proper M-ideal with grade $g(B)$ by Proposition 3.9.

In the sequel we assume w.l.o.g. that B has diameter 2 and that $0 \in B$. We put

$$\begin{aligned} K &:= \overline{\text{co}}\,\{(\theta y, \theta) \mid \theta \in \mathbb{K},\ |\theta| = 1,\ y \in B\} \\ &= \overline{\text{aco}}\, B \times \{1\} \end{aligned}$$

(the closures being taken with respect to the product topology) and let $|\,.\,|$ be the associated Minkowski functional. (To see that K is absorbing, use Proposition 1.4 to obtain

$$\text{int}\, B_X \times \{0\} \subset \frac{1}{2}(B \times \{1\} - B \times \{-1\}) \subset K.) \qquad (*)$$

Since by $(*)$

$$|(y,r)| \le \|y\| + |r| \le 4 \cdot |(y,r)|,$$

$|\,.\,|$ is a norm which generates the product topology. Thus X is a Banach space.

ad a): It remains to prove (look at $(*)$)

$$K \cap (Y \oplus \{0\}) \subset B_Y \times \{0\}. \qquad (**)$$

So let $(y, 0) \in K$. We may write

$$\begin{aligned} (y,0) &= \lim(y_n, r_n) \\ r_n &= \sum_k \mu_{n,k}\,, \qquad \sum_k |\mu_{n,k}| \le 1 \\ y_n &= \sum_k \mu_{n,k} y_{n,k}\,, \qquad y_{n,k} \in B. \end{aligned}$$

If y_B^{**} denotes the centre of $\overline{B}^{w*}$ in Y^{**}, then (cf. Theorem 1.6)

$$\begin{aligned}\|y_n\| &\leq \left\|\sum_k \mu_{n,k}(y_{n,k} - y_B^{**})\right\| + \left|\sum_k \mu_{n,k}\right| \|y_B^{**}\| \\ &\leq 1 + |r_n|\ \|y_B^{**}\|.\end{aligned}$$

Since $r_n \to 0$ and $\|\,.\,\|$ and $|\,.\,|$ are equivalent on Y we get

$$\|y\| = \lim \|y_n\| \leq 1,$$

i.e. $y \in B_Y$.

ad b): Consider $x = (0, -1)$. A moment's reflection reveals $d(x, Y \oplus \{0\}) = 1$ and $P_{Y\oplus\{0\}}(x) = \{(y,0) \mid (y,1) \in K\}$. (For the proof take into account that $|r| \leq |(y,r)|$ for all $(y,r) \in X$.) Therefore, $B \times \{0\} \subset P_{Y\oplus\{0\}}\,(x)$. For the converse inclusion consider $y \in Y$ such that $(y,1) \in K$. We write

$$(y,1) = \lim (y_n, r_n)$$

with

$$\begin{aligned} r_n &= \sum_k \lambda_{n,k}\theta_{n,k}\,, \qquad \lambda_{n,k} \geq 0,\ \sum_k \lambda_{n,k} = 1,\ |\theta_{n,k}| = 1 \\ y_n &= \sum_k \lambda_{n,k}\theta_{n,k}y_{n,k}\,, \qquad y_{n,k} \in B. \end{aligned}$$

We have

$$\hat{y}_n \ := \ y_n + \sum_k \lambda_{n,k}(1-\theta_{n,k})y_{n,k} \ \in \ B$$

and

$$\hat{y}_n \to y$$

thanks to the following lemma. This proves $y \in B$ as desired.

ad c): We remark that $P_{Y\oplus\{0\}}((y,r)) = -r(B-y) \times \{0\}$, hence it is a pseudoball in $Y \oplus \{0\}$ with radius $|r| = d((y,r), Y \oplus \{0\})$. Thus c) is a consequence of Proposition 1.3(b). □

So, the proof of Theorem 3.10 will be completed as soon as the following lemma is proved.

Lemma 3.11 *For $m \in \mathbb{N}$, $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda_k = 1$, and $\theta_k \in \mathbb{C}$ with $|\theta_k| = 1$ we have*

$$\sum_{k=1}^m \lambda_k |1-\theta_k| \leq \sqrt{2\sqrt{\varepsilon}} + 2\sqrt{\varepsilon}$$

provided that $|\sum \lambda_k(1-\theta_k)| \leq \varepsilon$.

PROOF: We define

$$\begin{aligned} N_1 &:= \{k \in \{1,\dots,m\} \mid \operatorname{Re}\theta_k > 1-\sqrt{\varepsilon}\}, \\ N_2 &:= \{k \in \{1,\dots,m\} \mid \operatorname{Re}\theta_k \le 1-\sqrt{\varepsilon}\}, \\ \lambda &:= \sum_{k\in N_1} \lambda_k, \end{aligned}$$

and we claim that

$$1-\lambda \le \sqrt{\varepsilon}, \tag{1}$$

$$|1-\theta_k| \le \sqrt{2\sqrt{\varepsilon}} \quad \text{for every } k \in N_1. \tag{2}$$

ad (1): By assumption we have $|1-z| \le \varepsilon$ and therefore $\operatorname{Re} z \ge 1-\varepsilon$, where $z := \sum \lambda_k\theta_k$. Let $z_1 := \sum_{k\in N_1}(\lambda_k/\lambda)\theta_k$ and $z_2 := \sum_{k\in N_2}(\lambda_k/(1-\lambda))\theta_k$. The convexity of $\{w \in \mathbb{C} \mid |w| \le 1,\ \operatorname{Re} w \le 1-\sqrt{\varepsilon}\}$ implies that $\operatorname{Re} z_2 \le 1-\sqrt{\varepsilon}$. Hence

$$1-\varepsilon \le \operatorname{Re} z = \operatorname{Re}(\lambda z_1 + (1-\lambda)z_2) \le \lambda + (1-\lambda)(1-\sqrt{\varepsilon}),$$

and this gives $1-\lambda \le \sqrt{\varepsilon}$.
ad (2): For $k \in N_1$ we have $1-\operatorname{Re}\theta_k < \sqrt{\varepsilon}$ and thus

$$|1-\theta_k|^2 = 2(1-\operatorname{Re}\theta_k) < 2\sqrt{\varepsilon}.$$

The lemma follows by combining (1) and (2):

$$\begin{aligned} \sum_{k=1}^m \lambda_k|1-\theta_k| &= \sum_{k\in N_1}\lambda_k|1-\theta_k| + \sum_{k\in N_2}\lambda_k|1-\theta_k| \\ &\le \lambda\sqrt{2\sqrt{\varepsilon}} + (1-\lambda)\cdot 2 \\ &\le \sqrt{2\sqrt{\varepsilon}} + 2\sqrt{\varepsilon}. \end{aligned}$$

□

We next aim at giving an example of a Banach space which can be a proper M-ideal, yet not an extreme M-ideal. (A geometric characterisation of extreme M-ideals is contained in the Notes and Remarks section.)
First we present a result on finite dimensional subspaces of proper M-ideals which will eventually lead to a necessary condition for an M-ideal to be extreme.

Proposition 3.12 *Suppose Y is a proper M-ideal in X with* $\dim X/Y = 1$ *and* $\alpha := g(Y,X) > 0$*. Then, given $\varepsilon > 0$ and a finite dimensional subspace $E \subset Y$, one may find $y_0 \in Y$ satisfying*

$$\frac{\alpha}{1+\varepsilon}\max\{\|e\|,|\lambda|\} \ \le\ \|e+\lambda y_0\| \ \le\ (1+\varepsilon)\max\{\|e\|,|\lambda|\}$$

for all $e \in E,\ \lambda \in \mathbb{K}$.

PROOF: We recall that under our present assumptions the canonical operator

$$I_{X,Y^*} : X \to Y^{**},\ \langle I_{X,Y^*}(x), y^* \rangle = \langle x, y^* \rangle$$

is an (into-) isomorphism whose inverse has norm α^{-1}. To get started, consider the decomposition

$$X^{**} = Y^{\perp\perp} \oplus_\infty \operatorname{lin}\{x_0^{**}\}$$

in which we assume $\|x_0^{**}\| = 1$. Let $\delta > 0$ and choose with the help of the principle of local reflexivity an injective operator

$$T :\ \operatorname{lin}(E \cup \{x_0^{**}\}) \to X$$

such that

$$\|T\| \cdot \|T^{-1}\| \le 1 + \delta$$

and

$$Te = e \ \text{ for all } \ e \in E.$$

For $x_0 := Tx_0^{**}$ we obtain

$$\frac{1}{1+\delta} \max\{\|e\|, |\lambda|\} \ \le \ \|e + \lambda x_0\| \ \le \ (1+\delta) \max\{\|e\|, |\lambda|\}$$

for all $e \in E,\ \lambda \in \mathbb{K}$.

We have to "push" x_0 into Y. As a first step, we push it into Y^{**} by means of $y_0^{**} := I_{X,Y^*}(x_0)$. It follows easily

$$\frac{\alpha}{1+\delta} \max\{\|e\|, |\lambda|\} \le \|e + \lambda y_0^{**}\| \le (1+\delta) \max\{\|e\|, |\lambda|\}$$

for all $e \in E,\ \lambda \in \mathbb{K}$. As a second step, we again apply the principle of local reflexivity to obtain an injection

$$S :\ \operatorname{lin}(E \cup \{y_0^{**}\}) \to Y$$

with $\|S\| \cdot \|S^{-1}\| \le 1 + \delta$ which extends the identity on E.

With an appropriate choice of δ and $y_0 := S(y_0^{**})$ we finally achieve the desired result. $\square$

Proposition 3.12 claims that there are uniformly M-orthogonal directions in a proper M-ideal Y for every finite dimensional subspace with the uniformity constant depending only on the grade. If Y is even an extreme M-ideal (i.e. $\alpha = 1$), then an obvious induction process shows that Y contains arbitrarily good copies of $\ell^\infty(n)$ "everywhere", i.e. the induction process can be started with an arbitrary $y \in Y$. Moreover, these subspaces are even nested so that Y contains $(1+\varepsilon)$-copies of c_0, for every $\varepsilon > 0$. This result extends to all proper M-ideals as will be shown using a different approach in Theorem 4.7.

The following corollary supports the point of view that elements of (proper) M-ideals should – loosely speaking – vanish at infinity.

Corollary 3.13 *If Y is an extreme M-ideal, then*

$$0 \in \overline{\mathrm{ex}}^{w*} B_{Y^*}.$$

PROOF: Let $y_1, \ldots, y_n \in Y$, $\|y_i\| = 1$, and $\delta > 0$. We have to produce some $p \in \mathrm{ex}\, B_{Y^*}$ such that

$$|p(y_i)| \leq \delta \quad \text{for all } i.$$

To this end consider $E := \mathrm{lin}\,\{y_1, \cdots, y_n\}$ and $\varepsilon = \delta/3 > 0$. Choose y_0 according to 3.12. In particular (since $\alpha = 1$ here) $\|y_0\| \geq 1/(1+\varepsilon)$ so that there exists $p \in \mathrm{ex}\, B_{Y^*}$ with $p(y_0) \in \mathbb{R}$ and $p(y_0) > 1/(1+\varepsilon) - \varepsilon$. Hence for $i = 1, \ldots, n$ and suitable scalars θ_i with modulus 1

$$\begin{aligned} |p(y_i)| &= p(\theta_i y_i + y_0) - p(y_0) \\ &\leq \|\theta_i y_i + y_0\| - p(y_0) \\ &\leq 1 + \varepsilon - \left(\frac{1}{1+\varepsilon} - \varepsilon\right) \\ &\leq \delta. \end{aligned}$$

□

The preceding corollary enables us to prove the existence of nonextreme M-ideals. (The example of Remark 3.8(c) does not give this.)

