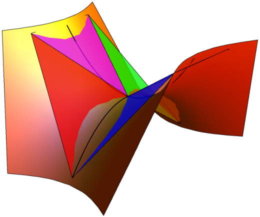
Berlin Mathematical School

CGL

Freie Universität Berlin



(Improved) Optimal Triangulation of Saddle Surfaces

Computational Geometric Learning (CGL) supported by EU FET-Open grant
Transregio-SFB *Discretization in Geometry and Dynamics* (DGD)

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Freie Universität Berlin,
Rijksuniversiteit Groningen

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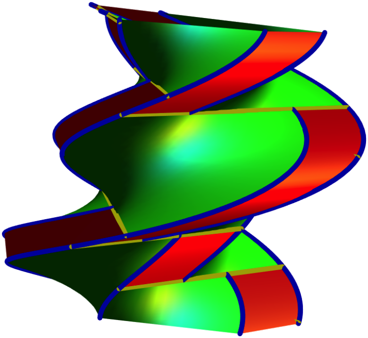
Schloss Schley, November 2013

Motivation

- Smooth surface is locally approximated by a quadratic patch.
- Euclidean motion transforms the quadratic patch to graph of a bi-variate polynomial.
- → approximate graphs of quadratic polynomials!

$$\{(x, y, z) : z = F(x, y)\}$$



• H. Pottmann, R. Krasauskas, B. Hamann, K. Joy, and W. Seibold: On piecewise linear approximation of quadratic functions. *Journal for Geometry and Graphics* **4** (2000), 31–53.

Outline

Introduction

Interpolating Approximation

Non-interpolating Approximation

Vertical Distance

- We are interested in a neighborhood of some point.
- Make the surface normal vertical.
- The direction in which Hausdorff distance is measured becomes almost vertical.

Definition (Vertical Distance, L_∞ Distance)

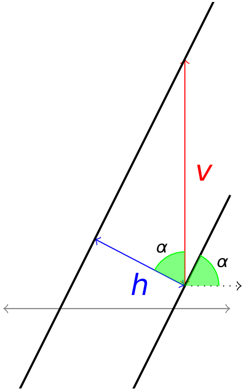
Given two domains $D_1, D_2 \subset \mathbb{R}^2$ and two graphs $f\colon D_1 \to \mathbb{R}$ and $g\colon D_2 \to \mathbb{R}$ then the *vertical distance* is

$$\operatorname{dist}_V(f,g) = \max_{(x,y)\in D_1\cap D_2} |f(x,y) - g(x,y)|$$

Lemma

Let $A, B \subset \mathbb{R}^3$ be two sets with equal projection to the plane. Then

$$\text{dist}_H(A, B) \leq \text{dist}_V(A, B)$$



V-Distance of Quadratic Functions

Lemma (Every two points are the same)

Let S be the graph of a quadratic function.
For every point $p \in S$, there is an affine transformation $\mathcal{T}_p : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies the following:

- $\mathcal{T}_p(p) = \vec{0}$
- $\mathcal{T}_p(S) =$ *a quadratic graph $\tilde{S}$ with a homogeneous polynomial of the form*

$$\tilde{F}(x, y) = ax^2 + bxy + cy^2 \qquad (*)$$

- *For all $q, r \in \mathbb{R}^3$ on a vertical line,*

$$|q - r| = |\mathcal{T}_p(q) - \mathcal{T}_p(r)|.$$

- $\mathcal{T}_p(p)$ *on the first two coordinates is a translation in $\mathbb{R}^2$.*

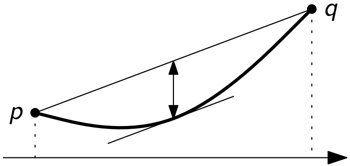
If S is negatively curved, the maximum distance to a triangle never occurs in the interior.

