Pseudotriangulations

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Day 5

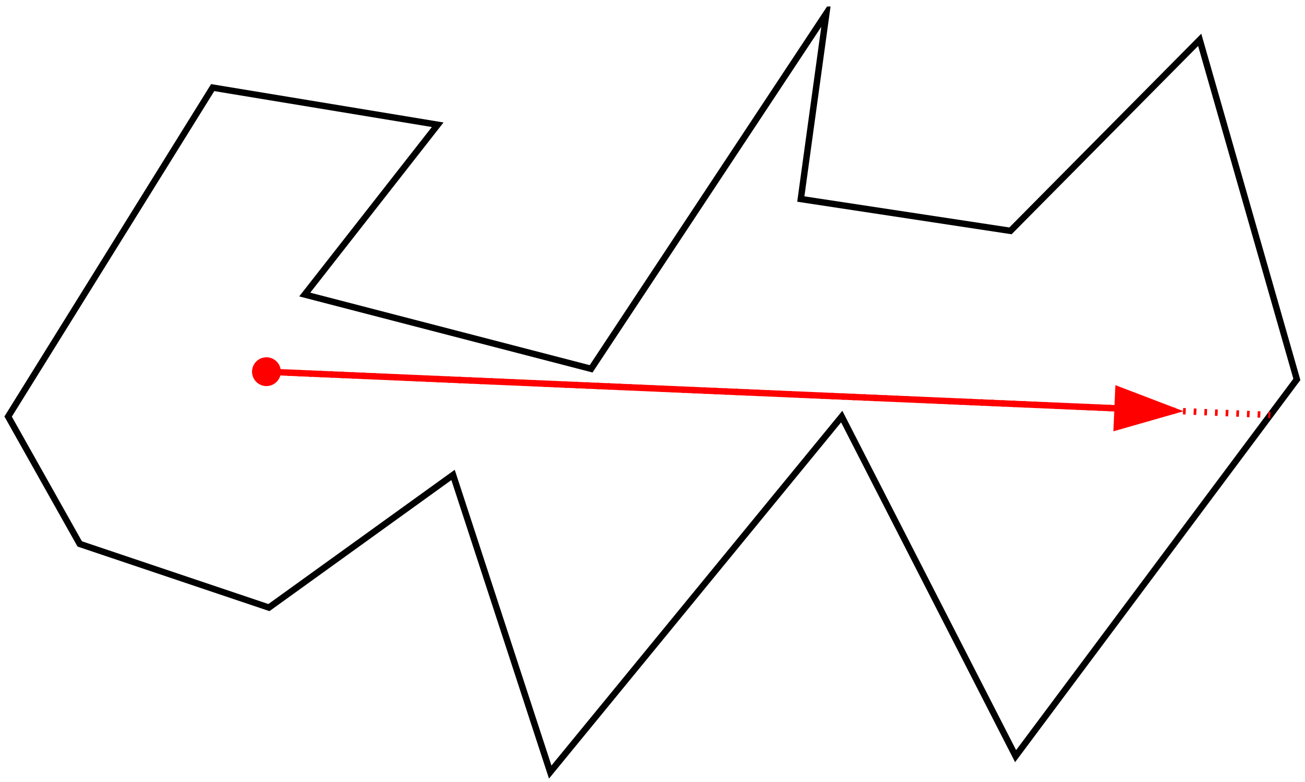
Literature:

Pseudo-triangulations — a survey.
G.Rote, F.Santos, I.Streinu, 2008

Outline

1. Motivation: ray shooting
2. Pseudotriangulations: definitions and properties
3. Rigidity, Laman graphs
4. Rigidity: kinematics of linkages
5. Liftings of pseudotriangulations to 3 dimensions

1. Motivation: Ray Shooting in a Simple Polygon



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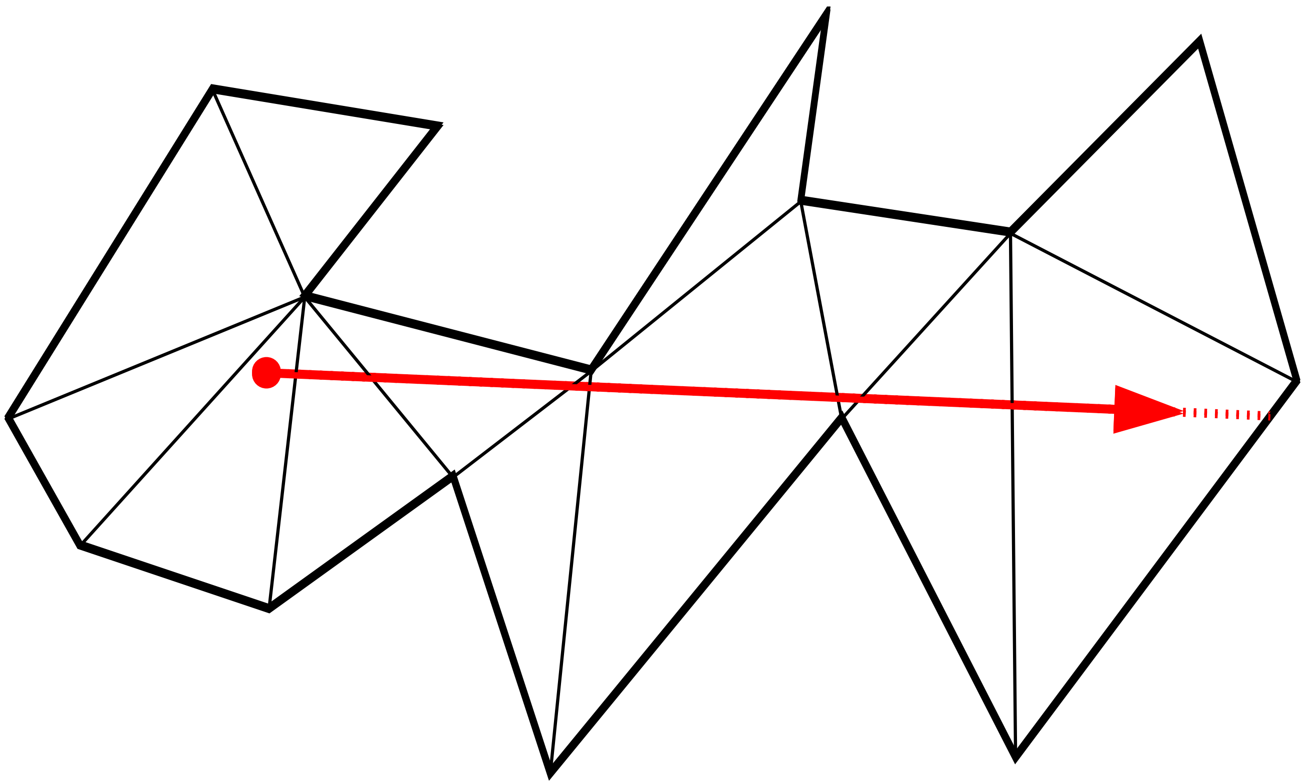

Walking in a triangulation:

Walk to starting point. Then walk along the ray.

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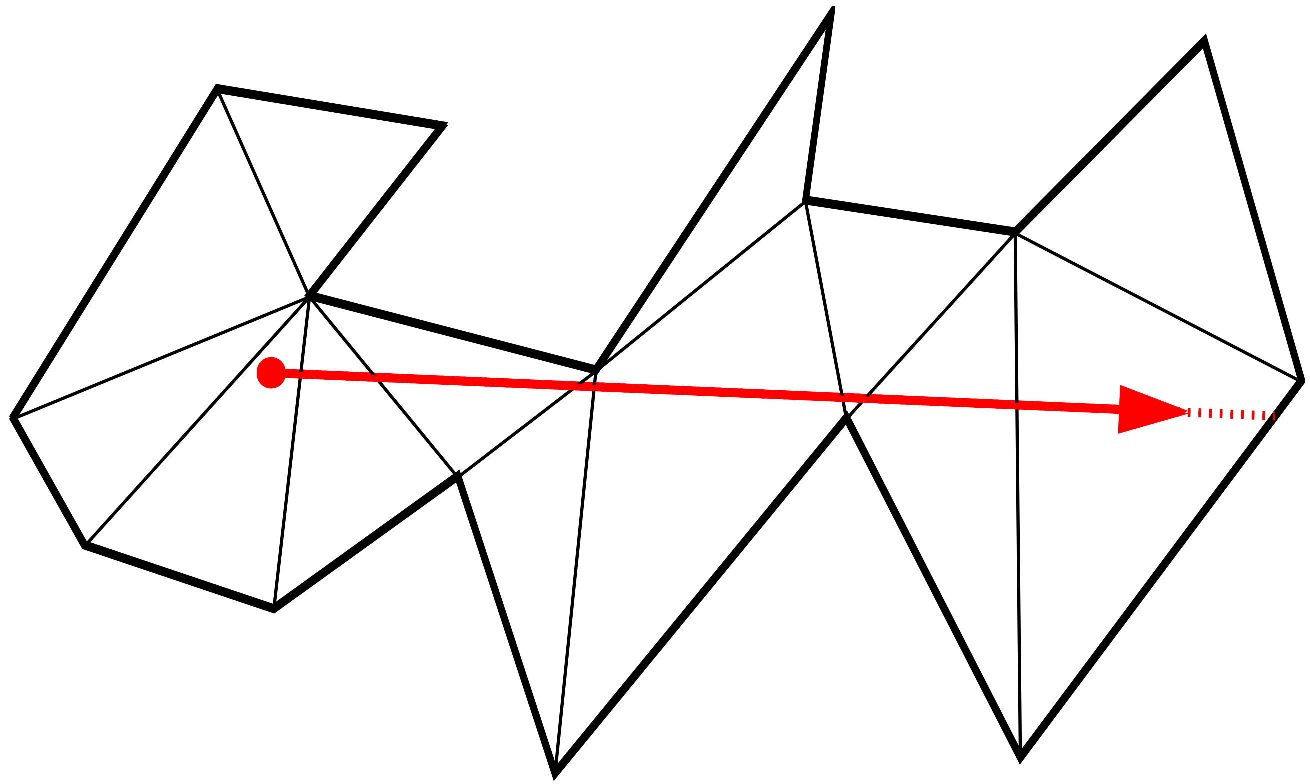
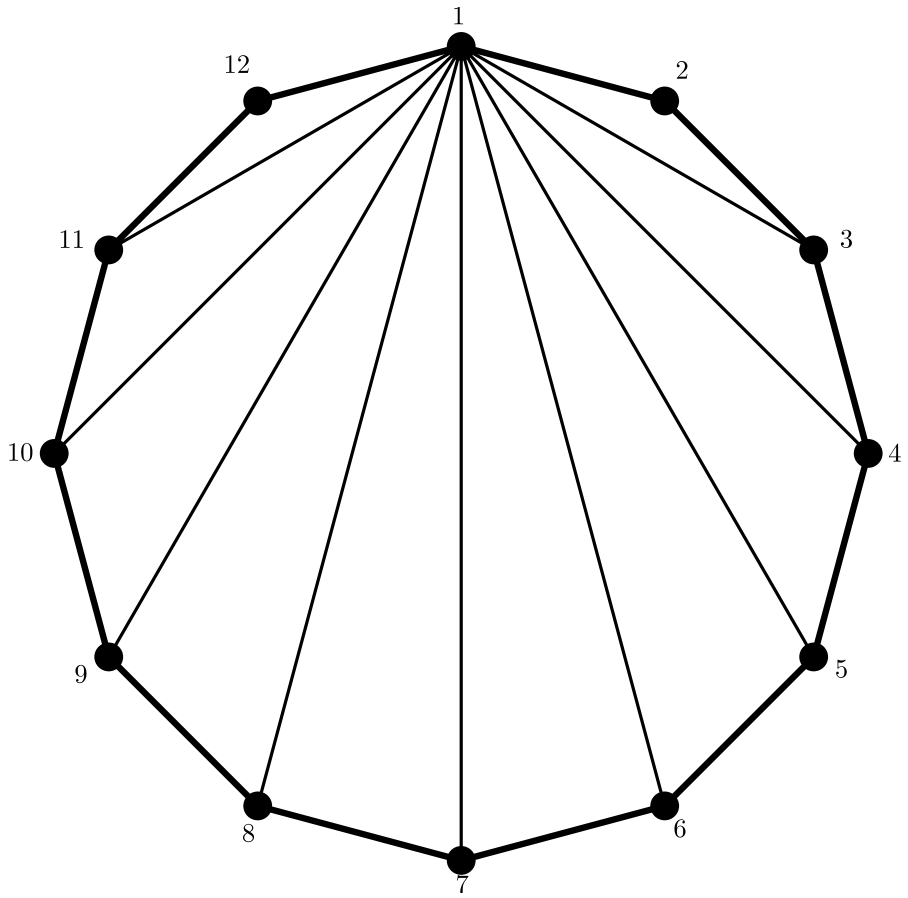


Walking in a triangulation:

Walk to starting point. Then walk along the ray.