Example 3.14 *Consider the sup-normed spaces*

$$\begin{aligned} X &= \{(s_n)_{n\geq 0} \in c \mid s_0 = s_1 + 2s_\infty\} \\ Y &= \{(s_n) \in X \mid s_0 = 0\}. \end{aligned}$$

(Here, $s_\infty := \lim s_n$.) Then

(a) *Y can be a proper M-ideal – in fact Y is a proper M-ideal in X – but*

(b) *Y cannot be an extreme M-ideal in any superspace.*

PROOF: A straightforward application of the 3-ball property (Theorem I.2.2) shows that Y is an M-ideal in X. To show that it is proper, let $\delta_m : (s_n) \mapsto s_m$ be the evaluation functional on X. It is quickly seen that $2\delta_n + \delta_1$ attains its norm on Y so that $2\delta_n + \delta_1 \in Y^*$ ($\subset X^*$, cf. Remark I.1.13). But $(2\delta_n + \delta_1)$ tends to $\delta_0 \in Y^\perp$ with respect to $\sigma(X^*, X)$, hence Y^* is not weak* closed in X^*. This proves (a). To prove (b) we use Corollary 3.13. We have

$$\mathrm{ex}\, B_{Y^*} \subset \{\lambda \cdot \delta_n|_Y \mid n = 1, 2, \ldots, \infty,\ |\lambda| = 1\}$$

by the Krein-Milman theorem (more precisely Milman's converse to it) since the latter set is norming. But it is also $\sigma(Y^*, Y)$-closed, therefore $0 \notin \overline{\mathrm{ex}}^{w*} B_{Y^*}$. □

II.4 The intersection property IP

In Theorem 3.10 we gave an internal characterisation of Banach spaces which can (or cannot) appear as proper M-ideals. In this section we are going to discuss an easy to check intersection property whose presence in a Banach space X entails that X fails to be a proper M-ideal.

Definition 4.1 *A Banach space X has the intersection property (IP for short) if, given $\varepsilon > 0$, there are a finite family $\{x_i \mid i = 1, \dots, n\} \subset S_X$ and $\delta > 0$ such that*

$$\bigcap_{i=1}^{n} B(\pm x_i, 1+\delta) \subset B(0, \varepsilon).$$

An equivalent formulation, which saves one existence quantifier, is of course

$$\forall \varepsilon > 0 \;\; \exists n \in \mathbb{N} \;\; \exists x_1, \dots, x_n \in \operatorname{int} B_X \;\; \forall x \in X \;(\forall_i \|x \pm x_i\| \le 1 \Rightarrow \|x\| \le \varepsilon).$$

It is easily verified that c_0 fails the IP while $C(K)$ has it. (For the latter, take $n = 1$ and $x_1 = \mathbf{1}$, independently of ε.) The $C(K)$-proof admits immediate generalisation. Recall that $x \in B_X$ is called a strong extreme point of B_X if

$$x_k \in X, \;\; \|x \pm x_k\| \to 1 \quad \Rightarrow \quad x_k \to 0.$$

Taking $n = 1$ and x_1 a strong extreme point in Definition 4.1 we conclude that Banach spaces X for which B_X contains a strong extreme point enjoy the IP. It is, however, worth remarking at this point that we shall eventually encounter a strictly convex space which fails the IP (see the remarks following Example 4.6).
Let us collect more examples of spaces with IP.

Proposition 4.2 *Each of the following properties is sufficient for a Banach space X to have the IP:*

(a) *B_X contains a strong extreme point,*
(b) *X is a (real or complex) unital Banach algebra (e.g. $X = L(E)$),*
(c) *there exists $x_0 \in X$ such that $|p(x_0)| = 1$ for all $p \in \operatorname{ex} B_{X^*}$,*
(d) *$\overline{\operatorname{ex}}^{w*} B_{X^*} \subset S_{X^*}$ (e.g. X is a C_Σ-space),*
(e) *B_X is dentable (e.g. B_X contains a strongly exposed point),*
(f) *X has the Radon-Nikodým property (e.g. X is reflexive or a separable dual space),*
(g) *X contains a nontrivial L-summand (e.g. X is an L^1-space),*
(h) *X is the space $C_\Sigma(S^m)$.*

The definition of a C_Σ-space will be recalled in the next section (p. 83); for the notions employed in (e) and (f) we refer to the monograph [158].
PROOF: (a) was observed above.

(b) The complex case follows from Theorem 5, p. 38 in [84], where it is shown that the unit of a Banach algebra is a strong extreme point of the unit ball, and (a). The real case can be reduced to the complex case by a complexification procedure (p. 68ff. in [85]).

(c) is a consequence of (a), since such an x_0 is a strong extreme point.

(d) We first claim:

For each $\varepsilon > 0$ there is a finite family $\{x_i \mid i \leq n\} \in \operatorname{int} B_X$ such that

$$(*) \qquad \max_i |p(x_i)| > 1 - \varepsilon \qquad \text{for all } p \in \operatorname{ex} B_{X^*}.$$

If this were false, we could find a net (p_F) in $\operatorname{ex} B_{X^*}$, indexed by the finite subsets of $\operatorname{int} B_X$, such that

$$\max_{x \in F} |p_F(x)| \leq 1 - \varepsilon_0,$$

for a suitable $\varepsilon_0 > 0$. Then also

$$|x^*(x)| \leq 1 - \varepsilon_0 \qquad \text{for all } x \in \operatorname{int} B_X,$$

where x^* is any weak*-accumulation point of the net (p_F). Hence $x^* \in (\overline{\operatorname{ex}}^{w*} B_{X^*}) \backslash S_{X^*}$, which contradicts (d). (By the way, it is not hard to show that conversely $(*)$ implies the condition in (d).) Now let $\varepsilon > 0$ and (x_i) as in $(*)$. We wish to show

$$\|x \pm x_i\| \leq 1 \ \text{ (all } i) \ \Rightarrow \ \|x\| \leq \sqrt{2\varepsilon},$$

which will yield the IP.

Let $p \in \operatorname{ex} B_{X^*}$. Then by assumption

$$|p(x) \pm p(x_i)| \leq 1 \qquad (\text{all } \ i).$$

But $|p(x_i)| > 1 - \varepsilon$ for a certain i, so that

$$|p(x)| \leq \sqrt{2\varepsilon}$$

and, consequently,

$$\|x\| = \sup_{p \in \operatorname{ex} B_{X^*}} |p(x)| \leq \sqrt{2\varepsilon}.$$

That C_Σ-spaces have the property in (d) is contained in [385, p. 72f.]; as a matter of fact, for C_Σ-spaces X, $\operatorname{ex} B_{X^*}$ is weak* closed.

(e) Suppose X fails the IP. We will show that there exists $\varepsilon > 0$ with

$$x \in \overline{\operatorname{co}}\,(B_X \backslash B(x, \varepsilon/2)) \ \text{ for all } \ x \in S_X,$$

i.e., the unit ball is not dentable.

By assumption there exists $\varepsilon > 0$ such that for all $x \in S_X$ and $n \in \mathbb{N}$ there is $z_n \in X$, $\|z_n\| > \varepsilon$, with $\|x \pm z_n\| \leq 1 + \frac{1}{n} =: r_n$.

Then

$$\frac{1}{2}\Big((x + z_n)r_n + (x - z_n)r_n\Big) \to x$$

and

$$\|(x \pm z_n)r_n - x\| > \varepsilon/2$$

for large n.

(f) follows from (e), by well-known results ([158, Chap. V]).

(g) Suppose $X = X_1 \oplus_1 X_2$ and consider any $x_i \in X_i$, $\|x_i\| = 1$. We claim

$$B(\pm x_1, 1+\varepsilon) \cap B(\pm x_2, 1+\varepsilon) \subset B(0, 2\varepsilon).$$

So let y be a member of the left hand side and decompose

$$y = y_1 + y_2 \in X_1 \oplus_1 X_2.$$

Then

$$\|y_1 \pm x_1\| + \|y_2\| \le 1+\varepsilon,$$

hence

$$\|x_1\| + \|y_2\| \le 1+\varepsilon,$$

so that

$$\|y_2\| \le \varepsilon.$$

Analogously, $\|y_1\| \le \varepsilon$ and so

$$\|y\| = \|y_1\| + \|y_2\| \le 2\varepsilon.$$

(h) As usual, S^m is the Euclidean sphere in $\mathbb{R}^{m+1}$ and

$$C_\Sigma(S^m) = \{f \in C(S^m) \mid f(s) = -f(-s) \quad \forall\, s \in S^m\}.$$

This is a C_Σ-space and consequently enjoys the IP by (d). The upshot of this example is, however, that the number n appearing in Definition 4.1 must be larger than m for $X = C_\Sigma(S^m)$. So let us prove:

(∗) For $f_1, \ldots, f_m$ in the open unit ball of $C_\Sigma(S^m)$ there is $g \in C_\Sigma(S^m)$ with

$$\|g \pm f_i\| \le 1 \ \text{ (all } i\text{)}, \ \text{ but } \|g\| > 1/3.$$

The decisive tool to prove this assertion is a corollary to the Borsuk-Ulam theorem [6, p. 485, Satz VIII] according to which any m functions in $C_\Sigma(S^m)$ have a common zero, say $f_1(s_0) = \ldots = f_m(s_0) = 0$. To construct g choose a neighbourhood U of s_0 such that

$$U \cap -U = \emptyset \ \text{ and } \ |f_i(s)| < 1/2 \ \text{ for } \ s \in U,\ i \le m.$$

Let $h : S^m \to [0,1]$ be a continuous function vanishing outside U with $h(s_0) = 1$. Put $g(s) = \frac{1}{2}(h(s) - h(-s))$ and $V = -U \cup U$. Then $g \in C_\Sigma(S^m)$, $\|g\| = |g(s_0)| = 1/2$, and g vanishes off V. It follows easily $\|g \pm f_i\| \le 1$. ◻

One cannot expect many permanence properties for the IP. For example, $c_0 \oplus_1 c_0$ has the IP by Proposition 4.2(g), but the one-complemented subspace c_0 fails it. Also, ℓ^2 has the IP while the injective tensor product $\ell^2 \widehat{\otimes}_\varepsilon \ell^2$ fails it. This follows from Theorem 4.4 below and Example I.1.4(d). We stress, however, that in connection with M-ideals positive results can be achieved.

Proposition 4.3 *Suppose X has the IP and J is an M-ideal in X. Then X/J has the IP. In particular, the IP is inherited by M-summands.*

PROOF: For every $\varepsilon > 0$, we find by assumption $x_1, \ldots, x_n \in \operatorname{int} B_X$ such that

$$\|z \pm x_i\| \leq 1 \quad (\text{all } i) \qquad \Rightarrow \qquad \|z\| \leq \varepsilon.$$

We claim that $q(x_1), \ldots, q(x_n)$ will do for X/J, where $q : X \to X/J$ is the quotient map. So let

$$\|q(x) \pm q(x_i)\| \leq 1 \qquad (\text{all } i).$$

Then all the balls $B(x \pm x_i, 1)$ meet J since J is proximinal (Proposition 1.1), and their intersection contains a small ball around x, since $\|x_i\| < 1$. Apply Theorem I.2.2(v) to obtain $y \in J$ such that

$$\|x \pm x_i - y\| \;\;\leq\;\; 1 \qquad\quad (\text{all } i).$$

By assumption we have $\|x - y\| \leq \varepsilon$, i.e. $\|q(x)\| \leq \varepsilon$. □

The following theorem furnishes a large class of Banach spaces which fail the IP. Note that by Proposition 4.2 the failure of the IP is connected to some kind of "vanishing at infinity" (especially (c) and (d) suggest this point of view). So we think that Theorem 4.4 is quite a natural result.