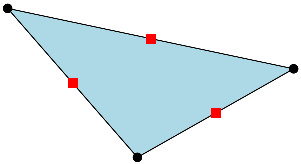
Lemma

For a line segment $\overline{pq}$ between two points $p = (p_x, p_y, p_z)$ and $q = (q_x, q_y, q_z)$ on a quadratic graph S,

$$\mathrm{dist}_V(\overline{pq}, S) = \frac{1}{4}\left|\tilde{F}(q_x - p_x, q_y - p_y)\right|$$

- *$\tilde{F}(x, y)$ is the homogeneous polynomial $(*)$.*
- *The max. vertical distance is attained at the midpoint.*





Setup

From now on,

$$S = \{ (x, y, z) : z = xy \}$$

(by a linear transformation of the *x*-*y*-plane)

Goal

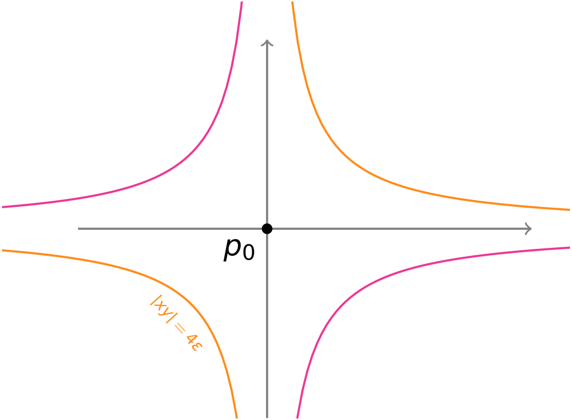
Given $\varepsilon > 0$, find a triangle T with vertices $p_0, p_1, p_2 \in S$ of *largest area* such that

$$\mathrm{dist}_V(T, S) \leq \varepsilon$$

Translated and reflected copies of T have the same error and tile the plane:

max. AREA ⟺ min. NUMBER of triangles

Maximize the Area of Planar Triangles



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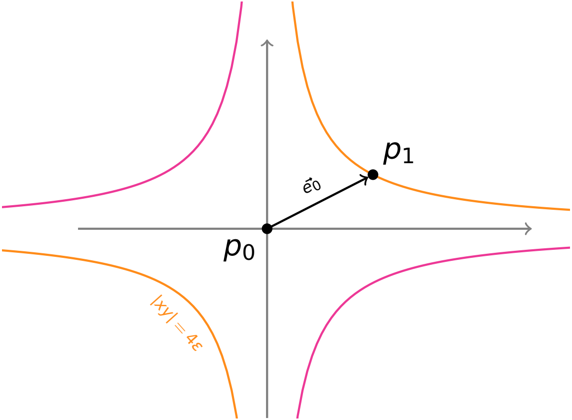

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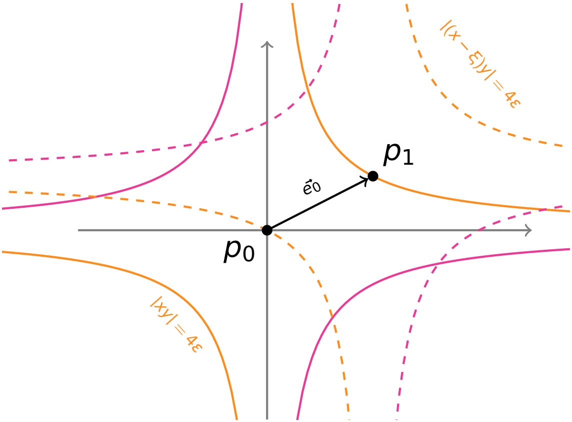

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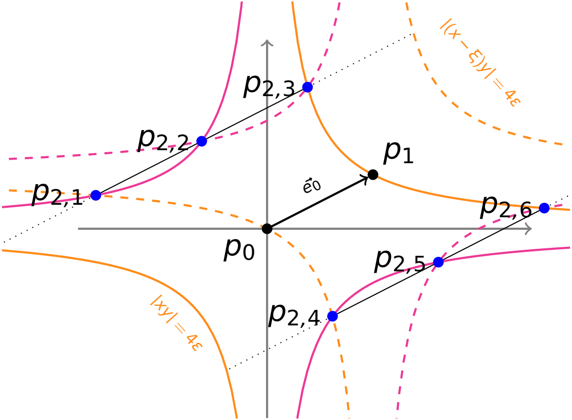