$O(n)$ steps in the worst case.

Triangulations of a *convex* polygon



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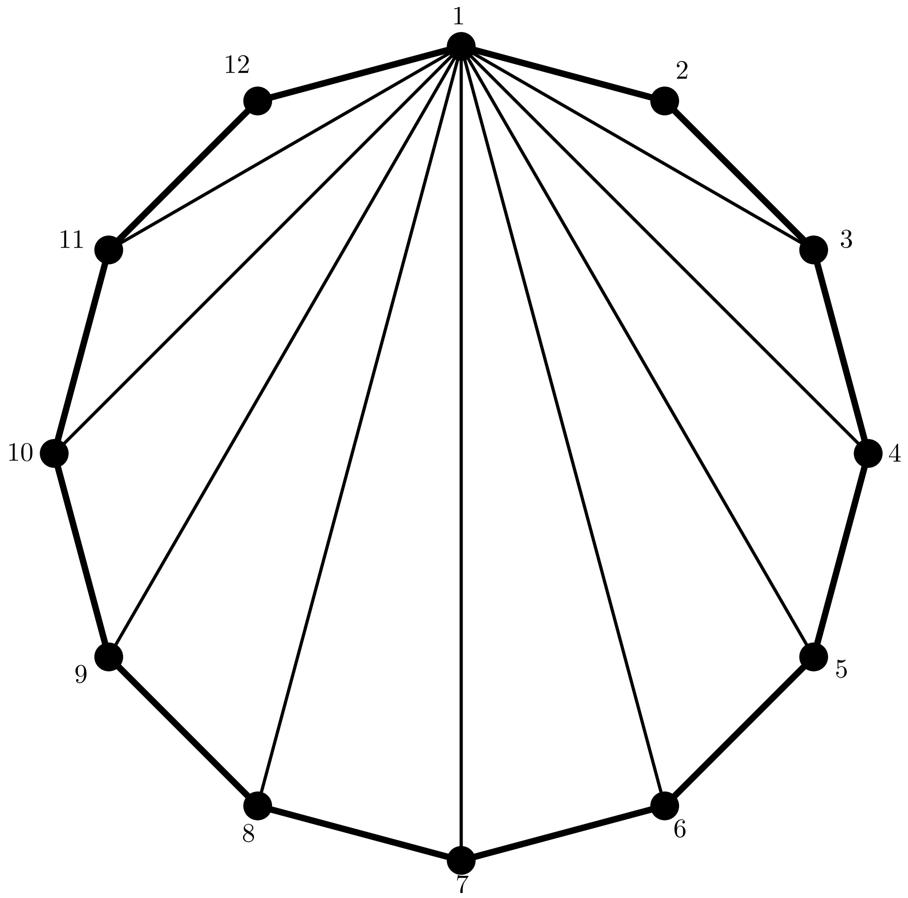

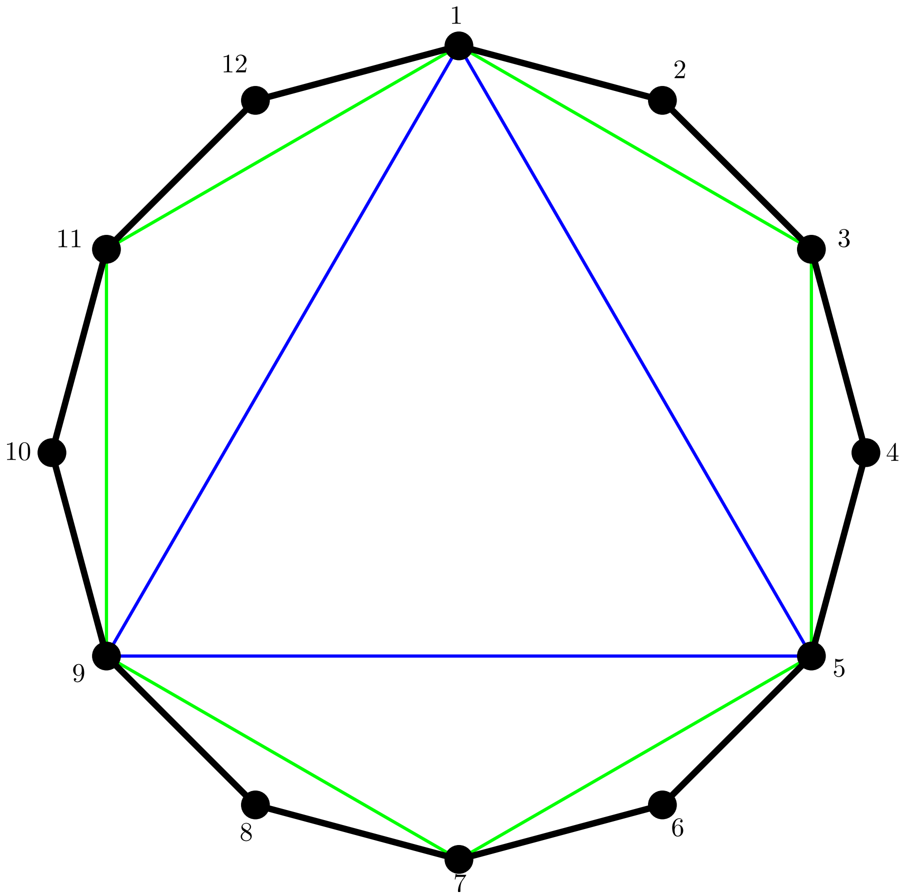

balanced triangulation
A path crosses $O(\log n)$
triangles.

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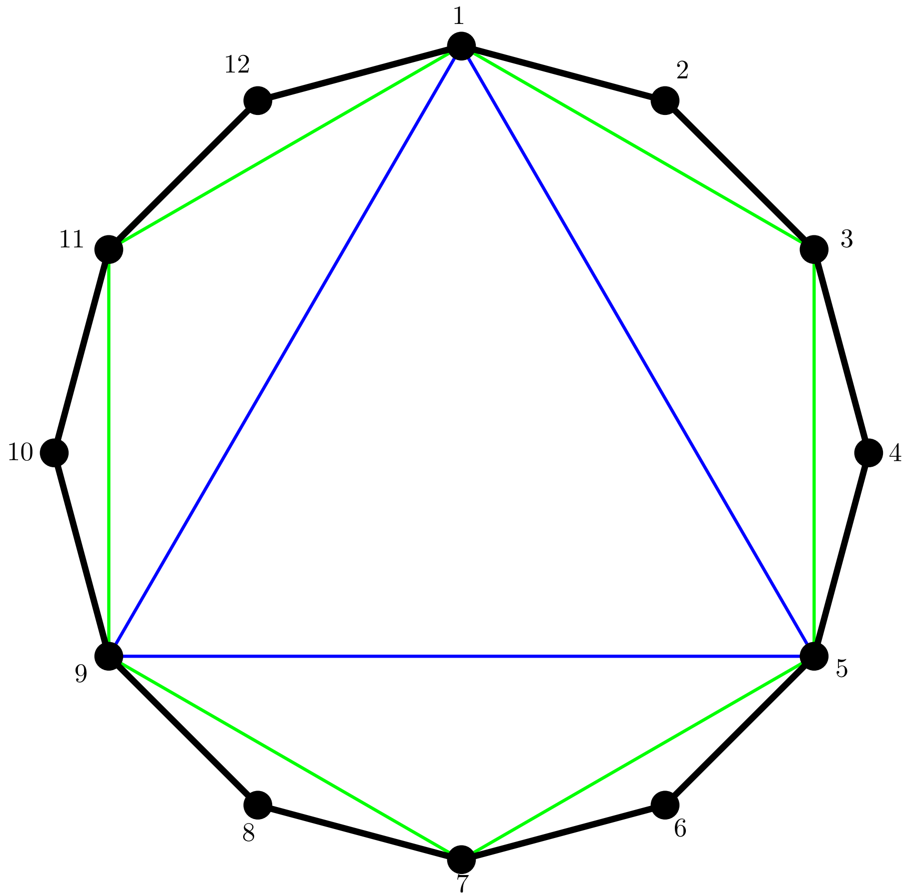
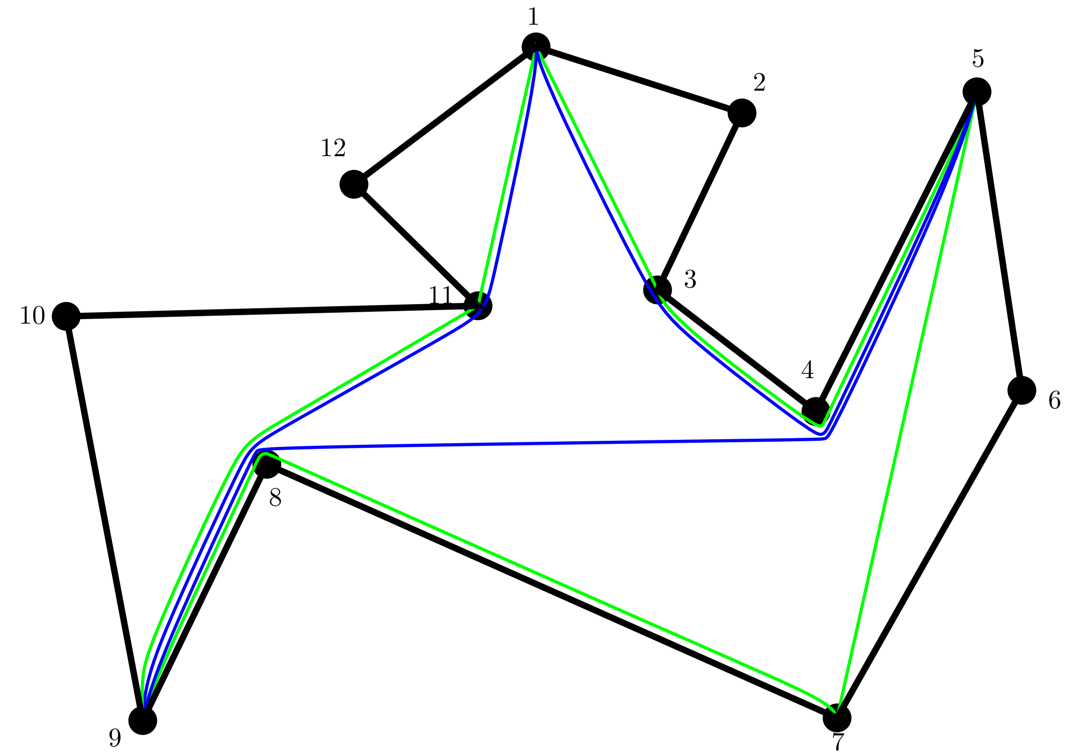