Theorem 4.4 *A proper M-ideal X fails the IP.*

PROOF: We know from Theorem 3.10 (or Corollary 1.7 if you prefer) that X contains a pseudoball B with radius 1 which is not a ball so that

$$\alpha := d(x^{**}, X) > 0$$

where x^{**} is the centre of $\overline{B}^{w*}$ in X^{**}. Let $\varepsilon < \alpha$ and $x_1, \ldots, x_n \in \operatorname{int} B_X$. We wish to produce $z \in X$ with $\|z\| \geq \varepsilon$ and $\|z \pm x_i\| \leq 1$ for all i, thus showing that X fails the IP. First of all choose $\eta > 0$ such that

$$(1 + \eta) \max \|x_i\| < 1 \quad \text{and} \quad \varepsilon < \alpha/(1 + \eta)^2.$$

The defining property of pseudoballs provides us with an $x \in X$ such that $x \pm (1+\eta)x_i \in B$ and thus $\|x \pm (1 + \eta)x_i - x^{**}\| \leq 1 \quad (i = 1, \ldots, n)$. We now use the principle of local reflexivity to obtain an injection

$$T : \; E := \operatorname{lin} \{x^{**}, x, x_1, \ldots, x_n\} \;\to\; X$$

with $\|T\|, \|T^{-1}\| \leq 1 + \eta$ which is the identity on $E \cap X$. Hence

$$\|x - Tx^{**} \pm (1 + \eta)x_i\| \leq 1 + \eta$$

and

$$\|x - Tx^{**}\| \geq \|x - x^{**}\|/(1 + \eta) \geq \alpha/(1 + \eta).$$

Therefore $z = (x - Tx^{**})/(1 + \eta)$ has the required properties. □

The converse of Theorem 4.4 does not hold. To prepare a counterexample we prove:

Lemma 4.5 *Suppose $X_1, X_2, \ldots$ are Banach spaces and put*

$$X = X_1 \oplus_\infty X_2 \oplus_\infty \cdots = \{(x_m) \mid x_m \in X_m, \|(x_m)\| := \sup_m \|x_m\| < \infty\}.$$

Then:

(a) *If no X_m can be a proper M-ideal, then neither can X.*
(b) *If X has the IP, then each X_m has the IP. Moreover, the number $n = n(\varepsilon)$ in the definition of the IP can be chosen to be the same for X and each X_m.*

PROOF: (a) Let P_m be the M-projection from X onto X_m (we consider X_m as naturally embedded in X). Suppose X is an M-ideal in some superspace Z. Then X_m is an M-ideal in Z (Proposition I.1.17(b)) so that by assumption on X_m there is an M-projection Q_m from Z onto X_m. Proposition I.1.2(a) shows $P_m = Q_m|_X$.
Define $Q : Z \to Z$ by $Qz = (Q_m z)_{m\in\mathbb{N}}$. We have shown that Q is a contractive projection onto the M-ideal X, hence X cannot be a proper M-ideal by Corollary I.1.3.
(b) can immediately be verified. □

Example 4.6 *There is a Banach space which fails the IP, yet cannot be a proper M-ideal.*

PROOF: Let $X_m = C_\Sigma(S^m)$ (cf. 4.2(h)) and consider the ℓ^∞-sum $X = X_1 \oplus_\infty X_2 \oplus_\infty \cdots$. Then X_m has the IP (Proposition 4.2(h)), thus cannot be a proper M-ideal (Theorem 4.4). Therefore X cannot be a proper M-ideal either (Lemma 4.5(a)).
On the other hand, to verify the IP for X_m we need at least $m+1$ vectors in Definition 4.1 as was pointed out in the proof of Proposition 4.2(h). Therefore X fails the IP (Lemma 4.5(b)). □

As a consequence of Theorem 4.4 and Proposition 4.2(a) one obtains that the extreme point structure of the unit ball of a proper M-ideal is quite weak; at any rate there are no strong extreme points, let alone strongly exposed points. However, it is shown in [303] that $B_X = \overline{\mathrm{co}}\,\mathrm{ex}\,B_X$ for $X = K(\ell^p)$, $1 < p < \infty$, $p \neq 2$. Since $K(\ell^p)$ is a proper M-ideal in $L(\ell^p)$ for these p (this will be proved in Example VI.4.1), none of these extreme points can be strongly extreme. Even better: The quotient space $C(\mathbb{T})/A$ ($\mathbb{T} =$ unit circle, $A =$ disk algebra) is strictly convex and an M-ideal in its bidual (Remark IV.1.17). Though all the points with norm one are extreme, none of them is strongly extreme.
As another consequence of 4.4 and 4.2 we observe that a proper M-ideal must fail the RNP. This also follows from the following result which was already announced in the previous section, where a special case was treated (see the discussion following Proposition 3.12).

Theorem 4.7 *A Banach space X which fails the IP contains a subspace isomorphic to c_0. In particular, every proper M-ideal contains c_0.*

PROOF: A by now classical theorem due to Bessaga and Pełczynski [421, Prop. 2.e.4] asserts that X contains a copy of c_0 if and only if there is a weakly unconditionally

Cauchy series which does not converge. So, our aim is to produce a sequence (x_i) such that

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < 1 \qquad \text{for all } n \in \mathbb{N} \text{ and all } \varepsilon_i = \pm 1,$$

yet

$$\inf \|x_i\| > 0.$$

Let $\alpha > 0$ be such that for all finite families $\{z_1, \ldots, z_n\} \subset \operatorname{int} B_X$ there is $z \in X$ with $\alpha < \|z\| < 1$ and $\|z \pm z_i\| < 1$ for all i. Such an α exists by assumption on X. (The inequality $\|z\| < 1$ can be obtained upon adding 0 to $\{z_1, \ldots, z_n\}$.) Now the desired sequence can easily be defined inductively: Start with an arbitrary x_1 satisfying $\alpha < \|x_1\| < 1$. Suppose $x_1, \ldots, x_n$ have been found such that

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < 1 \ \text{ for all choices of signs } \varepsilon_i = \pm 1, \tag{1}$$

$$\|x_i\| > \alpha \ \text{ for } i = 1, \ldots, n. \tag{2}$$

Apply the above version of non-IP to the finite family $\{\sum_1^n \varepsilon_i x_i \mid \varepsilon_i = \pm 1\}$ to obtain x_{n+1} extending (1) and (2) to $n+1$.
This completes the proof of Theorem 4.7. □

James' distortion theorem (e.g. [421, Prop. 2.e.3]) permits us to infer that there are even $(1+\varepsilon)$-isomorphic copies of c_0, for every $\varepsilon > 0$. One might wonder if Theorem 4.7 can be strengthened so as to even yield isometric copies of c_0; after all, the assumptions of 4.7 are of isometric type as well. This, however, is impossible; the strictly convex space $C(\mathbb{T})/A$, which is an M-ideal in its bidual, serves as a counterexample (Remark IV.1.17). Another counterexample will be presented in Corollary III.2.12 where we will exhibit a smooth space X which is an M-ideal in X^{**}. Another enlightening counterexample concerning $(1+\varepsilon)$-isomorphisms will be presented in Proposition III.2.13.

In the last part of this section we will consider a question of isomorphic nature, namely what conditions imply that a Banach space can be renormed so as to become a proper M-ideal. It will turn out that in the isomorphic setting an example such as 4.6 does not exist. The key idea is contained in the following notion.

Definition 4.8 *A closed bounded convex subset S of a Banach space is called a* quasiball *if*

(a) $\operatorname{int}(S - S) \neq \emptyset$,

(b) $\bigcap_{i=1}^{n}(x_i + S) \neq \emptyset$ *for any finite family of vectors* $x_1, \ldots, x_n \in \frac{1}{2}(S - S)$.

A quasiball is called proper *if it is not symmetric.*

Proposition 1.4 shows that every pseudoball is a quasiball. But unlike the definition of a pseudoball, the notion of a quasiball is only dependent on the topology of X and not on the particular choice of an equivalent norm. On the other hand, a quasiball S will be a pseudoball for the equivalent norm $|\,.\,|$ on X whose unit ball is $\frac{1}{2}\overline{S - S}$. (By the way, the remark following Proposition 1.3 thus implies that it would be enough to require (b) of Definition 4.8 for $n = 3$.) Moreover, S is proper if and only if it is not a ball for $|\,.\,|$. (This follows for example from Theorem 1.6.)

Theorem 4.9 *For a Banach space X, the following assertions are equivalent:*

(i) *X can be renormed to become a proper M-ideal.*

(ii) *X can be renormed to fail the IP.*

(iii) *X contains an isomorphic copy of c_0.*

(iv) *X contains a proper quasiball.*

PROOF: (iv) $\Rightarrow$ (i): This follows from the above considerations together with Theorem 3.10.

(i) $\Rightarrow$ (ii) is Theorem 4.4.

(ii) $\Rightarrow$ (iii) is Theorem 4.7.

(iii) $\Rightarrow$ (iv): Let Y be subspace of X isomorphic to c_0. Then Y contains a proper quasiball S_0 since c_0 does. We let $S = \overline{S_0 + B_X}$. Obviously, S is closed, convex, bounded, and $\operatorname{int}(S - S) \neq \emptyset$. Moreover, S is not symmetric. In fact, an easy application of the Hahn-Banach theorem reveals

$$y + B_X \subset S \;\Rightarrow\; y \in S_0. \tag{$*$}$$

Thus, should S be symmetric, w.l.o.g. with centre 0, $(*)$ implies that S_0 is symmetric around 0, too.
It is left to establish (b) from Definition 4.8. We first observe

$$\begin{aligned}
\operatorname{int} \frac{1}{2}(S - S) \quad &\subset \quad \operatorname{int} \frac{1}{2}\overline{(S_0 + B_X) - (S_0 + B_X)} \\
&= \quad \operatorname{int} \left(\overline{\frac{1}{2}(S_0 - S_0) + B_X}\right) \\
&= \quad \operatorname{int}_Y \frac{1}{2}(S_0 - S_0) \;+\; \operatorname{int} B_X.
\end{aligned}$$

[Proof of this equality: Let us abbreviate $\frac{1}{2}(S_0 - S_0)$ by V. Then obviously

$$C := \operatorname{int}_Y V + \operatorname{int} B_X \subset \operatorname{int} \overline{V + B_X} =: C^{'}$$

and also $C^{'} \subset \overline{C}$ since

$$V + B_X = \overline{\operatorname{int}_Y V} + \overline{\operatorname{int} B_X} \subset \overline{C}.$$

Thus C and $C^{'}$ are open convex sets satisfying

$$C \subset C^{'} \subset \overline{C}.$$

An application of the Hahn-Banach theorem now shows $C = C'$.]
Now let $x_i \in \operatorname{int} \frac{1}{2}(S - S) \quad (i = 1, \dots, n)$. Write

$$x_i = y_i + z_i \in \operatorname{int}_Y \frac{1}{2}(S_0 - S_0) + \operatorname{int} B_X.$$

Then, since $-z_i \in B_X$,

$$\begin{aligned}\bigcap_{i=1}^{n}(x_i+S) &\supset \bigcap_{i=1}^{n}(y_i+z_i)+(S_0+B_X)\\ &\supset \bigcap_{i=1}^{n}(y_i+S_0)\\ &\neq \emptyset.\end{aligned}$$

This completes the proof of Theorem 4.9. □

II.5 L^1-preduals and M-ideals

This final section is devoted to the study of L^1-predual spaces (i.e. Banach spaces X whose duals are isometrically isomorphic to spaces of type $L^1(\mu)$) and certain of their subclasses by means of M-structure methods. We shall restrict ourselves to *real* Banach spaces, although several of the results presented here have direct analogues for complex spaces, owing to the refinements in complex integral representation theory made in the 70s.

Our first result connects the M-ideals in an L^1-predual space with certain faces in the unit ball of L^1-spaces. Let us call a subset H of B_Y, Y a Banach space, a *biface* if $H = \mathrm{co}\,(F \cup -F)$ for some face F in B_Y.

Proposition 5.1 *In an L^1-predual space X, there is a one-to-one correspondence between M-ideals in X and weak* closed bifaces in B_{X^*}: $J \subset X$ is an M-ideal if and only if J is the annihilator of a weak* closed biface in B_{X^*}.*

Proof: Let H be a weak* closed biface in B_{X^*}. It is proved in [385, p. 220] that

(a) H equals the unit ball of its linear span $\mathrm{lin}\, H$.

(Note that (a) does not hold for bifaces in arbitrary Banach spaces, as shown by two-dimensional examples.) Using similar arguments we wish to show:

(b) If $p \in H$ and $\|p\| = \|p-q\| + \|q\|$, then $q \in H$.