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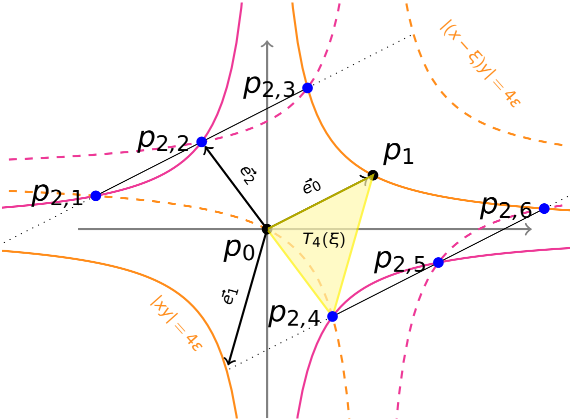
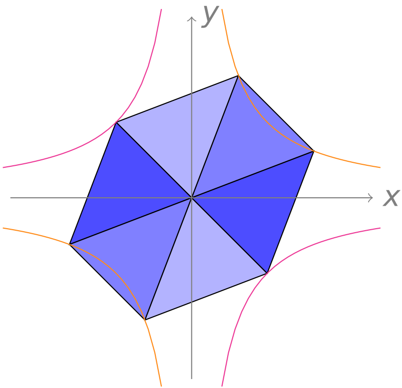


Optimize the Shape of Planar Triangles

Secondary criterion: Maximize the smallest angle



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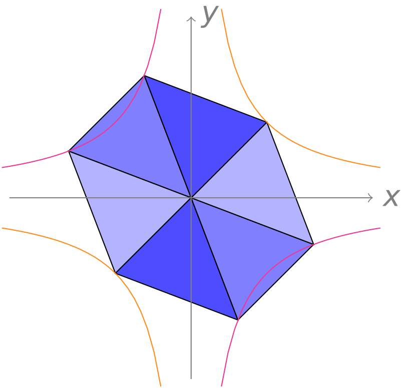
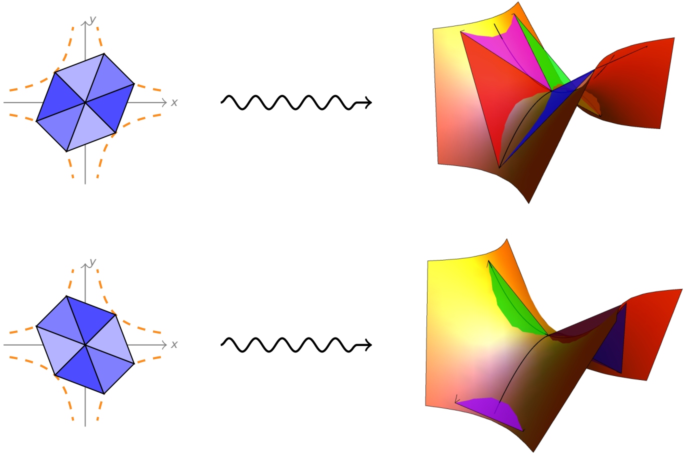


Lift the planar triangulation to the surface



Can We Do Better?

What do we have?

Given an $\varepsilon > 0$ and a saddle surface S, we can find a family $\mathcal{T}$ of triangles which interpolate the surface and

- have maximum area,
- maintain $\text{dist}_V(S, T) \leq \varepsilon$ for all $T \in \mathcal{T}$.

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Question...

- Can this be improved by allowing *non-interpolating* triangles?
- Pottmann et al. (2000) conjectured NO.

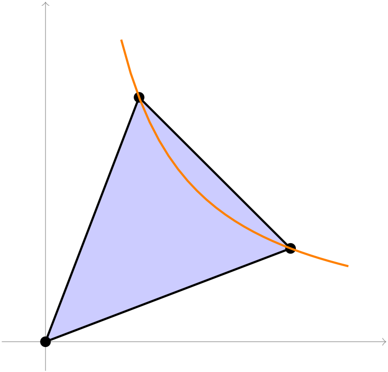
This question is easy for *convex* approximation.