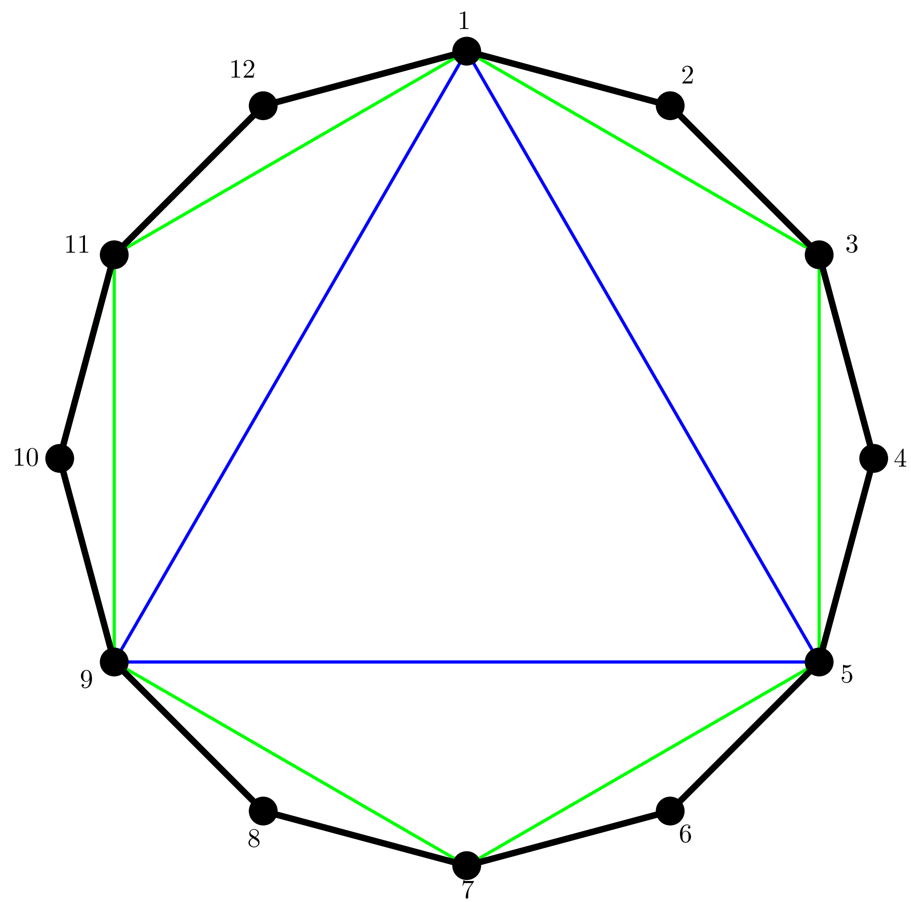


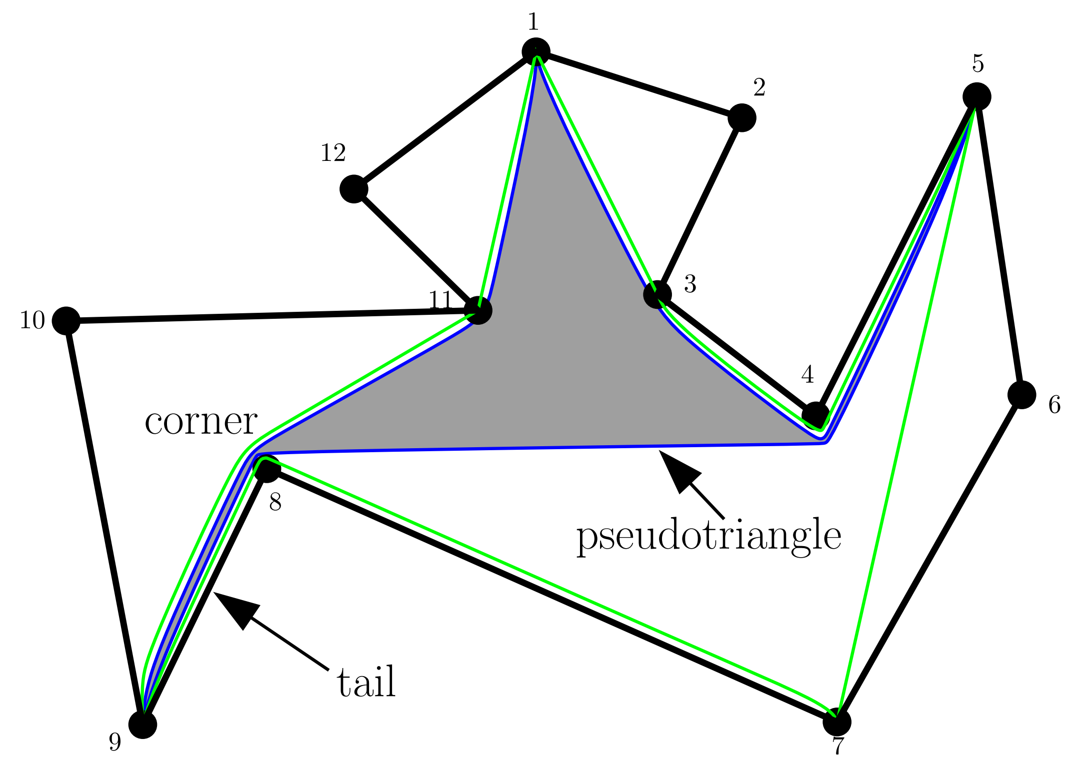
balanced triangulation:
An edge crosses $O(\log n)$
triangles.

balanced *geodesic* triangulation:
An edge crosses $O(\log n)$
pseudotriangles.

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]

Triangulations of a *simple* polygon



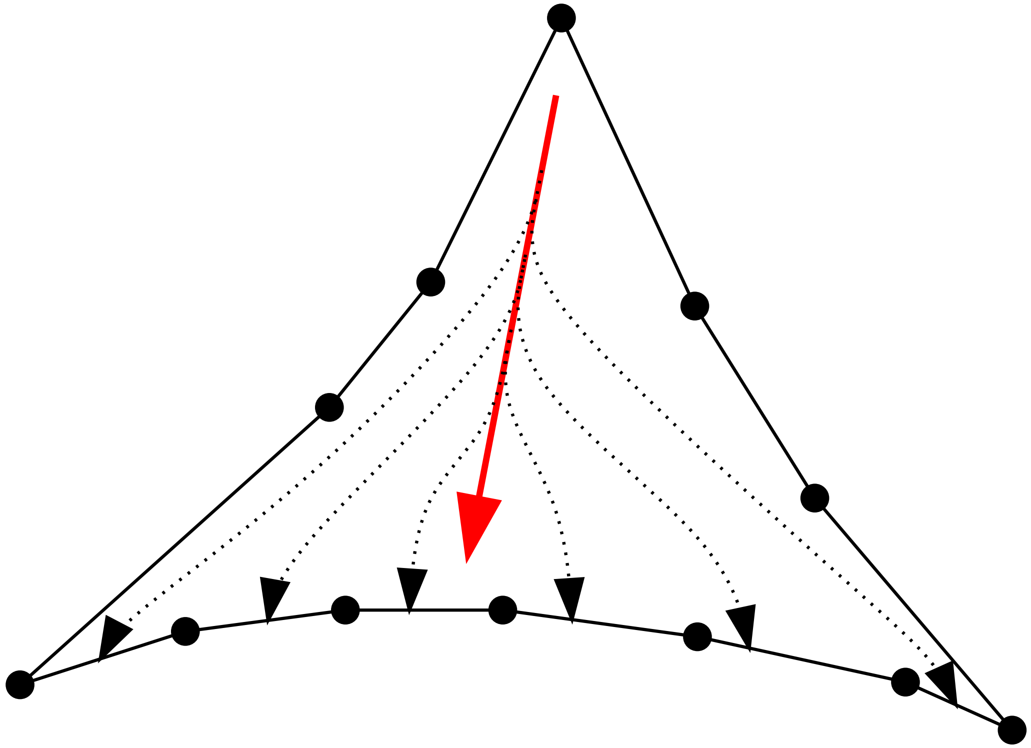


balanced triangulation: An edge crosses $O(\log n)$ triangles.

balanced *geodesic* triangulation: An edge crosses $O(\log n)$ pseudotriangles.

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]

Going through a single pseudotriangle



balanced binary tree for each pseudo-edge:
$\rightarrow$ $O(\log n)$ time per pseudotriangle
$\rightarrow O(\log^2 n)$ time total

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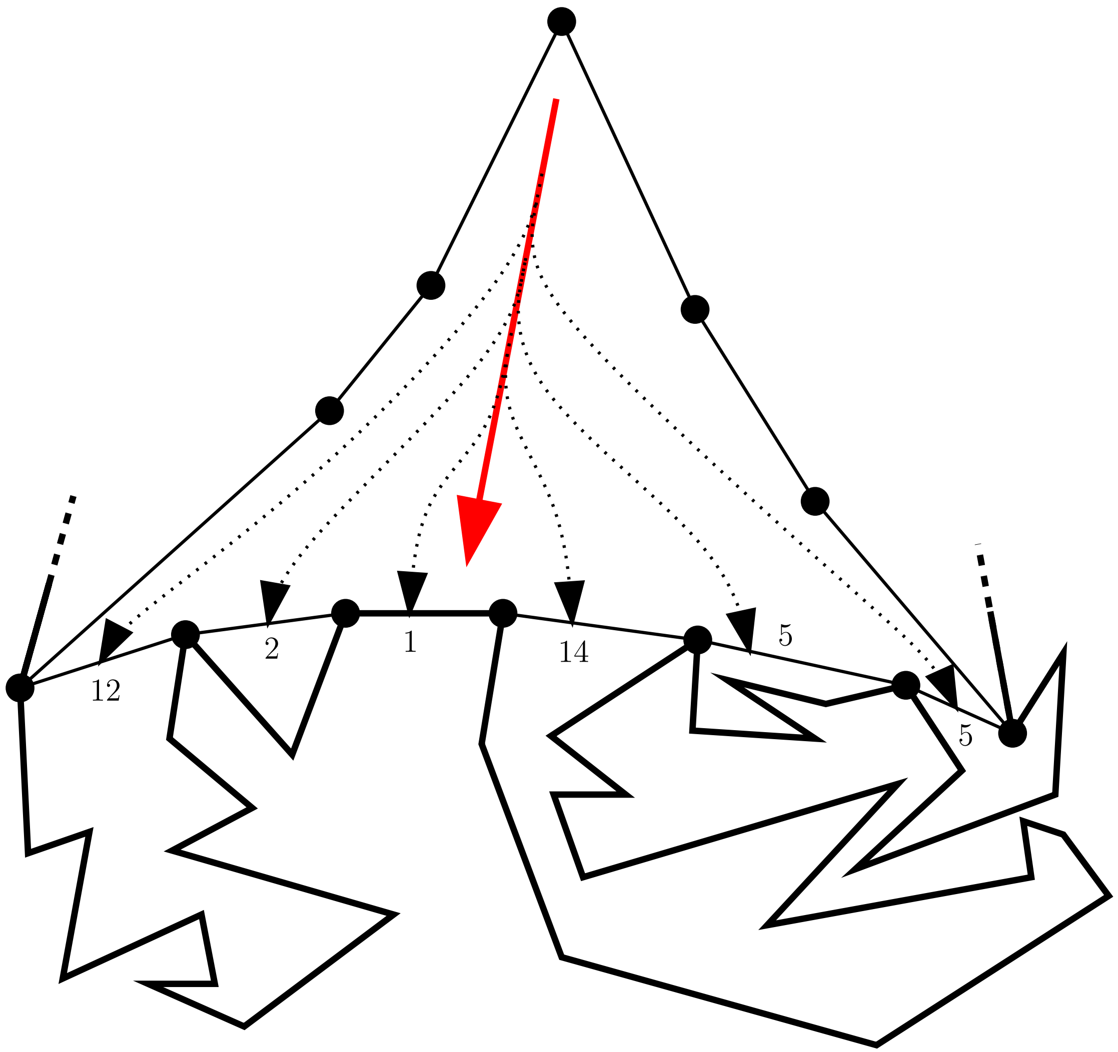


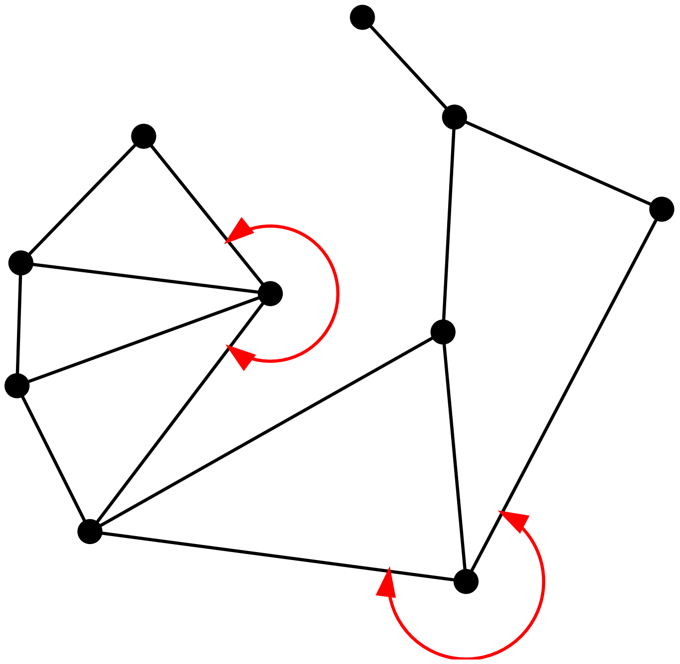
balanced binary tree for each pseudo-edge:
$\rightarrow$ $O(\log n)$ time per pseudotriangle
$\rightarrow O(\log^2 n)$ time total

weighted binary tree:
$\rightarrow O(\log n)$ time total

2. Pseudotriangulations: Basic definitions and properties

Pointed Vertices

A *pointed* vertex is incident to an angle $> 180^\circ$ (a *reflex* angle or *big* angle).



A straight-line graph is pointed if all vertices are pointed.