To prove (b) we may assume that $H \neq B_{X^*}$ and that X^* is isometric to some L^1-space in such a way that F corresponds to a subface of

$$\{u \in L^1 \mid \|u\| = 1,\ u \geq 0\}$$

where $\geq$ is the usual order of L^1 (see [385, p. 171]).

Claim: If $0 \leq u \leq v,\ u \neq 0$ and $\dfrac{v}{\|v\|} \in F$, then $\dfrac{u}{\|u\|} \in F$.

In fact, writing $v = u + w$ with $w \geq 0$ we obtain from the additivity of the L^1-norm on positive elements

$$\|v\| = \|u\| + \|w\|$$

and

$$\frac{v}{\|v\|} = \frac{\|u\|}{\|v\|}\,\frac{u}{\|u\|} + \frac{\|w\|}{\|v\|}\,\frac{w}{\|w\|} \in F,$$

hence the claim.

Now the norm equality in (b) is equivalent to

$$|p| = |p-q| + |q| \qquad \text{a.e.}$$

which in turn means

$$p^+ \geq q^+, \qquad p^- \geq q^-.$$

(p^+ is the positive part of the L^1-function p.) By assumption on p we may write

$$p = \lambda u_1 - (1-\lambda)u_2$$

with some $0 \leq \lambda \leq 1,\ u_i \in F$. Hence

$$\lambda u_1 \geq p^+ \geq q^+,$$

and the claim applies to show that $q^+ = 0$ or else $\dfrac{q^+}{\|q^+\|} \in F$. An analogous statement applies to q^- so that

$$\begin{aligned} q &= \|q^+\|\frac{q^+}{\|q^+\|} - \|q^-\|\frac{q^-}{\|q^-\|} + (1-\|q\|)\cdot 0 \\ &\in \operatorname{co}(F \cup -F \cup \{0\}) \\ &= \operatorname{co}(F \cup -F) \\ &= H. \end{aligned}$$

(a) and (b) together yield:

(c) $\operatorname{lin} H$ is a weak* closed order ideal in L^1 and hence an L-summand.

Weak* closedness follows from (a) and the Krein-Smulian theorem. If $|q| \leq |p|$ and $p \in H$, then by (b) $p^+ \in H$ and $p^- \in H$, hence $|p| \in H$. By the same argument q^+ and q^- are in H, hence $q \in \operatorname{lin} H$. For the equivalence of order ideals and L-summands cf. Example I.1.6(a). Thus, we have shown that

$$J := \{x \in X \mid p(x) = 0 \ \text{ for all } \ p \in H\}$$

is an M-ideal in X.

Conversely let J be an M-ideal in X. Representing X^* as $L^1(\mu)$ we see that there is a measurable set A such that (Example I.1.6(a) again)

$$J^\perp = \{p \in L^1 \mid p|_A = 0\}.$$

Now

$$F = \left\{ p \in L^1 \;\middle|\; \|p\| \leq 1,\ \int_{\complement A} p\, d\mu = 1 \right\}$$

defines a face in B_{L^1} such that

$$J^\perp = \operatorname{lin}(\operatorname{co}(F \cup -F)) \ \ (= \operatorname{lin} F). \qquad \square$$

We wish to consider the following subclasses of L^1-predual spaces.

- X is a G-space if there are a compact Hausdorff space K and $(s_i, t_i, \lambda_i) \in K \times K \times \mathbb{R}$, i in some index set I, such that X is isometric to
$$\{x \in C(K) \mid x(s_i) = \lambda_i \cdot x(t_i) \ \text{ for all } \ i \in I\}.$$
- X is a C_σ-space if there are a compact Hausdorff space K and an involutory homeomorphism $\sigma : K \to K$ (i.e. $\sigma^2 = id_K$) such that X is isometric to
$$\{x \in C(K) \mid x(t) = -x(\sigma(t)) \ \text{ for all } \ t \in K\}.$$
- X is a C_Σ-space if it is a C_σ-space for some fixed point free involution σ on some K.

Obviously, a C_σ-space is a G-space. Lindenstrauss [413, p. 79ff.] showed that G-spaces are L^1-preduals, but that the converse does not hold, thus disproving a conjecture of Grothendieck who had introduced the class of G-spaces – now termed after his initial – in [281]. (Previously, Stone [583, p. 465ff.] and Kakutani [363, p. 1005] had shown that the closed subalgebras (resp. sublattices) of a real $C(K)$-space are exactly those spaces isometric to G-spaces with $\lambda_i \in \{0, 1\}$ (resp. $\lambda_i \geq 0$) throughout.) A detailed analysis of these spaces can be found in [389] and [423], see also Lacey's monograph [385].

Below we shall present characterisations of G- and C_σ-spaces in terms of their M-structure properties. This will also lead to a (in our context) natural argument to show that the class of G-spaces is strictly smaller than the class of L^1-preduals.
We first show:

Proposition 5.2 *Let*

$$X = \{x \in C(K) \mid x(s_i) = \lambda_i x(t_i) \ \textit{ for all } \ i \in I\}$$

be a G-space. Then $J \subset X$ is an M-ideal if and only if there is a (not necessarily uniquely determined) closed set $D \subset K$ such that

$$J = \{x \in X \mid x|_D = 0\}.$$

Proof: The "only if" part is a consequence of Proposition I.1.18. To prove the "if" part we verify the 3-ball property (Theorem I.2.2(iv)). Given $x \in B_X$, $y_1, y_2, y_3 \in B_J$ we consider the function

$$y = \max\{y_1, y_2, y_3, -x\} + \min\{y_1, y_2, y_3, -x\} + x$$

(defined pointwise) which of course vanishes on D. Also, $y \in X$ by [413, p. 79], and the desired inequality $\|x + y_i - y\| \le 1 + \varepsilon$ is elementary to verify pointwise (even with $\varepsilon = 0$), since for real numbers the implication

$$|a_i| \le 1 \quad (\text{all } i) \qquad \Longleftrightarrow \qquad \left| a_i - (\max_j a_j + \min_j a_j) \right| \le 1 \quad (\text{all } i)$$

holds. □

Corollary 5.3 *In a G-space the intersection of an arbitrary family of M-ideals is an M-ideal.*

There is a partial converse to this result.

Theorem 5.4 *Let X be a separable L^1-predual with the property that the intersection of an arbitrary family of M-ideals is an M-ideal. Then X is a G-space.*

PROOF: The proof relies on the following fact (cf. e.g. [214] or [600]). An L^1-predual space X is a G-space if and only if

$$\overline{\text{ex}}^{w*} B_{X^*} \subset [0,1] \cdot \text{ex}\, B_{X^*}.$$

To provide a proof of Theorem 5.4 we consider $x_0^* \in \overline{\text{ex}}^{w*} B_{X^*}$. To show that $x_0^* \in [0,1] \cdot \text{ex}\, B_{X^*}$ it will be enough to show that $\text{lin}\,\{x_0^*\}$ is an L-summand, equivalently that $\ker x_0^*$ is an M-ideal (recall Lemma I.1.5). Since X is separable, B_{X^*} is weak* metrizable so that there is a sequence of distinct points (x_n^*) in $\text{ex}\, B_{X^*}$ with $x_0^* = w^*\text{-}\lim x_n^*$. (We are assuming $x_0^* \notin \text{ex}\, B_{X^*}$.) Now fix $m \in \mathbb{N}$ and consider the subspace

$$U_m = \overline{\text{lin}}^{\|\ \|} \{x_0^*, x_{m+1}^*, x_{m+2}^*, \ldots\} \subset X^*.$$

We wish to show that U_m is a weak* closed L-summand.
Define a linear operator $T : X \to c$ by $Tx = (\langle x_{n+m}^*, x\rangle)_{n\in\mathbb{N}}$. Let $e_k^* : (s_n) \mapsto s_k$ be the k^{th} evaluation functional on c and $e_0^* : (s_n) \mapsto \lim s_n$ be the limit functional. We then have

$$T^*(e_k^*) = x_{m+k}^* \qquad \text{and} \qquad T^*(e_0^*) = x_0^*.$$

The x_{m+k}^*, being linearly independent extreme points in some L^1-space, generate a subspace isometric to ℓ^1 as do the e_k^*. From this we conclude that $U_m = \text{ran}(T^*)$ is norm closed, and thus, by a classical result, it is weak* closed. Hence

$$U_m \;=\; \overline{\text{lin}}^{w*} \{x_{m+1}^*, x_{m+2}^*, \ldots\} \;=\; \left(\bigcap_{k\in\mathbb{N}} \ker x_{m+k}^* \right)^{\perp}.$$

But $\ker x_{m+k}^*$ is an M-ideal, and the assumption of Theorem 5.4 now yields that so is $\bigcap_{k\in\mathbb{N}} \ker x_{m+k}^*$. Therefore, U_m is in fact a weak* closed L-summand.
It remains to observe that $\text{lin}\,\{x_0^*\} = \bigcap_{m\in\mathbb{N}} U_m$ and to apply Proposition I.1.11(a) to obtain that $\ker x_0^*$ is an M-ideal which happens only if x_0^* is a multiple of an extreme functional (Lemma I.1.5). □

We remark that an L^1-predual always has infinitely many M-ideals (cf. Proposition 5.1) so that nontrivial intersections of M-ideals exist. It is an open problem as of this writing if one can dispense with the separability assumption in Theorem 5.4. However, it would be enough to require that $\operatorname{ex} B_{X^*}$ is sequentially dense in its weak* closure. As the example of the disk algebra shows (Example I.1.4(b)), one cannot weaken the assumption "X is an L^1-predual" to "$\ker p$ is an M-ideal for all $p \in \operatorname{ex} B_{X^*}$".

We next present an example of a separable L^1-predual which does not fulfill the conclusion of Corollary 5.3, hence cannot be a G-space by Theorem 5.4. (We admit that there are other ways of proving the existence of such spaces; the first example of this kind was constructed by Lindenstrauss [413, p. 78 and 81].)

Example 5.5 *Fix $t_0 \in [0,1]$ and let λ denote Lebesgue measure on $[0,1]$. Put*

$$X = \left\{ x \in C[0,1] \;\middle|\; x(t_0) = \int x \, d\lambda \right\}.$$

Then X is an L^1-predual, and for $t \neq t_0$

$$J_t := \{x \in X \mid x(t) = 0\}$$

is an M-ideal. However, if the sequence (t_n) converges to t_0 and $t_n \neq t_0$ for all n, the intersection $\bigcap_n J_{t_n}$ fails to be an M-ideal. Consequently, X is not isometric to a G-space.

PROOF: That X is an L^1-predual is a special case of results given in [46], but we wish to sketch a direct argument. Let μ denote the signed measure $\lambda - \delta_{t_0}$. We have to prove that $X^* = M[0,1]/\operatorname{lin}\{\mu\}$ is an L^1-space. Note

$$X^* = L^1(|\mu|)/\operatorname{lin}\{\mathbf{1}\} \oplus_1 M_{\text{sing}}(|\mu|) \tag{$*$}$$

where $M_{\text{sing}}(|\mu|)$ is the space of $|\mu|$-singular measures which is an L^1-space. Thus it is left to prove that $L^1(|\mu|)/\operatorname{lin}\{\mathbf{1}\}$ is an L^1-space, or equivalently [281] that its dual $U := \{f \in L^\infty(|\mu|) \mid \int f d\mu = 0\}$ is an L^∞-space. This will follow if there is a contractive projection P from $L^\infty(|\mu|)$ onto U. (Cf. e.g. [385, §§ 11 and 21] for these matters.) Now it is not hard to show that

$$Pf = f + \int f d\mu \cdot \chi_{\{t_0\}}$$

is such a projection.