Pseudo-Euclidean Transformations

- A λ-pseudo Euclidean map is given by:

$$(x, y) \mapsto (\lambda x, \tfrac{1}{\lambda} y)$$

- Vertical distance is preserved.
- Area (projected) is preserved.
- Surface $S = \{ z = xy \}$ is preserved.



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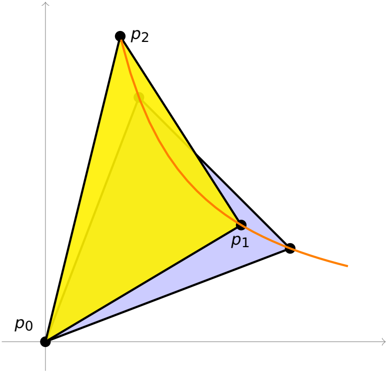

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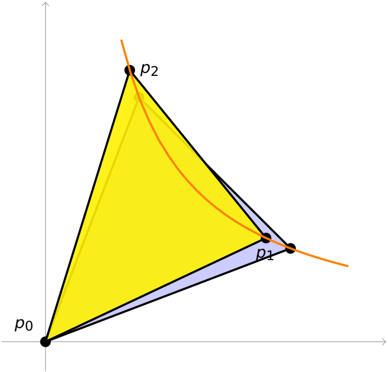

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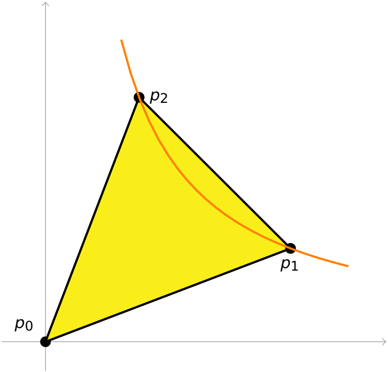

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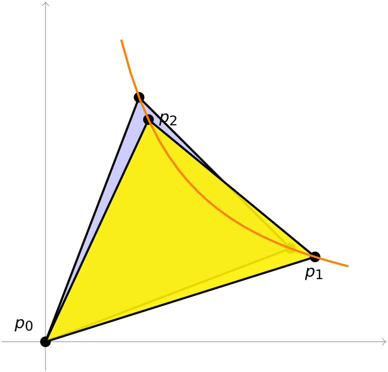

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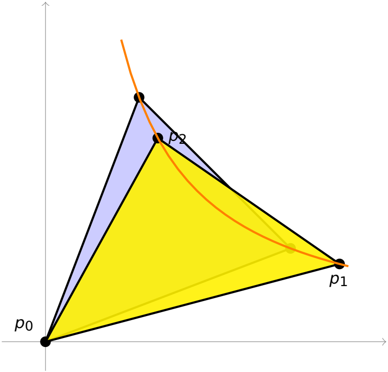
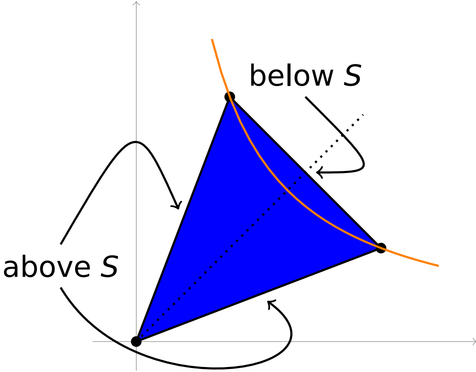


Fact

The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\varepsilon$.