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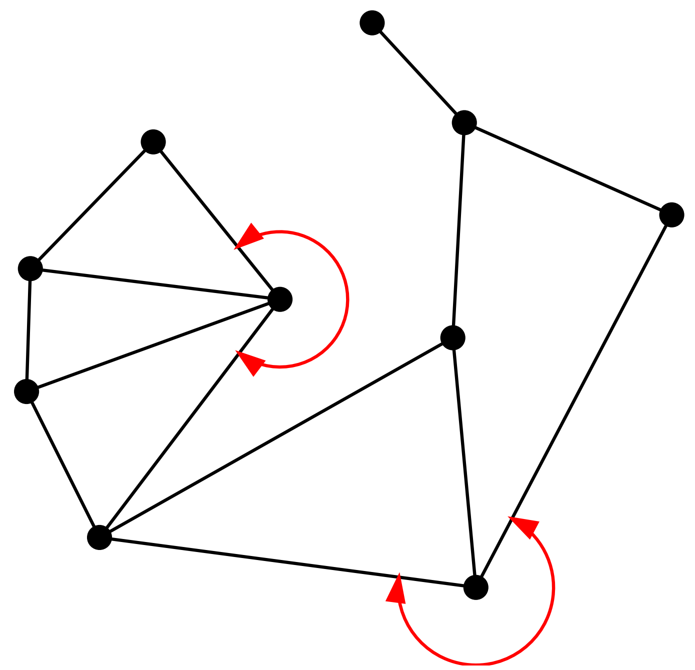

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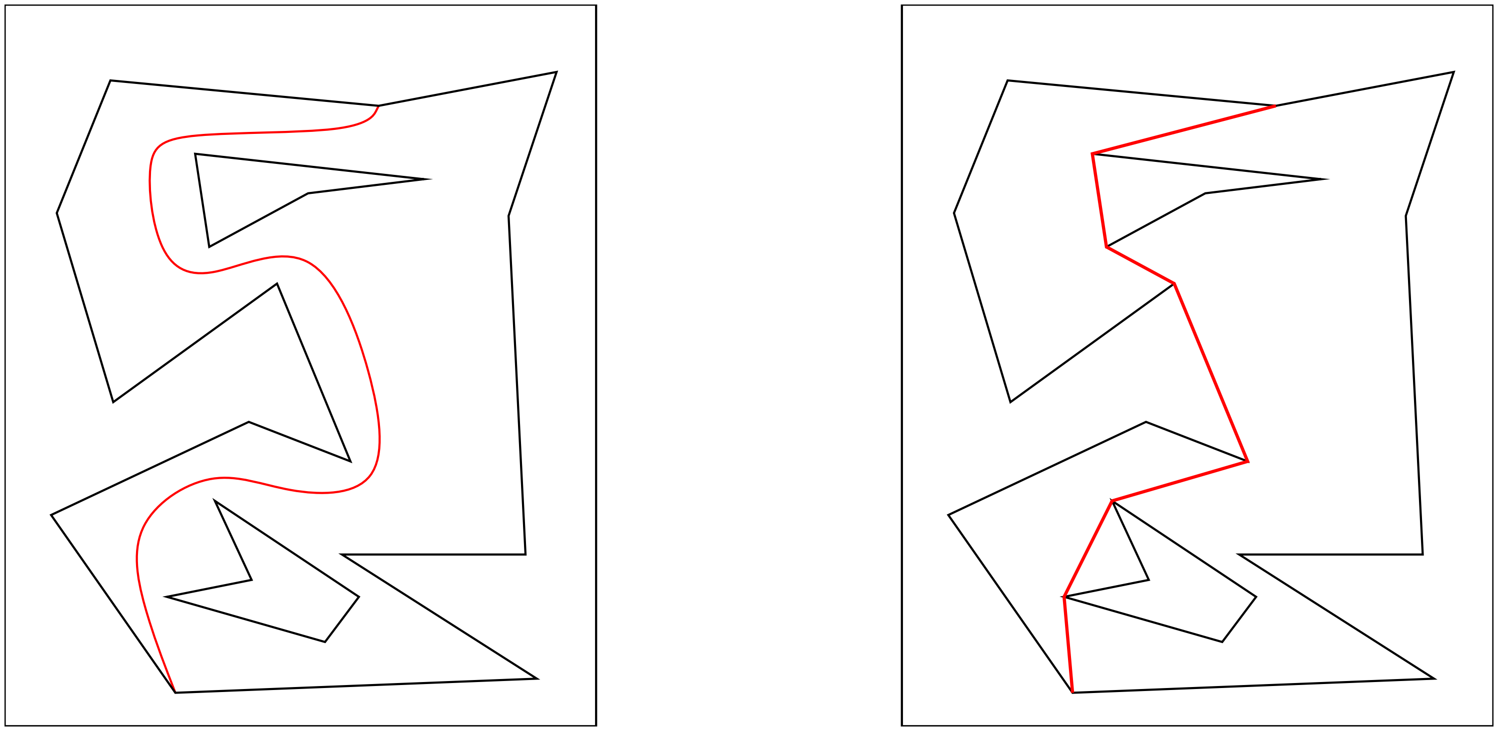
Where do pointed vertices arise?

Geodesic shortest paths

Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.



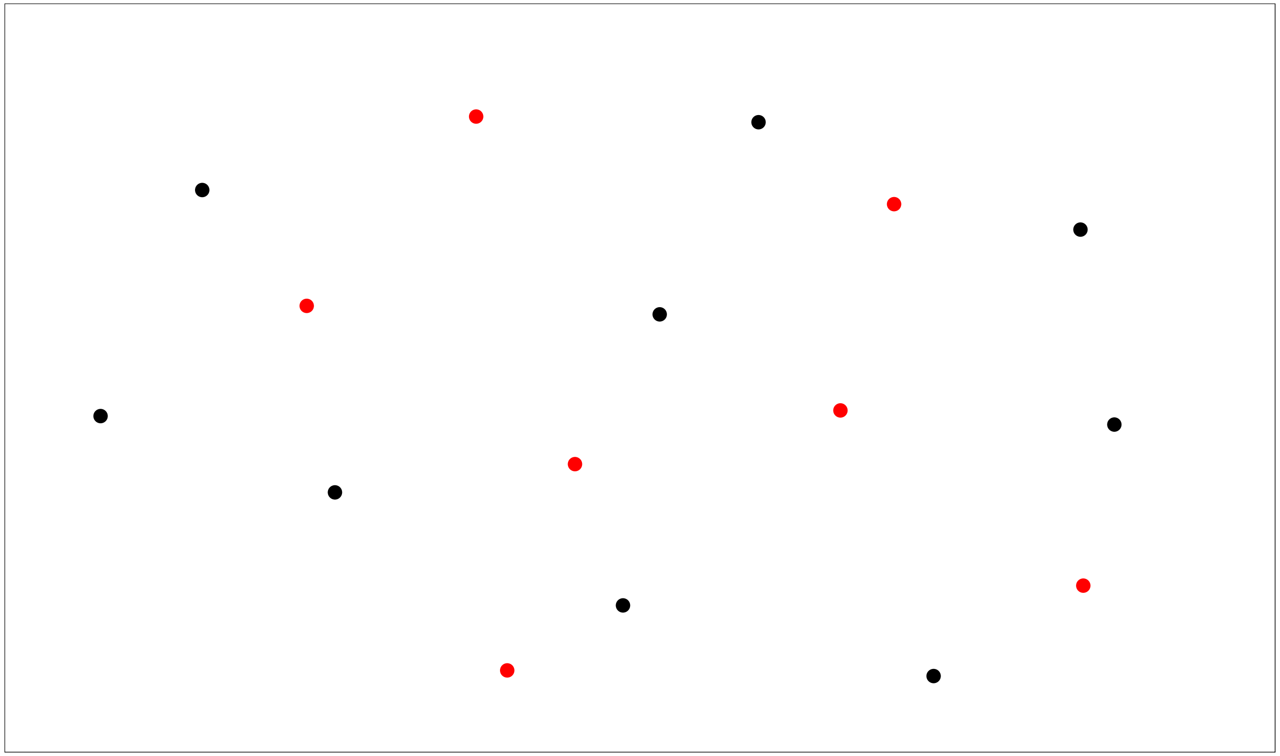
→ *geodesic* triangulations of a simple polygon

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink '94]

Pseudotriangulations

Given: A set V of vertices, a subset $V_p \subseteq V$ of *pointed vertices*.

A *pseudotriangulation* is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in V_p pointed.



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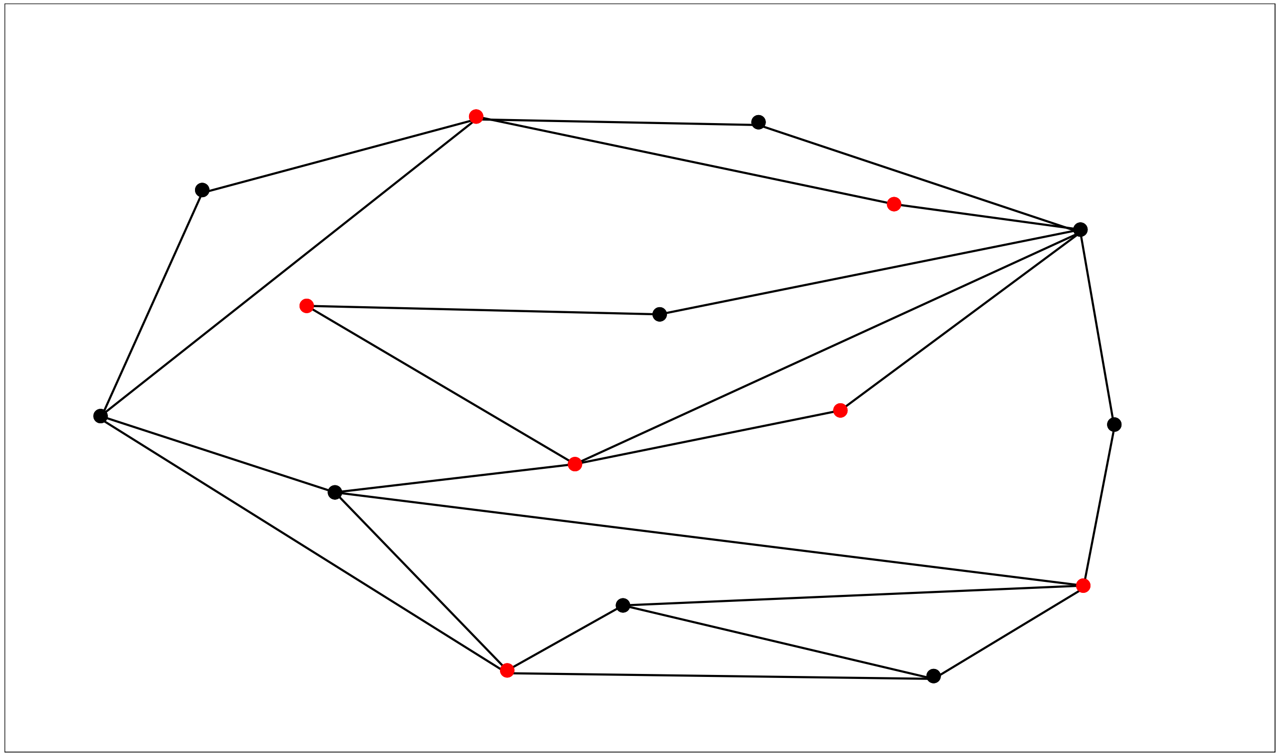

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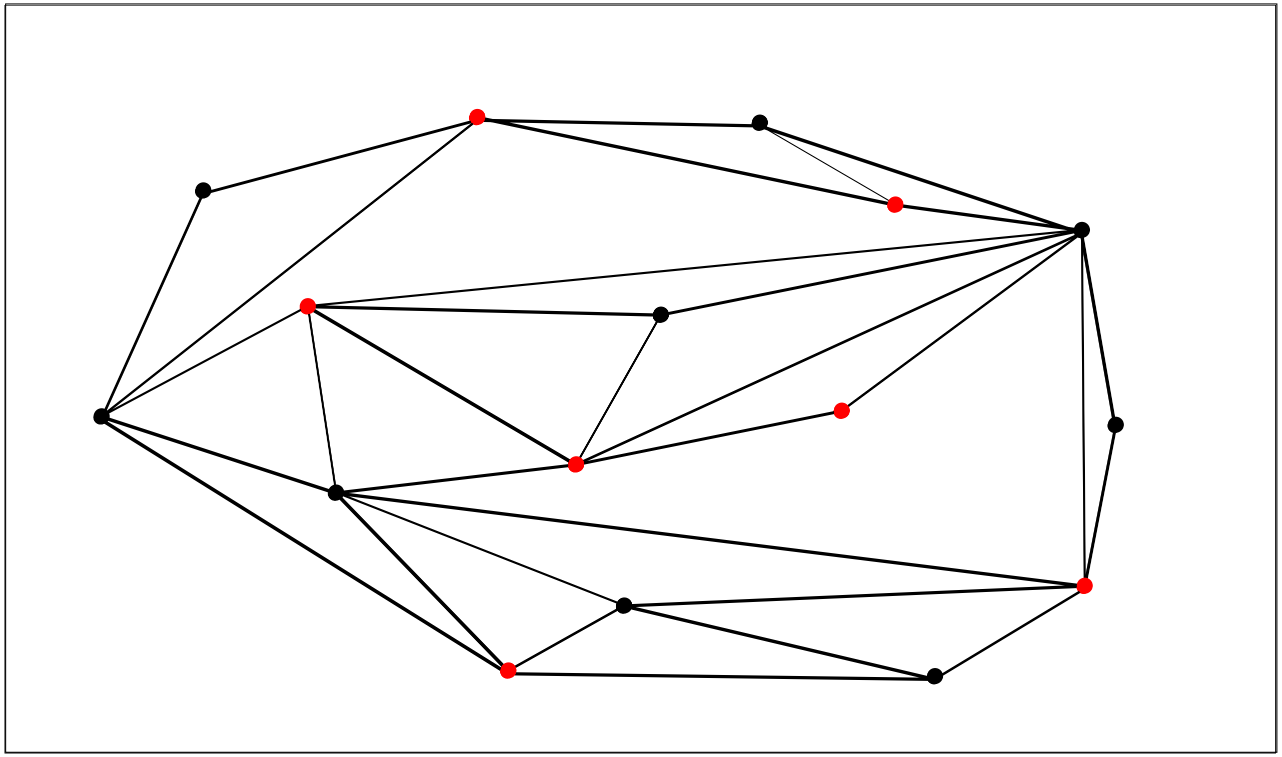