It is seen from $(*)$ that $\delta_t|_X \in \operatorname{ex} B_{X^*}$ for $t \neq t_0$, hence its kernel J_t is an M-ideal by Proposition 5.1. To refute that $J = \bigcap_n J_{t_n}$ is an M-ideal we refute the 3-ball (in fact 2-ball) property (Theorem I.2.2). We let $x = \mathbf{1}$, $\varepsilon > 0$ small enough ($\varepsilon < \frac{1}{10}$ will do) and will construct $y_1 \in B_J$ in such a way that *no* $y \in J$ satisfies

$$\|x \pm y_1 - y\| \leq 1 + \varepsilon. \tag{$**$}$$

The construction of such a y_1 is quite easy to understand but a bit cumbersome to describe in detail. Here is the idea: Choose an open neighbourhood U of t_0 of Lebesgue measure $\leq \varepsilon$. Also, choose open neighbourhoods U_n of those t_n which are not in U, whose total length does not exceed ε, either. The complement of these neighbourhoods consists of finitely many closed intervals. The function y_1 is defined in such a way that $|y_1(t)| = 1$ on a large part of these intervals (total length $\geq 1 - 3\varepsilon$ say), $\|y_1\| \leq 1$, $\int y_1 d\lambda = 0$, and

$y_1(t_n) = 0$ for all n so that $y_1 \in J$. Let y be any continuous function satisfying $(**)$. Then

$$\begin{aligned} |y(t) - 1| &\leq \varepsilon && \text{on a set of measure } \geq 1 - 3\varepsilon, \\ |y(t)| &\leq 2 + \varepsilon && \text{otherwise.} \end{aligned}$$

Consequently

$$\int y \, d\lambda \geq 1 - 10\varepsilon > 0$$

for sufficiently small ε. But if $y \in J$ we have

$$\int y \, d\lambda = y(t_0) = 0$$

so that indeed no $y \in J$ can satisfy $(**)$. □

The following result characterises G-spaces among all Banach spaces by a richness condition concerning the M-ideal structure. This condition will be given in terms of the structure topology on $\operatorname{ex} B_{X^*}$, which we defined in I.3.11. To appreciate it the reader is referred to the remarks following I.3.11. There E_X was defined to be the quotient space $\operatorname{ex} B_{X^*}/\!\sim$ with $p \sim q$ if and only if $p = q$ or $p = -q$.

Theorem 5.6 *For a Banach space X, the following conditions are equivalent:*
(i) *The structure topology on E_X is Hausdorff.*
(ii) *X is a G-space.*

We found it convenient to single out part of the argument for the implication (i) ⇒ (ii) in the following lemma.

Lemma 5.7 *If the structure topology on E_X is Hausdorff, then X is an L^1-predual.*

Proof: By a result of Lima's [399, Theorem 5.8] it is enough to prove
(a) If $p \in X^*$, $\|p\| = 1$ and $P(p) \in \{0, p\}$ for all L-projections P on X^*, then $p \in \operatorname{ex} B_{X^*}$.
(b) $\operatorname{lin}\{p\}$ is an L-summand in X^* for all $p \in \operatorname{ex} B_{X^*}$.

Note that (b) is equivalent with saying that singletons are closed in E_X, i.e. E_X is a T_1-space in its structure topology. Thus we are left with showing (a).

The Hausdorff property easily implies:

$(*)$ If $F \subset E_X$ is a structurally closed set containing at least two points, then there are structurally closed sets F_1, F_2 different from F such that $F = F_1 \cup F_2$.

With the help of the Krein-Milman and Krein-Smulian theorems, $(*)$ translates into the language of L-summands as follows:

$(**)$ If $N \subset X^*$ is a weak* closed L-summand with $\dim(N) \geq 2$, then there are weak* closed L-summands N_1, N_2 different from N such that $N = N_1 + N_2$.

Now let p be a point as in (a) and let N be the intersection of all weak* closed L-summands containing p. Then N itself is a weak* closed L-summand (cf. I.1.11(a)). If

$p \notin \operatorname{ex} B_{X^*}$ then $\dim(N) \geq 2$. Assuming this to be the case we could split N according to $(**)$. Either $p \in N_1$ or $p \in N_2$ contradicts the choice of N, thus, if P_i is the L-projection onto N_i, $P_1(p) = P_2(p) = 0$ by the very choice of p. Observe that $P = P_1 + P_2 - P_1P_2$ is the L-projection onto N so that $P(p) = 0$, contradicting $p \in N$. This completes the proof of Lemma 5.7. $\square$

We remark that Lemma 5.7 remains true if "E_X is Hausdorff" is replaced by "E_X fulfills $(*)$"; examine the above proof. (Property $(*)$ is quickly seen to imply the T_1-separation axiom.) The property $(*)$, which might be called "total reducibility" in accordance with [88, Chap. II.4], is not necessary for a Banach space X to be an L^1-predual, since the space of Example 5.5 fails it. [One can show, using the 3-ball property, that exactly the subspaces $\{x \in X \mid x|_D = 0\}$, where $D \subset [0,1]$ is a nonvoid closed set not containing t_0, are the nontrivial M-ideals in X. Consequently, E_X can be identified as a set with $[0,1]\backslash\{t_0\}$, the structurally closed sets corresponding to those D above and the trivial sets $\emptyset$ and $[0,1]\backslash\{t_0\}$. Thus $F = [0,1]\backslash\{t_0\}$ is a counterexample to $(*)$.]

Proof of Theorem 5.6:

(i) $\Rightarrow$ (ii): We already know from Lemma 5.7 that X must be an L^1-predual. By the result quoted in the proof of 5.4, it remains to show that the weak* closure of $\operatorname{ex} B_{X^*}$ is contained in $[0,1] \cdot \operatorname{ex} B_{X^*}$. So, let (p_i) be a net in $\operatorname{ex} B_{X^*}$ converging in the weak* sense to some $p \in B_{X^*}$. We may assume $p \neq 0$. Let N be the smallest weak* closed L-summand in X^* containing p. We now claim that (p_i) converges structurally to all $q \in \operatorname{ex} B_N$: Should this be false there would exist a structurally closed set $F \subset \operatorname{ex} B_{X^*}$, some $q \in (\operatorname{ex} B_N)\backslash F$ and a subnet $(p_j) \subset F$ of (p_i). Now F has the form $\operatorname{ex} B_L$ for some weak* closed L-summand, and $p = w^*\text{-}\lim p_j$ implies $p \in L$. Hence $N \subset L$ and therefore $q \in L$ which contradicts $q \notin F$. (We have relied on Lemma I.1.5 several times.) By the claim and since E_X is structurally Hausdorff we conclude that N must be one-dimensional, i.e. $p/\|p\| \in \operatorname{ex} B_N \subset \operatorname{ex} B_{X^*}$.

(ii) $\Rightarrow$ (i): Let $X \subset C(K)$ be a G-space, and let $[p] \neq [q]$ be two equivalence classes in E_X. We wish to find two M-ideals J_1, J_2 in X such that

$$p \notin J_2^\perp \quad \text{and} \quad q \notin J_1^\perp.$$

Observe that $p \in \operatorname{ex} B_{X^*}$ has the form

$$p(x) = \lambda x(s_1)$$

where $\lambda = \pm 1$, and we may assume $\lambda = 1$ since $[p] = [-p]$. In the same vein,

$$q(x) = x(s_2)$$

for some $s_2 \in K$. Since p and q are linearly independent, we may find $x_1,\ x_2 \in X$ such that

$$\begin{aligned} p(x_1) &= q(x_1) &&= 1 \\ p(x_2) &= -q(x_2) &&= 1. \end{aligned}$$

For the closed sets

$$\begin{aligned} D_1 &= \{s \in K \mid x_1(s)x_2(s) \geq 0\} \\ D_2 &= \{s \in K \mid x_1(s)x_2(s) \leq 0\} \end{aligned}$$

we consider the M-ideals (Proposition 5.2)

$$J_i = \{x \in X \mid x|_{D_i} = 0\}.$$

Let $x_\pm = x_1 \pm x_2 - (\max\{x_1, \pm x_2, 0\} + \min\{x_1, \pm x_2, 0\})$. Then $x_\pm$ is the pointwise median of $x_1, \pm x_2$ and 0, hence $x_\pm \in X$ by [413, p. 79]. It is left to observe

$$p(x_+) = x_+(s_1) \neq 0, \quad \text{yet} \quad x_+|_{D_2} = 0$$

and

$$q(x_-) = x_-(s_2) \neq 0, \quad \text{yet} \quad x_-|_{D_1} = 0$$

so that $p \notin J_2^\perp$, $q \notin J_1^\perp$, as requested. □

Next we shall discuss the centralizer of an L^1-predual. First we give examples of L^1-preduals having trivial centralizers. (The centralizer of a Banach space was introduced in I.3.7.)
Let X be the L^1-predual of Example 5.5. It is quickly checked that

$$\operatorname{ex} B_{X^*} = \{\pm\delta_t|_X \mid t \neq t_0\},$$

hence $T \in Z(X)$ (recall that we are considering real spaces here) if and only if there exists $h \in C[0,1]$ such that

$$(Tx)(t) = h(t)x(t) \qquad \forall t \in [0,1].$$

However, the only continuous functions with

$$hx \in X \qquad \text{for all } x \in X$$

are the constants, which is not hard to see. Consequently,

$$Z(X) = \mathbb{R} \cdot Id.$$

We pointed out after I.3.12 that X has a nontrivial M-ideal provided $Z(X)$ is nontrivial. The preceding example shows that there are separable spaces for which the converse is not true.
For the following example, which presents a nonseparable G-space yielding the same effect, one needs more elaborate arguments. First we give a general representation of the centralizer of a G-space. Let

$$X = \{x \in C(K) \mid x(s_i) = \lambda_i \cdot x(t_i) \quad \forall i \in I\}$$

be a G-space. It is proved in [214, Lemma 4] that

$$\operatorname{ex} B_{X^*} = \{\pm\delta_{s|_X} \mid \|\delta_{s|_X}\| = 1\}.$$

[Let us give a quick argument for this fact. The inclusion "$\subset$" is proved by the usual extension argument. For "$\supset$" we consider the M-ideal $\{x \in X \mid x(s) = 0\}$ (Proposition 5.2), obtain the decomposition $X^* = J^* \oplus_1 \operatorname{lin}\{\delta_{s|_X}\}$ and use Lemma I.1.5.]
Continuing our discussion, we may assume that $\{s \in K \mid \|\delta_{s|_X}\| = 1\}$ is dense in K. In fact, this is just a matter of representation, since X may be represented in an obvious way as a G-subspace of $C(\overline{\operatorname{ex}}^{w*} B_{X^*})$. Using the same method as above, one may associate to each $T \in Z(X)$ a bounded continuous function h on $K\backslash L$, where L is the set of common zeros of all the $x \in X$, with

$$(Tx)(t) = h(t)x(t).$$

In addition, h must be constant on each equivalence class pertaining to the equivalence relation

$$s \sim t \quad \Longleftrightarrow \quad \exists \lambda \neq 0 \quad \forall x \in X \quad x(s) = \lambda x(t)$$

on $K\backslash L$. Conversely such a function h defines an element in $Z(X)$ if multiplication by h leaves X invariant. (Cf. Example I.3.4(g) for a general view of this procedure.)

Proposition 5.8 *There exists a nonseparable G-space with a trivial centralizer.*

We remark that as a consequence of results by N. Roy [533], every separable G-space necessarily has a nontrivial centralizer, see the Notes and Remarks section.

PROOF: Let Y be any nonseparable Banach space, and let K denote its dual unit ball with the weak* topology. Define

$$X = \{x \in C(K) \mid x(\lambda y^*) = \lambda x(y^*) \quad \forall \lambda \in [-1,1],\ y^* \in K\}.$$

Obviously, X is a G-space, and Y can be identified with a subspace of X. In view of our above remark, we have to prove the following:

> If $h \in C^b(K\backslash\{0\})$ is constant on each punctured ray $\{\lambda y^* \mid 0 < |\lambda| \leq 1\}$ (where $\|y^*\| = 1$), then h is constant.