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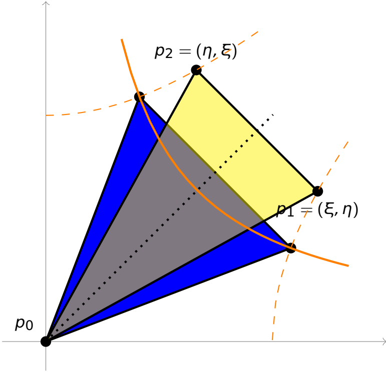

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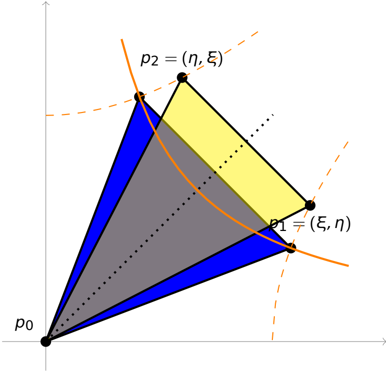

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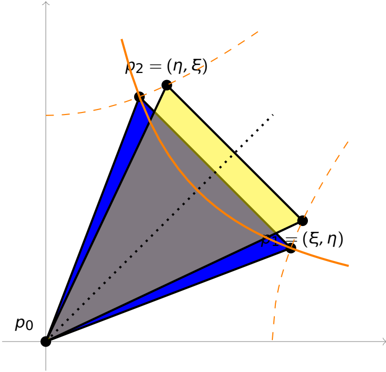

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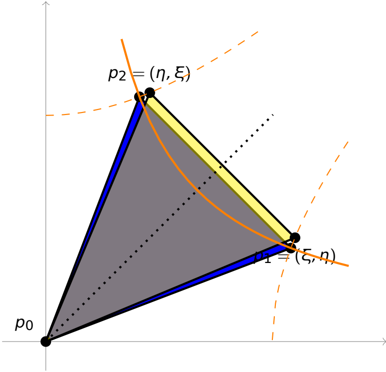

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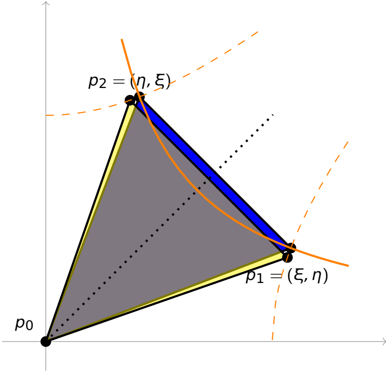

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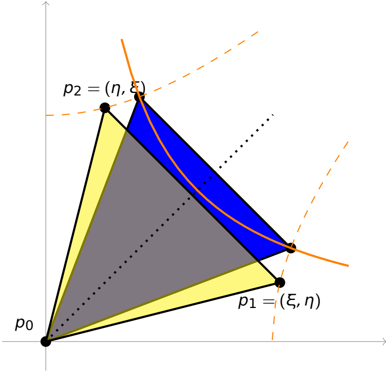

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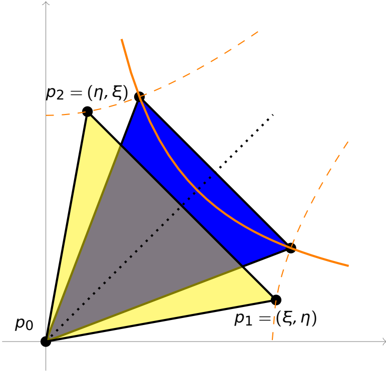
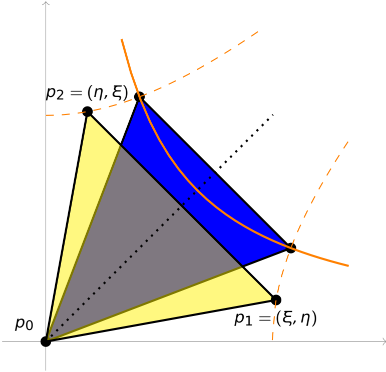
- one-parameter family of area preserving triangles



Fact

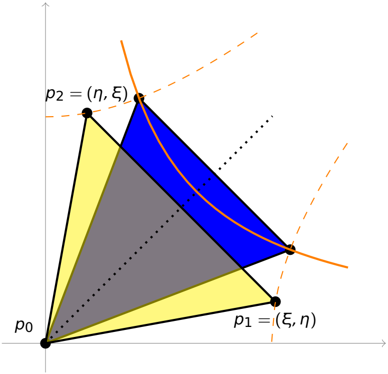
The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\varepsilon$.

- one-parameter family of area preserving triangles
- How should they be lifted?