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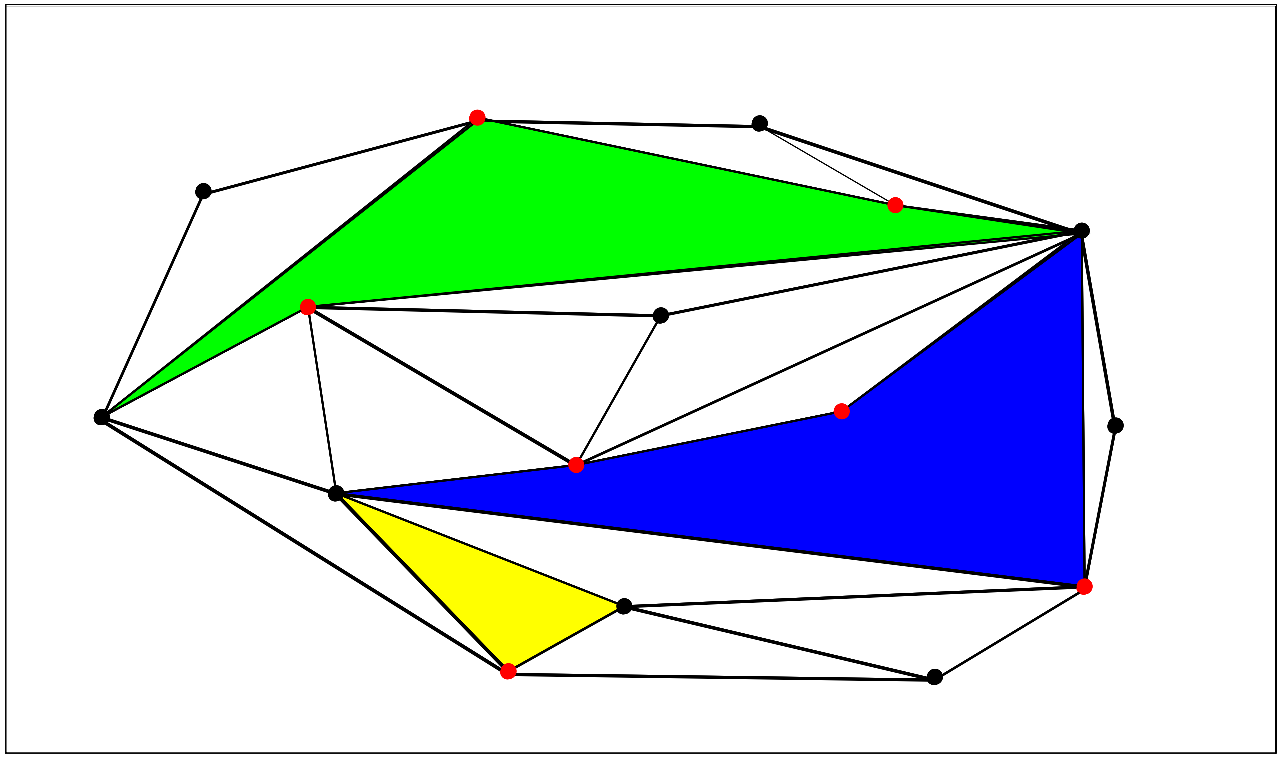
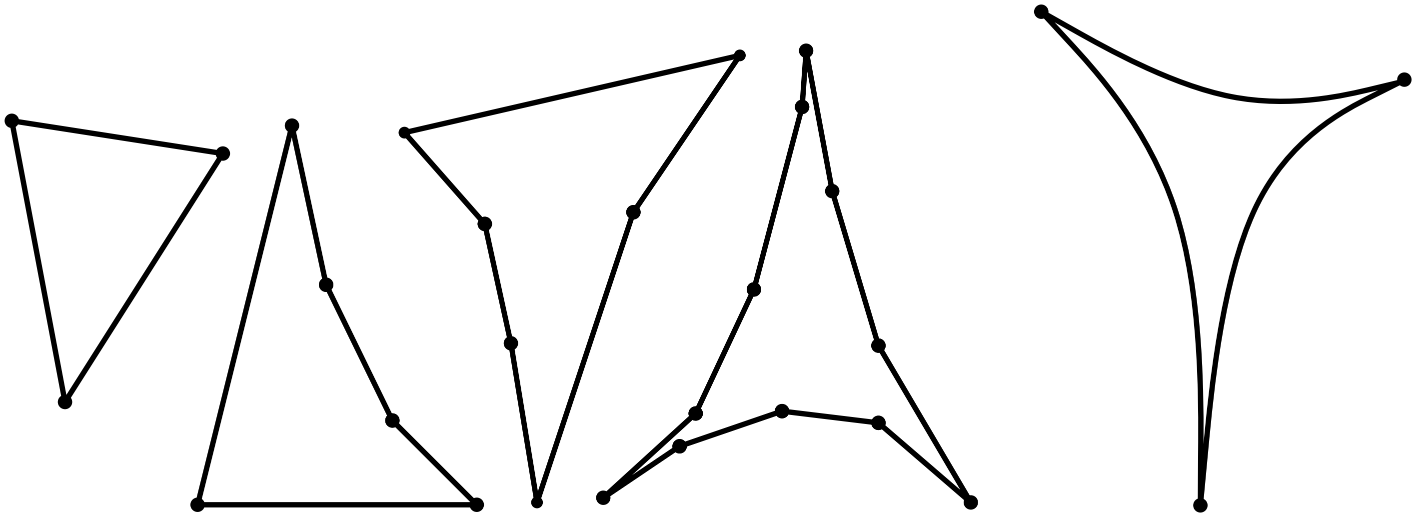


Pseudotriangles

A pseudotriangle has three convex *corners* and an arbitrary number of reflex vertices ($> 180^\circ$).



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Proof. $(1) \implies (2)$ All convex hull edges are in E.
$\rightarrow$ decomposition of the polygon into faces.
Need to show: If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Characterization of pseudotriangulations

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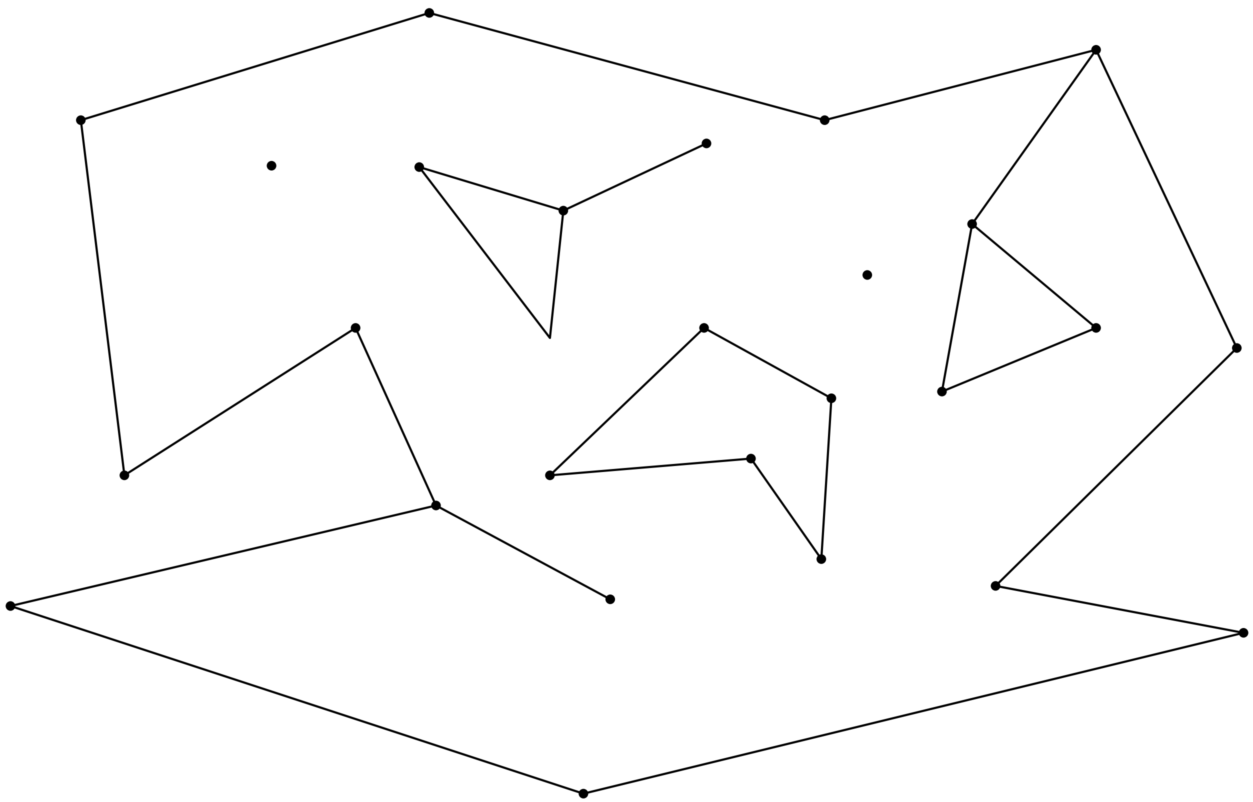
Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.



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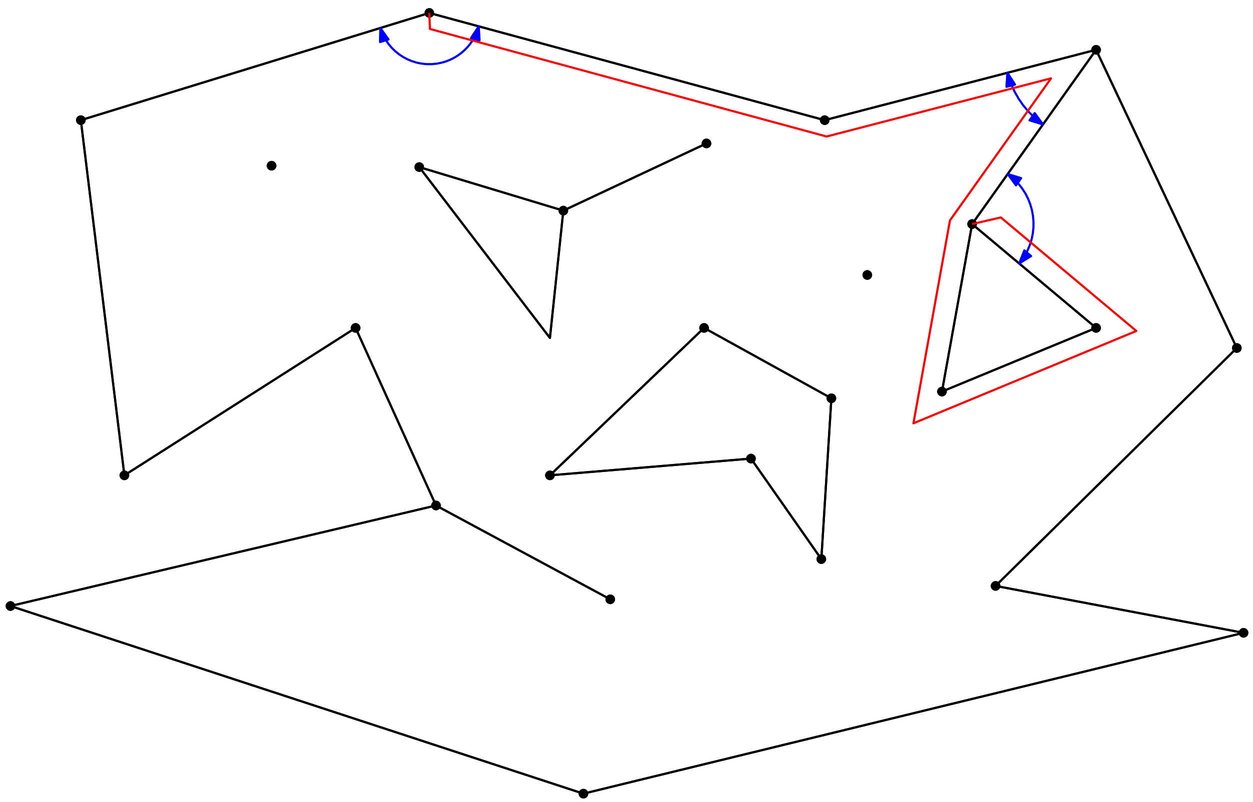