Suppose not. First of all, let us extend h to $Y^*\backslash\{0\}$ so as to be constant on each punctured one-dimensional subspace. This extension – still to be called h – is continuous with respect to the bounded weak* topology, i.e. weak* continuous on bounded sets. Recall [178, p. 427] that a neighbourhood base of 0 for the bounded weak* topology is given by absolute polars of sequences in Y tending strongly to 0. Now if h were not constant there would be $y_1^*,\ y_2^* \neq 0$ and scalars $a_1, a_2 \in \mathbb{R}$ such that

$$h(y_1^*) < a_1 < a_2 < h(y_2^*).$$

By continuity $h|_{y_1^*+U} < a_1$, where

$$U = \{y^* \mid |\langle y^*, y_n\rangle| \leq 1 \quad \forall n \in \mathbb{N}\}$$

with some $y_n \to 0$. In particular,

$$h(y_1^* + y^*) < a_1$$

for all $y^* \in (\mathrm{lin}\,\{y_n | n \in \mathbb{N}\})^\perp =: V_1^\perp$. Since V_1 is separable and Y is not, $V_1^\perp \neq \{0\}$. Hence, for all $y^* \in V_1^\perp \backslash \{0\}$, we obtain

$$h\Big(\frac{1}{n} y_1^* + y^*\Big) = h(y_1^* + n y^*) < a_1$$

so that

$$h(y^*) = \lim_{n\to\infty} h\Big(\frac{1}{n} y_1^* + y^*\Big) \leq a_1$$

by continuity. Analogously, for a certain separable subspace V_2 and all $y^* \in V_2^\perp \backslash \{0\}$

$$h(y^*) \geq a_2.$$

Since the separable subspace $V = \mathrm{lin}\,(V_1 \cup V_2)$ is different from Y, there exists $y^* \in V^\perp \backslash \{0\} = (V_1^\perp \cap V_2^\perp)\backslash\{0\}$, and for any such y^* we have

$$a_1 \geq h(y^*) \geq a_2,$$

contradicting $a_1 < a_2$. Thus, Proposition 5.8 is proven. □

It is interesting to compare Proposition 5.8 with Theorem 5.6 in view of the Dauns-Hofmann type Theorem I.3.12. Namely, the structure space E_X of the G-space X of 5.8 is a Hausdorff topological space, yet every real-valued continuous function on E_X is constant. Next we are going to characterise those Banach spaces which yield "sufficiently many" structurally continuous functions on E_X, equivalently whose centralizer is "sufficiently large". Recall the notation

$$Z_X = \overline{\mathrm{ex}}^{w*}\, B_{X^*} \backslash \{0\}$$

from Section I.3. Further, we identify $Z(X)$ with an algebra of functions on Z_X via $T \leftrightarrow a_T$. In the next theorem p and q are said to be nonantipodal if neither $p = q$ nor $p = -q$. (Recall that we assume the scalars to be real in this section.)

Theorem 5.9 *For a Banach space X, the following assertions are equivalent:*

(i) *$Z(X)$ separates nonantipodal points of Z_X.*

(ii) *X is a C_σ-space.*

Proof: (i) $\Rightarrow$ (ii): This is a corollary to Theorem I.3.10.
By assumption (i), the set of equivalence classes $\mathcal{F}(X, C_0(Z_X))$ (cf. the notation of Section I.3) consists exactly of the pairs $\{p, -p\}$, $p \in Z_X$ (note $a_T(p) = a_T(-p)$ for $T \in Z(X)$). Now Theorem I.3.10 states

$$\begin{aligned} X &= \{f \in C_0(Z_X) \mid f|_{\{p,-p\}} \in X|_{\{p,-p\}} \quad \forall p \in Z_X\} \\ &= \{f \in C_0(Z_X) \mid f(p) = -f(-p) \quad \forall p \in Z_X\} \end{aligned}$$

so that X is a C_σ-subspace of $C(\overline{\mathrm{ex}}^{w*}\, B_{X^*})$.
(ii) $\Rightarrow$ (i): By [385, p. 73] X is isometric to

$$\{f \in C_0(Z_X) \mid f(p) = -f(-p) \quad \forall p \in Z_X\}.$$

Also, $Z(X)$ acts on X by multiplication with functions in

$$\{h \in C^b(Z_X) \mid h(p) = h(-p) \quad \forall p \in Z_X\},$$

cf. the discussion preceding Proposition 5.8. To achieve the desired conclusion, choose, given nonantipodal $p_1, p_2 \in Z_X$, any $g \in C^b(Z_X)$ with

$$\begin{array}{rcccl} g(p_1) & = & g(-p_1) & = & 0, \\ g(p_2) & = & g(-p_2) & = & 1, \end{array}$$

and let $h(p) = g(p) + g(-p)$ for $p \in Z_X$. Then h “is” in $Z(X)$ and $h(p_1) \neq h(p_2)$. $\square$

In the final part of this section, which does not depend on the results presented so far, we propose the use of the intersection property IP introduced in Definition 4.1 in order to distinguish between C_σ- and C_Σ-spaces.
It is a well-known fact that a Banach space is a C_σ-space if and only if it is isometric to a norm one complemented subspace of a $C(K)$-space [385, p. 74]. It follows that the class of C_σ-spaces is stable with respect to forming ℓ^∞-sums, since the class of $C(K)$-spaces is. (This could also be deduced directly from the definition.) For the same reason, the class of C_σ-spaces is stable with respect to forming ultraproducts, cf. [299]. For the case of C_Σ-spaces, things look different.
Recall the definition of an ultraproduct of a family of Banach spaces X_i $(i \in I)$: Let $\mathcal{U}$ be a free ultrafilter on I, and let X be the $\ell^\infty(I)$-sum of the X_i. Then the quotient space X/N where

$$N = \{(x_i) \in X \mid \lim_{\mathcal{U}} \|x_i\| = 0\}$$

is called an ultraproduct of the X_i. Note that the norm in X/N is given by

$$\|(x_i) + N\| = \lim_{\mathcal{U}} \|x_i\|.$$

The reader is referred to [298] or [569] for a detailed account.

Proposition 5.10 *The class of C_Σ-spaces is not stable with respect to forming ℓ^∞-sums or ultraproducts.*

Proof: Let X be the Banach space of Example 4.6. By its very construction it is an ℓ^∞-sum of C_Σ-spaces. But X fails the IP so that X cannot be a C_Σ-space by Proposition 4.2(d).
We now turn to the assertion concerning ultraproducts. For that matter, fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then, with the above notation, X/N is an ultraproduct of the spaces $X_n = C_\Sigma(S^n)$ $(n \in \mathbb{N})$. We claim that X/N fails the IP and hence cannot be a C_Σ-space (Proposition 4.2(d) again).

Fix $\varepsilon_0 < \frac{1}{3}$. We wish to find, given equivalence classes $x^1 + N, \dots, x^m + N \in \operatorname{int} B_{X/N}$, a class $y + N$ such that

$$\|(y \pm x^k) + N\| \leq 1 \qquad (k = 1, \dots, m),$$

yet

$$\|y + N\| > \varepsilon_0,$$

thus refuting the defining condition for the IP.
In fact, we may assume that the representatives $x^k = (x_n^k)_{n\in\mathbb{N}}$ fulfill

$$\|x^k\| < 1 \qquad (k = 1, \dots, m),$$

i.e.

$$\|x_n^k\| < 1 \qquad (k = 1, \dots, m;\ n \in \mathbb{N}).$$

By $(*)$ in the proof of Proposition 4.2(h), there is, for each $n \geq m$, $y_n \in X_n$ such that

$$\|y_n \pm x_n^k\| \leq 1 \qquad (\text{all } k),$$

but

$$\|y_n\| > \frac{1}{3}.$$

Let $y_n = 0$ for $n < m$ and $y = (y_n)_{n\in\mathbb{N}} \in X$. Then

$$\|(y \pm x^k) + N\| \ \leq\ \sup_n \|y_n \pm x_n^k\| \ \leq\ 1 \qquad \text{for } k = 1, \dots, m$$

and

$$\|y + N\| \ =\ \lim_{\mathcal{U}} \|y_n\| \ \geq\ \frac{1}{3} \ >\ \varepsilon_0,$$

as desired. □

II.6 Notes and remarks

General remarks. The fundamental Proposition 1.1 provides a convenient tool for studying approximation theoretic properties by M-ideal methods. Various proofs of this proposition have appeared in the literature. Alfsen and Effros [11, Cor. I.5.6] derive it from their work on "dominated extensions" already alluded to in the Notes and Remarks to Chapter I. Independently it was obtained by Ando [19] as a consequence of a variant of our Lemma 2.5 which he proves without using any intersection properties (see also [318] for an account of this proof). We have drawn upon the simple proofs devised by Behrends [51, Prop. 6.5] and Yost [647]. Also Lau [387] gives a proof based on a variant of Lemma 2.5, which was later rediscovered in [240]. More specifically, he calls a subspace J of a Banach space X *U-proximinal* if there exists a positive function ϕ on $\mathbb{R}^+$ with $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$(B_X + J) \cap (1 + \varepsilon)B_X \subset B_X + \phi(\varepsilon)B_J \qquad \forall \varepsilon > 0$$

and proves that U-proximinal subspaces are in fact proximinal. He goes on to give examples of U-proximinal subspaces, among them M-ideals.

The fact that the set of best approximants in an M-ideal J algebraically spans J was first shown by Holmes, Scranton and Ward [320], our simple proof is due to Behrends [51, Prop. 6.6] and Lima [403]. The notion of a pseudoball is implicitly contained already in Lima's paper, too, but the explicit definition appears only in [67] where it is attributed to R. Evans. There it is also shown in addition to Theorem 1.6 that B is a pseudoball of radius 1 if and only if for every subspace $V \subset Y$ with $\dim Y/V < \infty$ there is some $y_0 + V \in Y/V$ such that $\operatorname{int} B(y_0 + V, 1) \subset q(B) \subset B(y_0 + V, 1)$ (where $q : Y \to Y/V$ denotes the quotient map). Other papers where the concept of a pseudoball is employed are [54], [267] and [479].

Also Proposition 1.8 comes from [320], the proof we have presented is a modification of the one in [647]. Apart from these authors, H. Fakhoury, too, has obtained Theorem 1.9 [216]. His argument is quite different and more in the spirit of the Alfsen-Effros paper. Corollary 1.7 and Proposition 1.10 seem to be new (with 1.10 essentially appearing in [291]) as does Proposition 1.11. This proposition answers a question raised in [320] where it is conjectured that an M-ideal is an M-summand as soon as its metric complement has an interior point. The concept of a grade (studied in [291], see also [67] and below) is the decisive tool to provide a complete solution.

The approximation theoretic properties of M-ideals have turned out to be of particular interest in the context of best approximation of bounded linear operators by compact operators. We refer to the Notes and Remarks to Chapter VI for a more detailed discussion.

As already mentioned in the text, Theorem 2.1 is due to Ando [20] and Choi and Effros [121]. For details of the development of this theorem see below. The classical Corollary 2.6 can be found in [173], for a nice geometric argument in the case that K is compact see [317, p. 103]. Corollary 2.7 comes from [486] and Corollary 2.8 from [444]. The strange proof of Sobczyk's theorem (Corollary 2.9) was suggested in [620]. Another argument for Proposition 2.10 is given in [651].

Proper M-ideals are studied in [67] and [293]. The fact that an M-ideal is proper if and only if the sets of best approximants are not balls (Proposition 3.2) was independently observed in [403] and [55]. Theorem 3.4 is due to Evans [205], with the present proof being essentially taken from [55]. Many proofs of Corollary 3.6 have appeared: see [205], [226], [399], [403], [476]. We can't help mentioning the simplest of these arguments, due to Payá [476]. Let $J^\perp \subset X^*$ be a weak* closed M-ideal so that there is an L-projection P from X^{**} onto $J^{\perp\perp}$. To show $Px \in J$ for $x \in X$, which is our aim, we calculate, given $y \in J$,

$$\begin{aligned} \|x-y\| &= \|Px-y\| + \|x-Px\| \\ &= \|Px-y\| + d(x, J^{\perp\perp}) \\ &= \|Px-y\| + d(x, J), \end{aligned}$$

where the natural identification $(X/J)^{**} \cong X^{**}/J^{\perp\perp}$ is used. Now take the infimum over all $y \in J$ to obtain

$$d(x, J) = d(Px, J) + d(x, J),$$

i.e. $d(Px, J) = 0$ and $Px \in J$. (One may note that this reasoning yields another proof of the fact, mentioned in the Notes and Remarks to Chapter I, that an F-ideal is an F-summand provided $(0, 1) \in \operatorname{ex} B_F$.) We have taken the chance to add another proof of Corollary 3.6 which ultimately builds on the simple Proposition I.1.2. The rest of Section II.3 is taken from [67] and [291], whereas Section II.4 relies on [67] and [293] save for Theorem 4.9 which appears in [654]. Concerning the intersection property of Section II.4 we would like to mention the problem that up to now no dual space is known which fails this property; however, we strongly suspect such a space to exist.