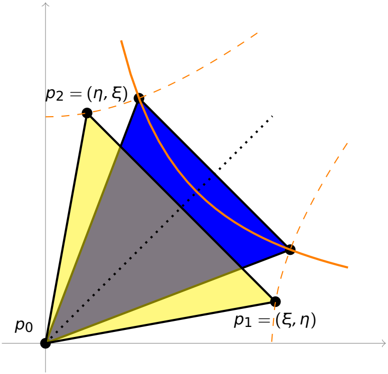
Vertical Perturbed Lifting

- Lift the triangle vertically such that the distance to S is minimized.



- Lift the triangle vertically such that the distance to S is minimized.
- Lift vertices off the surface by α:

$$S_\alpha = \{(x, y, z) : z = xy + \alpha\}$$



- Lift the triangle vertically such that the distance to S is minimized.
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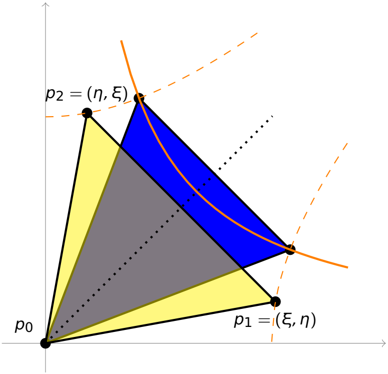
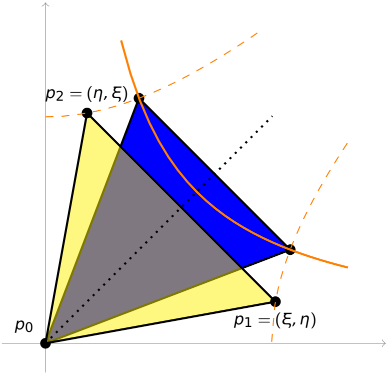
- Vertical distance is attained at midpoints.



- Vertical distances from edges to S are

$$\frac{\xi\eta}{4} + \alpha > 0$$
$$\frac{1}{4}(\xi - \eta)^2 - \alpha > 0$$

and have to be equal.



- Vertical distances from edges to S are

$$\frac{\xi\eta}{4} + \alpha > 0$$
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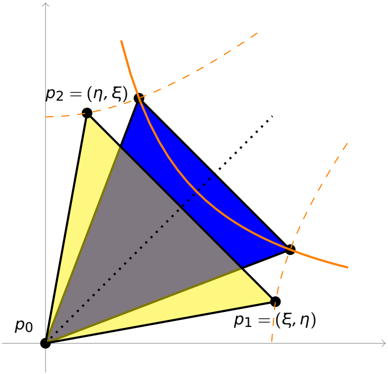
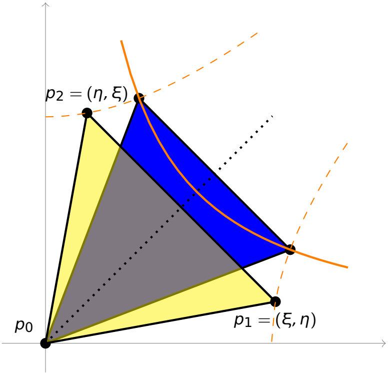
  and have to be equal.
- $\alpha = \frac{1}{8}(\xi^2 - 3\xi\eta + \eta^2)$



- The vertical distance is

$$\left|\frac{1}{8}(\xi^2 - \xi\eta + \eta^2)\right|$$



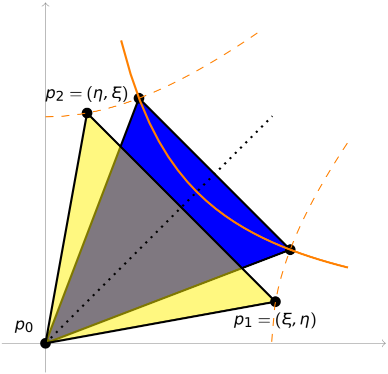
Vertical Perturbed Lifting (Cont.)