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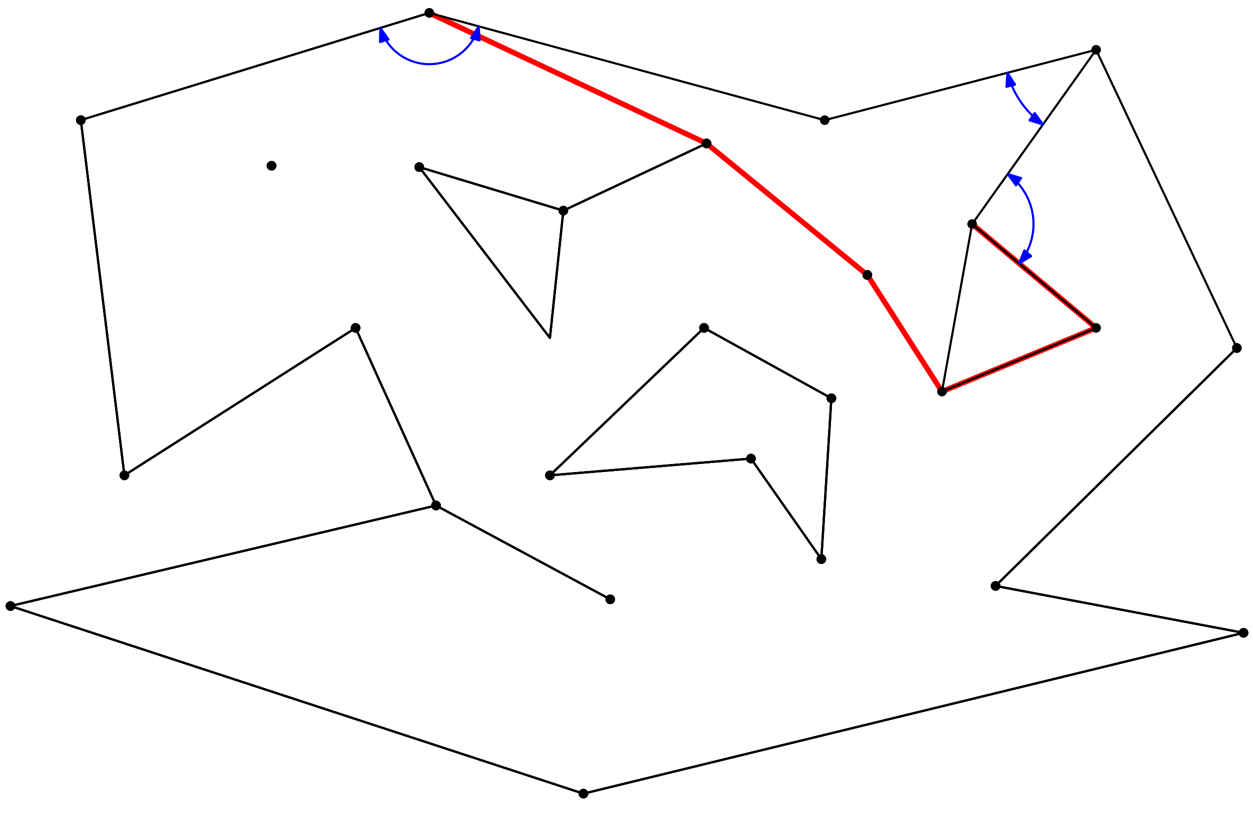
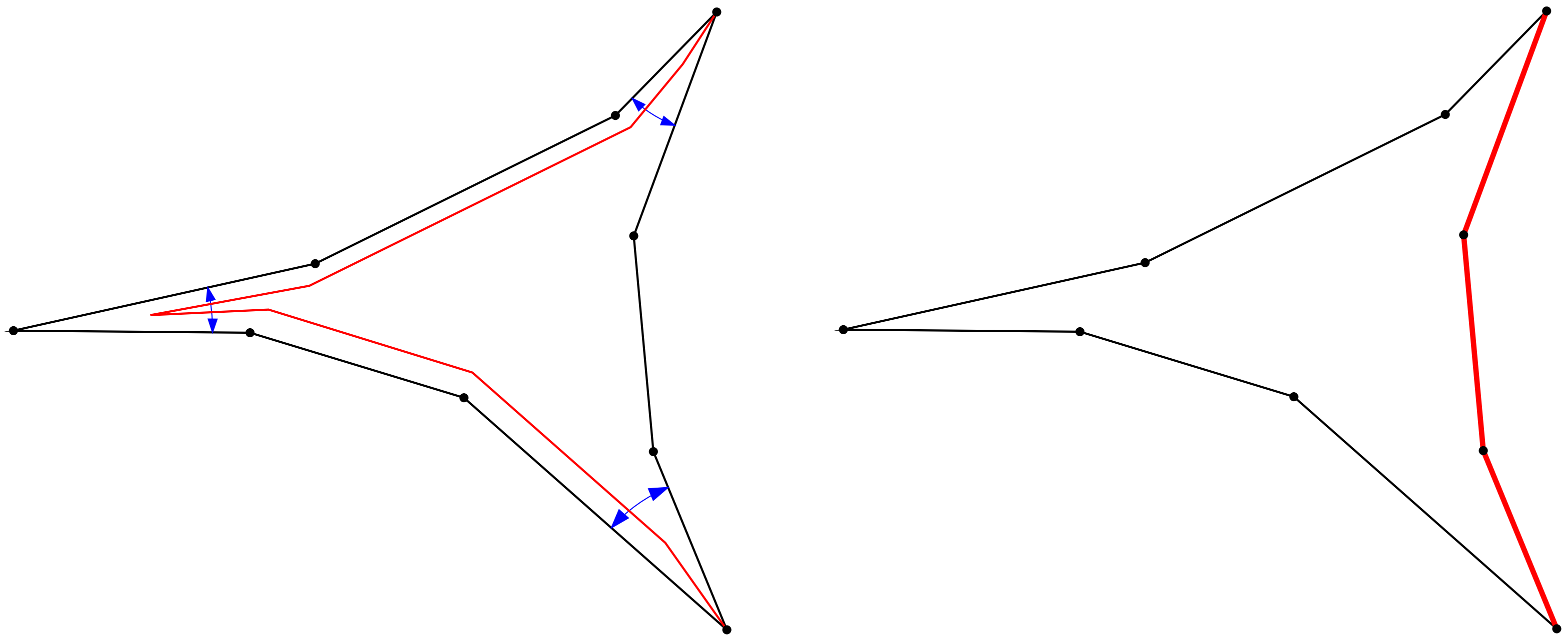


Characterization of pseudotriangulations continued

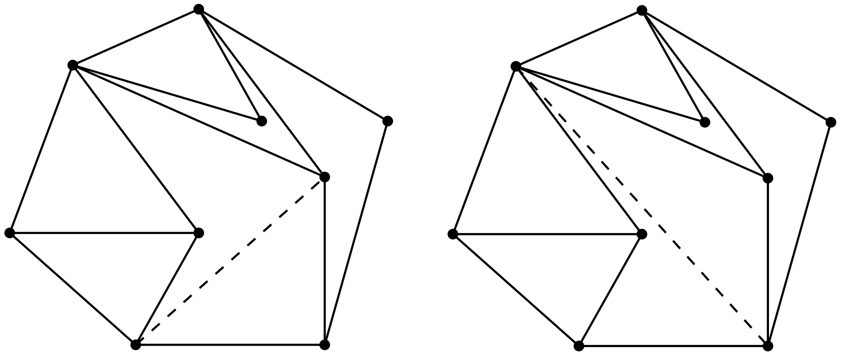
A new edge is always added, unless the face is already a pseudotriangle (without inner obstacles).



[Rote, C. A. Wang, L. Wang, Xu 2003]

Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.



before after

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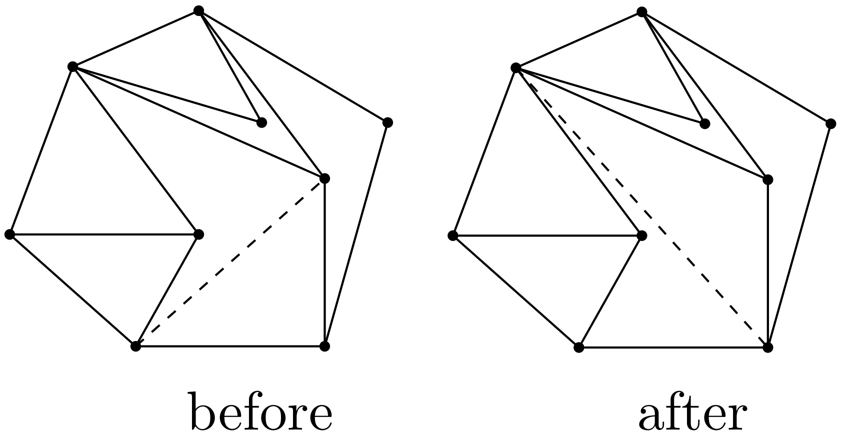

The flip graph is connected.
Its diameter is $O(n \log n)$. [Bespamyatnikh 2003]

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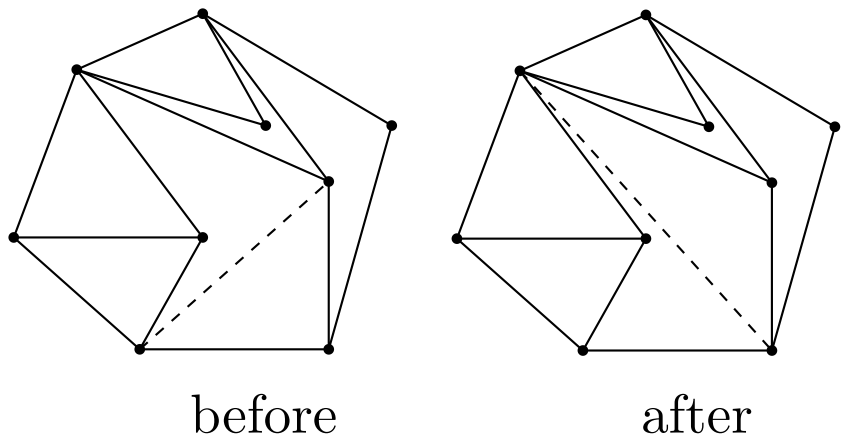

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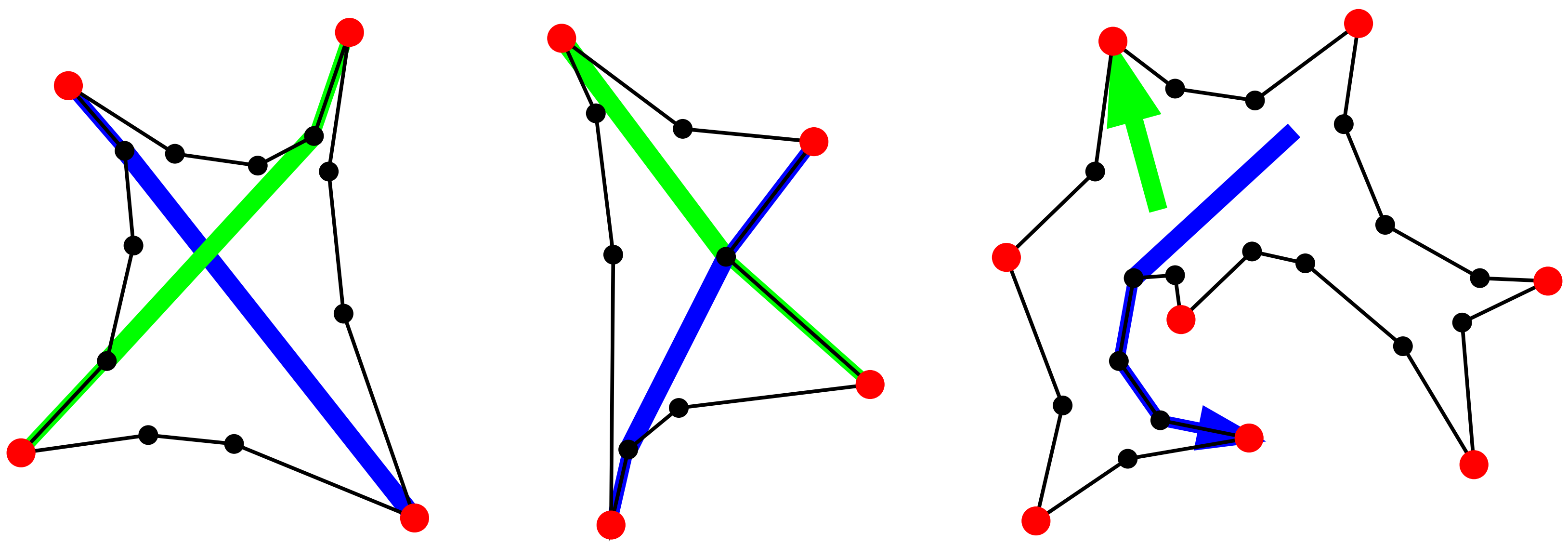
BETTER THAN TRIANGULATIONS!