As regards the material of Section II.5 let us note that the definition of a biface is due to Effros [189] who defines a structure topology on $\operatorname{ex} B_{X^*}$ by means of the weak* closed bifaces of B_{X^*}. By virtue of Proposition 5.1 this is the same as the structure topology of Section I.3 if X is a real L^1-predual space. (The use of bifaces in the general theory of L^1-preduals has proved useful e.g. in [214], [247], [386] or [429].) 5.2 – 5.7 are taken from Uttersrud's paper [607] with the exception of Theorem 5.4 which was first proved by N. Roy [535]. Simpler proofs of 5.4 are due to Lima, Olsen and Uttersrud [408] whose argument we have presented and to Rao [515]. Example 5.5 has already been used by Perdrizet [491, Section 6] for essentially the same purpose (up to the nomenclature), and a variant appears in Bunce's paper [100]. Uttersrud gives a detailed analysis of the M-structure of spaces of the form $\{x \in C(K) \mid x(t_0) = \int x \, d\mu\}$; the case $K = [0, 1]$, $t_0 = \frac{1}{2}$, $\mu = (\delta_0 + \delta_1)/2$ corresponds to Bunce's example. Theorem 5.6 was proved under the additional assumption that X is an L^1-predual by Effros [189] in the separable case and in the nonseparable case by Taylor [600] and Fakhoury [214] using arguments different from those presented here. That this additional assumption is in fact superfluous is shown in Lemma 5.7. It is not clear how much has to be added to the T_1-separation property of E_X in order to force X to be an L^1-predual. We have already remarked that the "splitting property" $(*)$ appearing in the proof of Lemma 5.7 is sufficient, but not necessary. Contributions to this problem can be found in [536] and [607].

In his paper [139] Cunningham suggested a method of representing a Banach space as a "function module" or a Banach bundle, cf. the Notes and Remarks to Chapter I. He proposed to call a Banach space *square* if all the fibres in such a representation are zero- or one-dimensional. (Admittedly, this notation is not too suggestive.) To put it another way, a square Banach space X is a sup-normed space of bounded scalar-valued functions on some compact Hausdorff space Ω such that, for $x \in X$ and $f \in C(\Omega)$, the pointwise product $f \cdot x$ belongs to X and the numerical function $|x(\cdot)|$ is upper semicontinuous. In [140] and [141] Cunningham shows that every square Banach space is a G-space, but that there are nonseparable G-spaces which fail to be square. To this end he proves Proposition 5.8. [To show that the nonseparable G-spaces of Proposition 5.8 are not square one has to take into account that the point evaluations $x \mapsto \pm x(\omega)$ are exactly the extreme functionals on a square space X [143] so that all the multiplication operators $x \mapsto f \cdot x$ for $f \in C(\Omega)$ belong to the centralizer $Z(X)$. In particular, the centralizer of a square Banach space cannot be trivial.] In contrast has N. Roy shown in [533] (see also [534]) that *separable* G-spaces are in fact square. She also proves that a Banach space X is square if and only if $Z(X)$ separates E_X, i.e. nonantipodal points of $\operatorname{ex} B_{X^*}$. This result should be compared to Theorem 5.6 and to Theorem 5.9 which comes from [629]. Note that for C_σ-spaces the equation $Z_X = \operatorname{ex} B_{X^*}$ holds; the above results do, of course, not imply that every square Banach space is a C_σ. Another result along these lines worth

mentioning is that a Banach space X is a C_σ-space if and only if the structure topology on $\operatorname{ex} B_{X^*}$ coincides with the weak* topology. This is proved (modulo Lemma 5.7) in [214].
Finally we remark that Proposition 5.10 was first proved, using topological arguments, in [300]. Our approach follows [293].

THE $1\frac{1}{2}$-BALL PROPERTY. Following D. Yost [647] we say that a closed subspace J of a Banach space X has the $1\frac{1}{2}$-*ball property* if the conditions

$$\|x\| < r_1 + r_2 \qquad \text{and} \qquad B(x, r_2) \cap J \neq \emptyset$$

imply that

$$B(0, r_1) \cap B(x, r_2) \cap J \neq \emptyset.$$

This is equivalent to requiring the (strict) 2-ball property subject to the restriction that one of the centres lies in J. The $1\frac{1}{2}$-ball property is, however, far less restrictive than the 2-ball property. Unlike the latter the $1\frac{1}{2}$-ball property is self-dual, i.e. J has it if and only if $J^\perp$ has it. Among the examples one finds apart from (semi) M-ideals

- L-summands and semi L-summands (for the definition cf. Section I.4),
- closed subalgebras of $C_{\mathbb{R}}(K)$,
- $K(L^1(\mu), \ell^1)$ as a subspace of $L(L^1(\mu), \ell^1)$,
- $K(C(S), C(T))$ as a subspace of $L(C(S), C(T))$ if S is scattered and T is extremally disconnected.

Furthermore $K(c)$ has the $1\frac{1}{2}$-ball property in $L(c)$ if c is considered to be the space of convergent sequences of real numbers, whereas $K(c)$ fails it if we consider the complex sequence space c! Also, closed subalgebras of $C_{\mathbb{C}}(K)$ need not have the $1\frac{1}{2}$-ball property. (All these examples are contained in Yost's papers [647] and [653].)
Among the complex Banach spaces with the $1\frac{1}{2}$-ball property the examples of $K(\ell^1)$ in $L(\ell^1)$ and its relatives were until recently the only known specimens except for M-ideals and L-summands. Now it is known that every complex Banach space can arise as a subspace with the $1\frac{1}{2}$-ball property of some superspace in a nontrivial way, i.e. without being an M-ideal or L-summand [440].
The importance of this notion stems from its approximation theoretic implications. In fact, the proof of Proposition 1.1 applies as well to show that subspaces with the $1\frac{1}{2}$-ball property are proximinal. Moreover, in analogy with Proposition 1.8 one can show that the metric projection onto a subspace with the $1\frac{1}{2}$-ball property is Lipschitz continuous, with the Lipschitz constant 2. Thus Theorem 1.9, too, extends to this greater generality. These results yield an interesting consequence, namely:

> *A Lipschitz continuous metric projection need not have a Lipschitz continuous selection.*

(A merely continuous selection always exists by Michael's theorem, cf. Theorem 1.9.) To see this let D denote the sup-normed space of all real valued functions on [0,1] which are

continuous at each irrational point, which are continuous from the right everywhere and which have limits from the left everywhere. Since D is isometric to a $C(K)$-space (e.g. by Gelfand's theorem) and $C = C_{\mathbb{R}}[0,1]$ is a closed subalgebra of D it has the $1\frac{1}{2}$-ball property, and the metric projection P_C is Lipschitz continuous on D. However, there cannot exist a Lipschitz selection of P_C because otherwise there would be a Lipschitz lifting for the quotient map from D onto D/C, contradicting a result by Aharoni and Lindenstrauss [4]. For this example see also [655], where a selfcontained proof is given.
In [649] Yost gives a new proof of the duality theorem for spaces with the 3-ball property (resp. 2-ball property) and L-summands (resp. semi L-summands). More specifically he proves that Y has the 2-ball property in X if and only if Y has $1\frac{1}{2}$-ball property and is Hahn-Banach smooth in X, and Y is a semi L-summand if and only if Y has the $1\frac{1}{2}$-ball property and is a Chebyshev subspace. By a theorem due to Phelps [494] this proves that J has the 2-ball property if and only if $J^\perp$ is a semi L-summand, and with a bit of extra work Yost obtains that J has the 3-ball property if and only if $J^\perp$ is an L-summand, i.e. the equivalence (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (v) of Theorem I.2.2.
Other papers dealing with the $1\frac{1}{2}$-ball property include [273], [471], [472] and [481].

Linear extension operators. The genesis of the important Theorem 2.1 lies in Pełczyński's papers [486] and [487]. There he proved the existence of a contractive linear extension operator from $X|_D$ to X under the following assumptions: X is a closed subspace of $C(K)$ where K is metrizable, $D \subset K$ is closed, $X|_D = C(D)$ and the pair $(X|_D, X)$ enjoys the bounded extension property. (In fact, his assumptions were formally stronger, but actually equivalent to those stated above, see [444].) His original proof was long and technical, and it was greatly simplified by Michael and Pełczyński in [444]. They were also able to drop the assumption that K be metrizable and $X|_D = C(D)$ in favour of the assumption that D be metrizable and $X|_D$ be a π_1-space; see also [514] for a proof. [Here a Banach space Z is called a π_1-space if and only if there is an increasing sequence of finite dimensional subspaces $E_1, E_2, \dots$ whose union is dense and each of which is the range of a contractive projection. This means that Z not only has the metric approximation property, but the metric approximation property is realized by an increasing sequence of (commuting) contractive finite rank projections; in particular such a space must be separable. It was shown in [445] that $C(D)$ is a π_1-space if D is a compact metric space. For a comparison of the various refinements of the BAP see the interesting recent paper [114] and its references.]
A considerable improvement in the development of linear extension theorems is due to Ryll-Nardzewski (see [490]) who replaced the π_1-property by the mere metric approximation property, still assuming separability of $X|_D$. The idea here is to reduce the problem to the previous case by switching to vector valued sequence spaces; this idea is used in the proof of Theorem 2.1, too. Note that the metric approximation property for $C(D)$ is far easier to check than the π_1-property and that the disk algebra yields an example of a space with the metric approximation property which fails to be a π_1-space [639].
In a similar vein, Lazar [388] proved the existence of a contractive linear extension operator from $A(F)$ to $A(K)$, where K denotes a Choquet simplex and F a closed metrizable face, as a consequence of his selection theorem, and Andersen [17] treated the same problem for a general compact convex set and a closed metrizable split face where he assumed

that $A(F)$ has the metric approximation property. This can be converted into a linear lifting theorem for quotients of C^*-algebras by closed two-sided ideals.

The first abstract theorem of this kind was proved by Ando [19] who supposed that J is an M-ideal in X and that X/J is a π_1-space to obtain a linear contractive lifting from X/J to X. Finally he showed in [20] that it is enough to assume that X/J is a separable space with the bounded approximation property in order to obtain a continuous linear lifting. All the results previously mentioned can be deduced from this theorem. It was also obtained by a different proof by Choi and Effros [121] (see also [120]), who point out the relevance of Theorem 2.1 for questions in the cohomology theory of C^*-algebras (see Section V.4). We have largely followed Ando's proof in the text.

Various authors have investigated which additional properties of a linear extension operator can be obtained, for instance positivity (see [17], [19], [388], [444]). Vesterstrøm has proved a rather general result in [613] using M-ideals in certain ordered Banach spaces. In the other direction we have already remarked that the assumptions in Theorem 2.1 are quite sharp; Proposition 2.3, which shows that the approximation assumption cannot be dispensed with, is related to an example in [149]. It follows from Theorem 1.9 that without metrizability there is always a nonlinear continuous normpreserving extension operator from $C(D)$ to $C(K)$; Benyamini [71] has shown that every number between 1 and ∞ may arise as the norm of a linear extension operator in this setting. It follows from the principle of local reflexivity that one can always obtain simultaneous extensions of finite dimensional subspaces of $C(D)$; see [215] for a more general result.