- The vertical distance is

$$\left|\frac{1}{8}(\xi^2 - \xi\eta + \eta^2)\right|$$

- Minimum is attained for

$$\xi_0 = \sqrt{2\sqrt{5}\varepsilon\frac{2+\sqrt{3}}{\sqrt{3}}}$$



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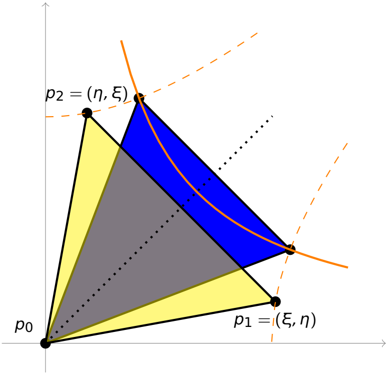
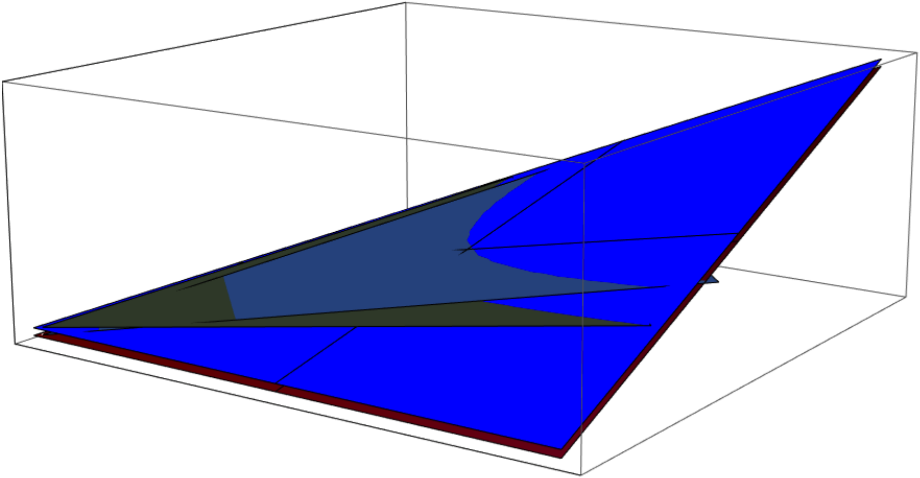
- and the vertical distance is

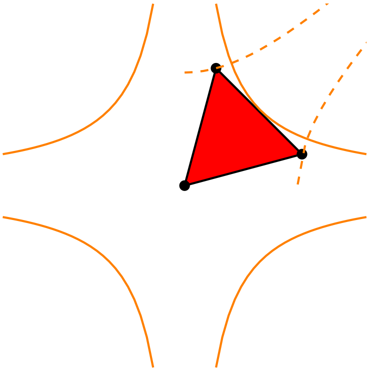
$$\frac{\sqrt{15}}{4}\varepsilon \approx 0.968246\varepsilon$$



Picture in Space

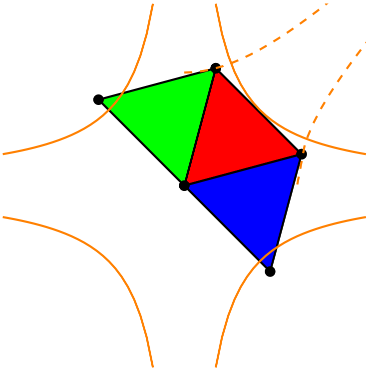


- Pseudo-euclidean motions give a one-parameter family of optimal triangles.



The Planar Super-Optimal Triangle

- Pseudo-euclidean motions give a one-parameter family of optimal triangles.
- Note the tangency property



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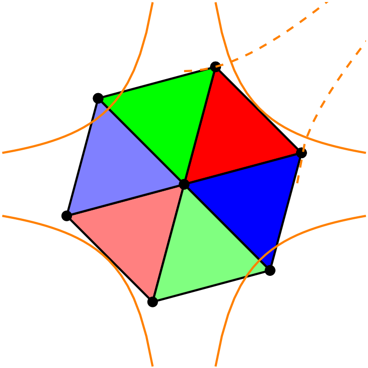

OPEN:
Lift vertices by *different* amounts?