Flipping

Every pseudoquadrangle has precisely two diagonals, which cut it into two pseudotriangles.

[*Proof.* Every *tangent ray* can be continued to a geodesic path running along the boundary to a corner, in a unique way.]



Vertex and face counts

Lemma. *A pseudotriangulation with* x *nonpointed and* y *pointed vertices has* $e = 3x + 2y - 3$ *edges and* $2x + y - 2$ *pseudotriangles.*

Corollary. *A pointed pseudotriangulation with* n *vertices has* $e = 2n - 3$ *edges and* $n - 2$ *pseudotriangles.*

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$$\underbrace{\sum\nolimits_t k_t + k_{\text{outer}}}_{2e} - 3|T| = y$$

$$e + 2 = (|T| + 1) + (x + y) \qquad \text{(Euler)}$$

Vertex and face counts

Lemma. *A pseudotriangulation with x nonpointed and y pointed vertices has $e = 3x + 2y - 3$ edges and $2x + y - 2$ pseudotriangles.*

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BETTER THAN TRIANGULATIONS!

Corollary. *A non-crossing pointed graph with* $n \geq 2$ *vertices has at most* $2n - 3$ *edges.*

Pseudotriangulations/Geodesic Triangulations

Applications:

- kinetics of bar frameworks, robot motion planning, the "Carpenter's Rule Problem" [Streinu 2000]
- data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]
- kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002]
- . . .

Pseudotriangulations/Geodesic Triangulations

Applications (continued):

- art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]
- locally convex surfaces, reflex-free hull
  [Aichholzer, Aurenhammer, Krasser, Braß 2003]
- pseudotriangulations on the sphere, smooth counterexample surface to a conjecture of A. D. Alexandrov [G. Panina 2005]

3. RIGIDITY, PLANAR LAMAN GRAPHS

What are the *graphs* of pseudotriangulations?

- planar
- $2n - 3$ edges
- . . . ?

Infinitesimal motions — rigid frameworks

A *framework* is a set of movable joints (vertices) connected by rigid *bars* (edges) of fixed length.

n points $p_1, \ldots, p_n$.

1. (global) *motion* $p_i = p_i(t)$, $t \geq 0$

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$$v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0)$$

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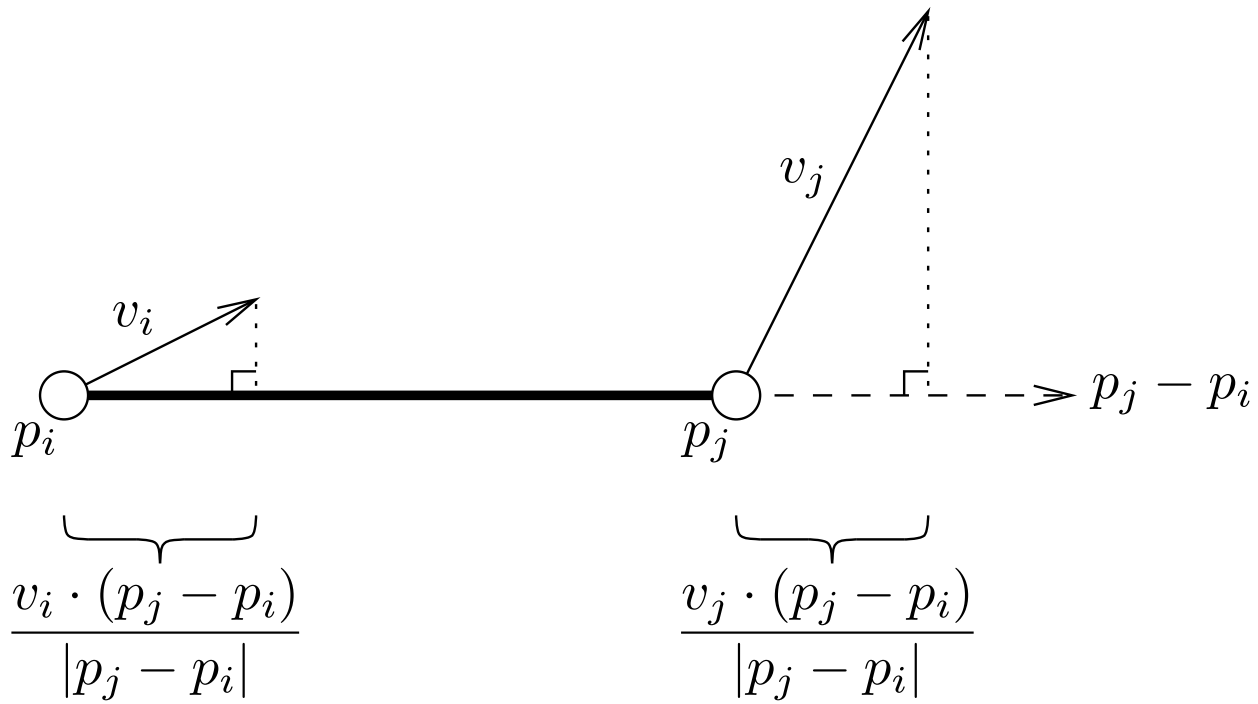
   velocity vectors $v_1, \ldots, v_n$.

3. constraints:
   $|p_i(t) - p_j(t)|$ is constant for every edge (bar) ij.

Expansion

$$\tfrac{1}{2} \cdot \frac{d}{dt}|p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle$$



expansion (or *strain*) of the segment ij

Infinitesimally rigid frameworks

A framework is *infinitesimally rigid* if the system of equations

$$\langle v_i - v_j, p_i - p_j \rangle = 0, \text{ for all edges } ij$$

in the vector variables $v_1, \ldots, v_n$ has only the trivial solutions: translations and rotations of the framework as a whole.

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in the vector variables $v_1, \ldots, v_n$ has only the trivial solutions: translations and rotations of the framework as a whole.

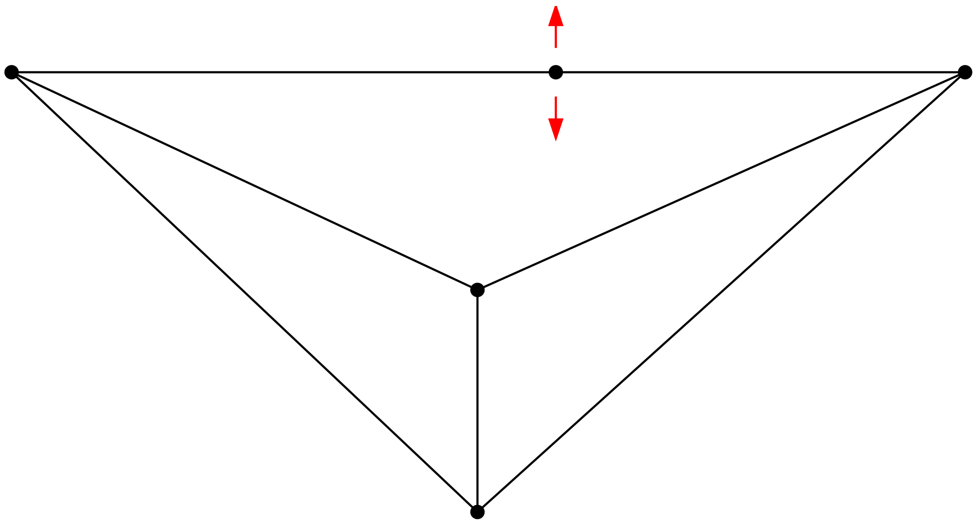
[Alternative: pin an edge ij by setting $v_i = v_j = 0$.

$\Longrightarrow$ only $(0, 0, \ldots, 0)$ is a trivial solution.]

Rigid frameworks

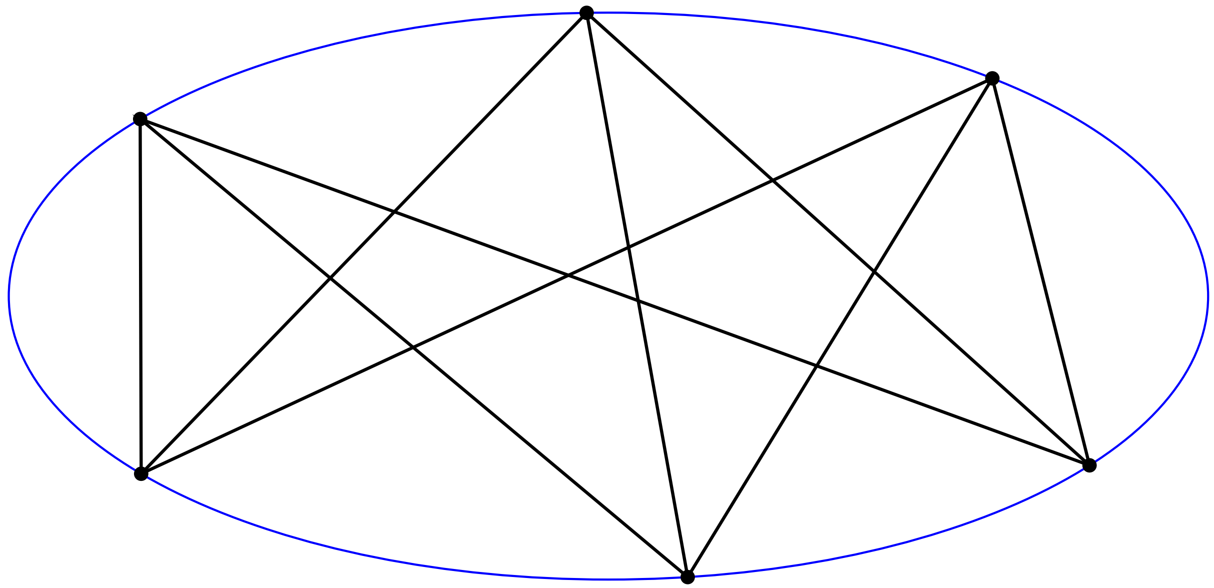
An infinitesimally rigid framework is rigid.

This framework is rigid, but not infinitesimally rigid:



Genercally rigid frameworks

A given graph can be rigid in most embeddings, but it may have special non-rigid embeddings:



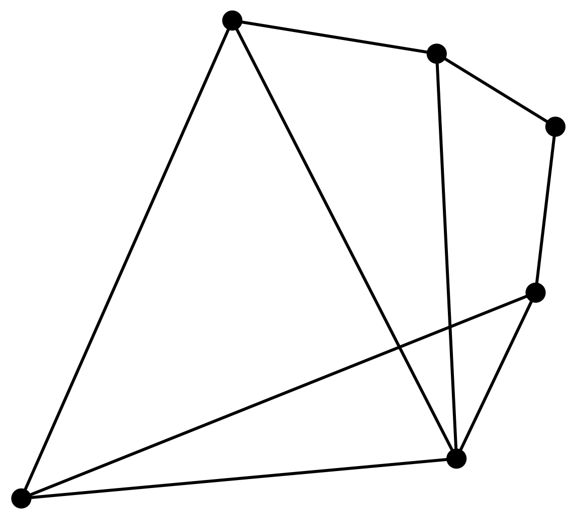
A graph is *generically rigid* if it is infinitesimally rigid in almost all embeddings.

This is a *combinatorial property* of the graph.