If one wants to apply Theorem 2.1 for function spaces $X \subset C(K)$, then a natural problem is to find conditions on D so that $X|_D = C(D)$. In this case D is called an interpolation set. Bishop [79] has shown that D is an interpolation set if $|\mu|(D) = 0$ for all $\mu \in X^\perp \subset M(K)$. The simplest proof of this proceeds as follows: Using the assumption of Bishop's theorem it is not hard to show that the map $\mu \mapsto (x \mapsto \int_D x \, d\mu)$ from $M(D)$ to X^* is an isometry. By duality this means that the restriction operator $x \mapsto x|_D$ is a quotient map, hence the result. But more is true: Since $J = J_D \cap X$ is an M-ideal in X (Cor. I.1.19) and thus proximinal, the restriction operator even maps the *closed* unit ball of X onto the closed unit ball of $C(D)$ so that we even get norm preserving extensions. This observation yields a proof of the Rudin-Carleson theorem mentioned in the proof of Corollary 2.7, since Bishop's condition is fulfilled by the F. and M. Riesz theorem. (The intimate connection between M-ideals and F. and M. Riesz type theorems will be discussed in Section IV.4; for elementary proofs of the Rudin-Carleson and the Riesz theorem we refer to [168] and its references.) Corresponding results under different orthogonality conditions (e.g. $\chi_D \mu = 0$ for all $\mu \in X^\perp$) are discussed e.g. in the papers [12], [94], [238], [248]; see Chapter 4 of the monograph [35] for a detailed account. For the topic of linear extension operators see also the memoir [488].

Finally we present an application of Theorem 2.1 to potential theory. Let $U \subset \mathbb{R}^n$ be open and bounded. We put

$$H(U) = \{f \in C(\overline{U}) \mid f \text{ is harmonic on } U\}.$$

The classical Dirichlet problem requires to find, given $\varphi \in C(\partial U)$, some $f \in H(U)$ such that $f|_{\partial U} = \varphi$. This is generally impossible. However, there is always a generalised solution, called the Perron-Wiener-Brelot solution. Those points $x_0 \in \partial U$ such that for all $\varphi \in C(\partial U)$ and corresponding Perron-Wiener-Brelot solutions f the relation $\lim_{x \to x_0} f(x) = \varphi(x_0)$ is valid are called regular boundary points. The set of all regular boundary points is denoted by $\partial_r U$, and it is classical that $\partial_r U = \partial U$ if the boundary of U is sufficiently smooth or if U is simply connected and $n = 2$. Concerning the weak solvability of the Dirichlet problem we now have:

> Proposition. *Let $U \subset \mathbb{R}^n$ be open and bounded, and let $E \subset \partial_r U$ be compact. Then there is a contractive linear operator $L : C(E) \to H(U)$ such that $(L\varphi)_{|E} = \varphi$ for all $\varphi \in C(E)$.*

This result will turn out to be almost obvious after we have reformulated it in terms of convexity theory. The space $H(U)$ is an order unit space and can hence be represented as a space of affine continuous functions $A(K)$. The crux of the matter is that here K is a Choquet simplex; see [81] or [193]. Moreover, it is known from [44] that $\partial_r U$ can be identified with the Choquet boundary of $H(U)$, that is $\operatorname{ex} K$. If we regard E as a compact subset of $\operatorname{ex} K$, then a corollary to Edwards' separation theorem states that every continuous function on E has a norm preserving extension to an affine continuous function on K [7, p. 91]. Let us denote $F = \overline{\operatorname{co}}\, E$ so that F is a split face since K is a simplex. Consequently, by Example I.1.4(c),

$$\{f \in A(K) \mid f_{|E} = 0\} = J_F \cap A(K) =: J$$

is an M-ideal in $A(K)$, and the quotient space $A(K)/J$ is isometric with $C(E)$ by the above. By Theorem 2.1 there is a linear contractive lifting L for the quotient map, and reidentifying $A(K)$ with $H(U)$ yields the desired solution operator from $C(E)$ to $H(U)$.

Grades of M-ideals. In Propositions 1.10 and 1.11 we attached two numbers $g^*(J, X)$ and $g_*(J, X)$ to an M-ideal J in X. According to Proposition 3.9 these numbers can be calculated as

$$\begin{aligned} g^*(J, X) &= \sup\{g(J, Y) \mid J \subset Y \subset X,\ \dim Y/J = 1\}, \\ g_*(J, X) &= \inf\{g(J, Y) \mid J \subset Y \subset X,\ \dim Y/J = 1\}. \end{aligned}$$

The interpretation is that J behaves like a very proper M-ideal in every direction if $g_*(J, X)$ is close to 1, while $g^*(J, X)$ close to 1 only yields the existence of one such direction. As was already announced in the text, for the familiar M-ideals J_D in $C(K)$ one obtains that $g^*(J_D, C(K)) = 0$ if D is clopen and $g^*(J_D, C(K)) = 1$ otherwise; and $g_*(J_D, C(K)) = 0$ if D contains an interior point and $g_*(J_D, C(K)) = 1$ otherwise. Also, it follows from Remark 3.8(d) that $g_*(Y, Y^{**}) = 1$ if Y is an M-ideal in its bidual.

One-codimensional M-ideals offer appealing geometric descriptions. If $Y = \ker(p)$ for some $p \in X^*$, $\|p\| = 1$, is an M-ideal in X and $L = \{x \in X \mid \|x\| = p(x) = 1\}$, then the unit ball of X is kind of cylindrically shaped with the set L being the lid of that cylinder, since one can show [291]

$$\operatorname{int} B_X \subset \operatorname{aco} L \subset B_X.$$

The idea of a cylindrical unit ball is perfect if J is even an M-summand, in which case L is the translate of a ball in J. The general case, however, produces a certain flaw in this idea because now L is merely the translate of a pseudoball in J. This flaw has already appeared in the example of Remark I.2.3(d) where we presented an M-ideal containing an extreme point of B_X, and it shows up again in the following characterisation of extreme M-ideals from [291]: J is an extreme one-codimensional M-ideal if and only if for each $x \in X$, $\|x\| > 1$, there is some $y \in \operatorname{co}(B_X \cup \{x\}) \cap J$ with $\|y\| > 1$.

Intersection properties and complex L^1-preduals. In his memoir [413] Lindenstrauss introduced, extending previous work by Hanner [286], the $n.k.$ intersection property as follows: A Banach space has the $n.k.$IP if, given a family of n closed balls each subfamily of which consisting of k balls has a point in common, the intersection of these n balls is nonempty. Then he proved that a *real* Banach space is an L^1-predual if and only if it satisfies the 4.2.IP. Moreover he shows that neither the 4.3.IP nor the 3.2.IP is sufficient to ensure that the space under consideration is an L^1-predual (every L^1-space is known to fulfill the 3.2.IP) and that, for each $n \geq 4$, the 4.2.IP is equivalent to the n.2.IP. This resembles, at least formally, our Theorem I.2.2 where M-ideals are characterised by an intersection condition. Unlike the latter theorem Lindenstrauss' theorem is clearly false for *complex* Banach spaces; not even the one-dimensional complex space $\mathbb{C}$ satisfies the 3.2.IP.
In order to provide a substitute for the n.2.IP, Hustad [327] proposed the following weak intersection property: A Banach space X is an $E(n)$-space if, for each family of n closed balls $B(x_1, r_1), \ldots, B(x_n, r_n)$, the implication

$$\bigcap_{i=1}^{n} B(x^*(x_i), r_i) \neq \emptyset \quad \forall x^* \in B_{X^*} \qquad \Longrightarrow \qquad \bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset$$

holds. It is readily seen that every Banach space is an $E(2)$-space (by virtue of the Hahn-Banach theorem) and that a real Banach space is an $E(n)$-space if and only if it satisfies the n.2.IP. Thus Lindenstrauss' result reads: A real Banach space X is an L^1-predual if and only if it is an $E(4)$-space. In [327] Hustad was able to prove that a complex Banach space is an L^1-predual if and only if it is an $E(7)$-space if and only if it is an $E(n)$-space for some $n \geq 7$.
The obvious question, which numbers smaller than 7 would suffice as well, was answered by Lima who at first proved that $n = 4$ is enough [398] and then obtained the final result that complex L^1-predual spaces are characterised by the $E(3)$-property [397, Appendix]. Simplifications of his proofs appear in [532] and [409]; let us remark that one variant of the final step of the proof that $E(3)$-spaces are L^1-preduals in the latter paper depends on a precise knowledge of the L-summands in the dual of an $E(3)$-space. Lima also shows in [398] that the 4.3.IP characterises complex L^1-preduals. Both these results suggest that in the complex case weaker intersection conditions than in the corresponding real situation suffice. With this philosophy at hand it was believed for many years that complex M-ideals might be characterised by the 2-ball property, but, as we have pointed out in the Notes and Remarks to Chapter I, this belief has only recently turned out to be faulty. – Papers related to this subject include [289], [326], [399], [400], [402], [404], [410], [411], [520].
The general theory of complex L^1-preduals advanced in the 70s after complex versions of the Choquet-Meyer representation theorem were proved, for which we refer to the

survey [495]. Substantial contributions are due to Effros [190] and Hirsberg and Lazar [312]. Using tools provided by these authors Olsen extended the Lazar-Lindenstrauss-Wulbert classification scheme ([389], [423]) of C_σ-spaces, G-spaces etc. to the complex setting which is not a straightforward task ([467], see also [455], [468] and [531]). Here we shall report on some results concerning L^1-preduals involving the L-structure of the dual space.

Apart from Lima's fundamental theorem [399, Th. 5.8] that a real or complex Banach space X is an L^1-predual if and only if

- if $p \in X^*$, $\|p\| = 1$ and $P(p) \in \{0, p\}$ for all L-projections P on X^*, then $p \in \operatorname{ex} B_{X^*}$,
- $\operatorname{lin}\{p\}$ is an L-summand in X^* for all $p \in \operatorname{ex} B_{X^*}$,

(we had occasion to use this in the proof of Lemma 5.7), the following results seem to be of interest. The authors of [202] show that a separable complex Banach space is an L^1-predual if and only if $\operatorname{lin}_{\mathbb{C}} \overline{\operatorname{aco}}^{w^*} K$ is an L-summand in X^* whenever $K \subset \operatorname{ex} B_{X^*}$ is weak* compact. By way of example they point out that the separability assumption cannot be dispensed with. In the general case they prove that a complex Banach space is an L^1-predual if and only if $\operatorname{lin}_{\mathbb{C}} F$ is an L-summand in X^* whenever F is a weak* closed face in B_{X^*}. (Both these results hold in the real case, too.) Rao [515] characterises complex L^1-preduals as follows: For all $x \in X$, $\|x\| = 1$, the set $\{p \in \operatorname{ex} B_{X^*} \mid |p(x)| = 1\}$ is structurally closed and $F = \{x^* \in B_{X^*} \mid x^*(x) = 1\}$ is a split face in $\overline{\operatorname{co}}^{w^*}(F \cup -iF)$. (In the real case the first condition is sufficient.) Moreover he points out that Proposition 5.2 extends to the complex case.

Our proofs of Proposition 5.2 and Theorem 5.6 rely essentially on the "max + min" operation which is restricted to the real case. It was believed to constitute remnants of a lattice structure in a real G-space [423, p. 337], and no complex analogue was contrived until recently when Uttersrud [608] succeeded in doing so, thus proving Theorem 5.6 for complex G-spaces as well. Uttersrud's idea is to view, given 3 real numbers r, s, t, $\frac{1}{2}(\max\{r, s, t\} + \min\{r, s, t\})$ as the centre of the smallest closed interval containing these three numbers. Accordingly he defines, given 3 complex numbers r, s, t, the complex number $c(r, s, t)$ as the centre of the smallest closed disk containing r, s and t. Using this notion he proves that a closed subspace X of $C_{\mathbb{C}}(K)$ is a G-space if and only if for $x, y \in X$ the centre function $c(x, y, 0)$ (defined pointwise) belongs to X if and only if the structure topology on E_X is Hausdorff.