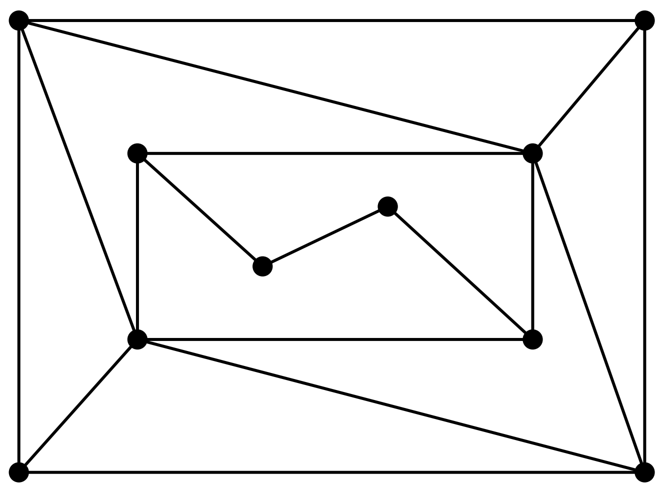
Minimally rigid frameworks

Theorem. A graph with n vertices is *minimally rigid* in the plane (with respect to $\subseteq$) iff it has the *Laman property*:

- It has $2n - 3$ edges.
- Every subset of $k \geq 2$ vertices spans at most $2k - 3$ edges.



$n = 6, \ e = 9$



$n = 10, \ e = 17$

[Laman 1961]

A pointed pseudotriangulation is a Laman graph

Proof: Every subset of $k \geq 2$ vertices is pointed and has therefore at most $2k - 3$ edges.

[Streinu 2001]

Every planar Laman graph is a pointed pseudotriangulation

Theorem. *Every planar Laman graph has a realization as a pointed pseudotriangulation. The outer face can be chosen arbitrarily.*

[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]

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Proof I: Induction, using *Henneberg constructions*

Proof II: via Tutte embeddings for directed graphs

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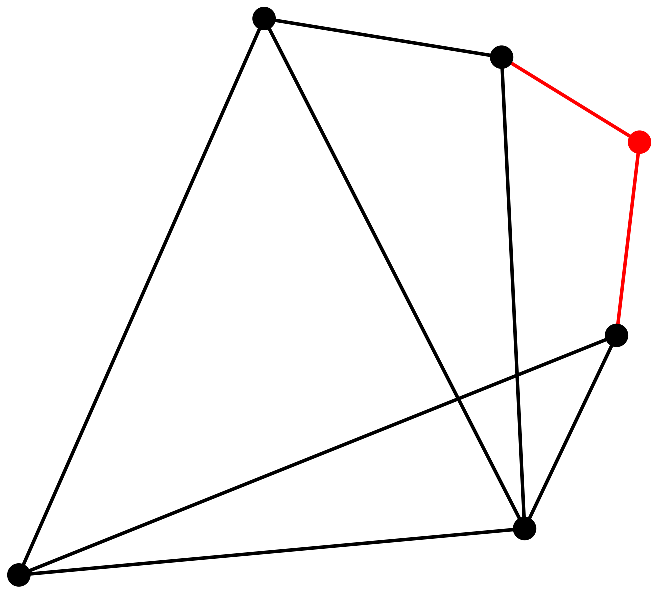
Proof I: Induction, using *Henneberg constructions*

Proof II: via Tutte embeddings for directed graphs

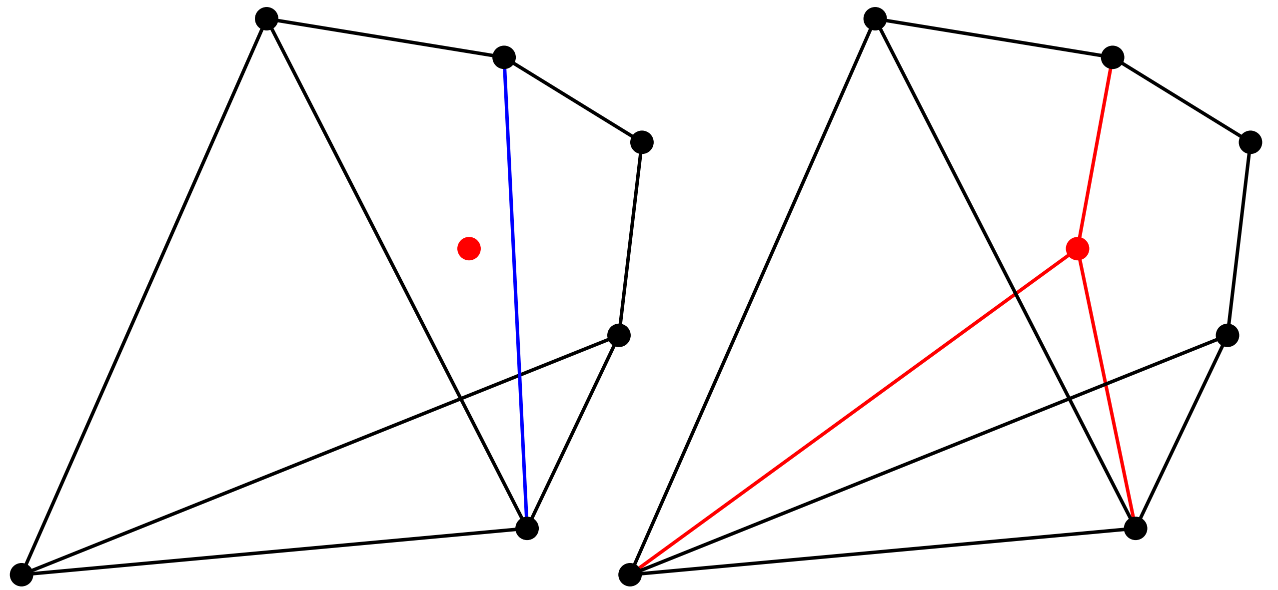
Theorem. *Every rigid planar graph has a realization as a pseudotriangulation (not necessarily pointed).*

[Orden, Santos, B. Servatius, H. Servatius 2003]

Henneberg constructions



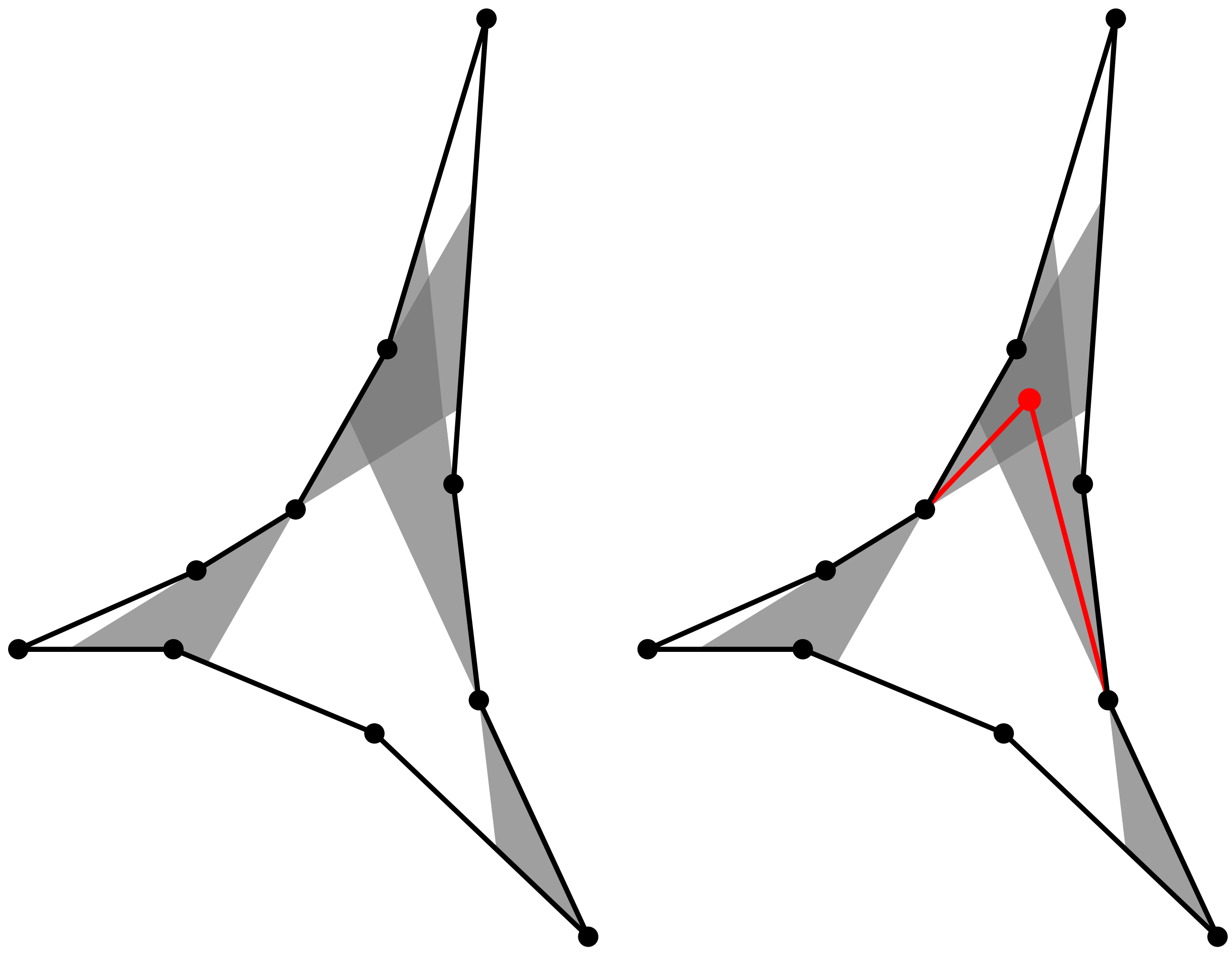
Type I



Type II

Every Laman graph can be built up by a sequence of Henneberg construction steps, starting from a single edge.

Proof I: Henneberg constructions



4. RIGIDITY AND KINEMATICS
Unfolding of polygons — expansive motions

The Carpenter's Rule Problem:

Theorem. Every polygonal arc in the plane can be brought into straight position, without self-overlap.

Every polygon in the plane can be unfolded into convex position. [Connelly, Demaine, Rote 2000], [Streinu 2000]

4. RIGIDITY AND KINEMATICS
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Proof outline:

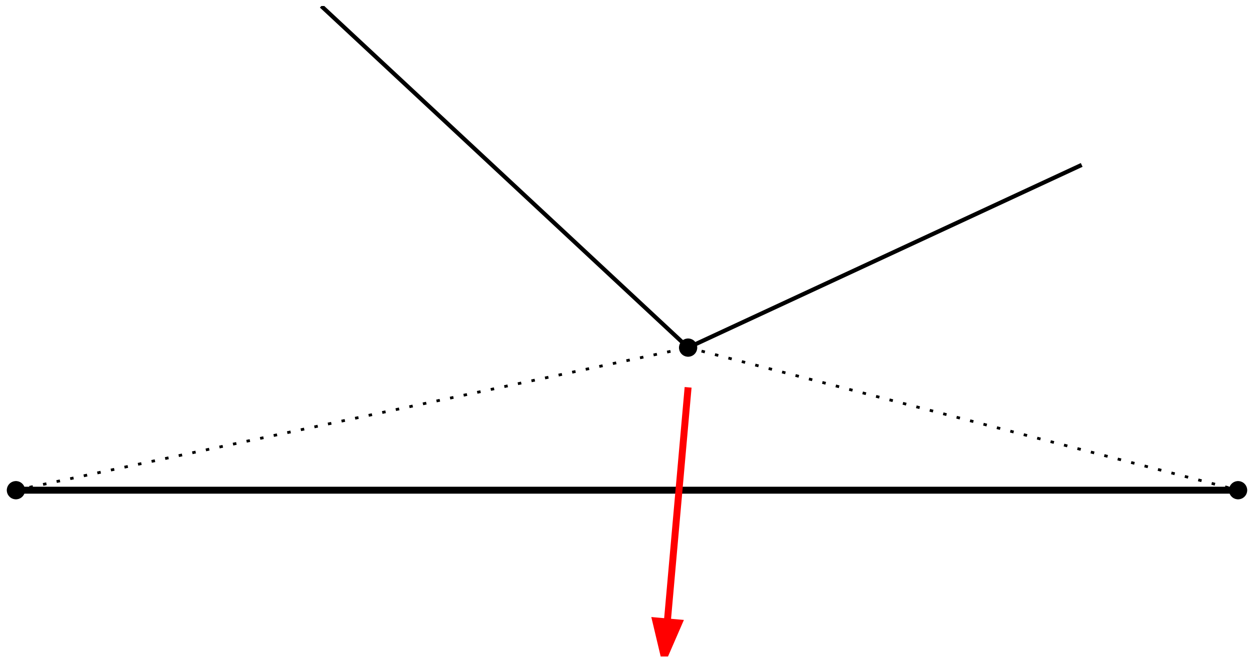
1. Find an *expansive* infinitesimal motion.

2. Find a global motion.

Expansive Motions

No distance between any pair of vertices decreases.

Expansive motions cannot lead to self-crossings.



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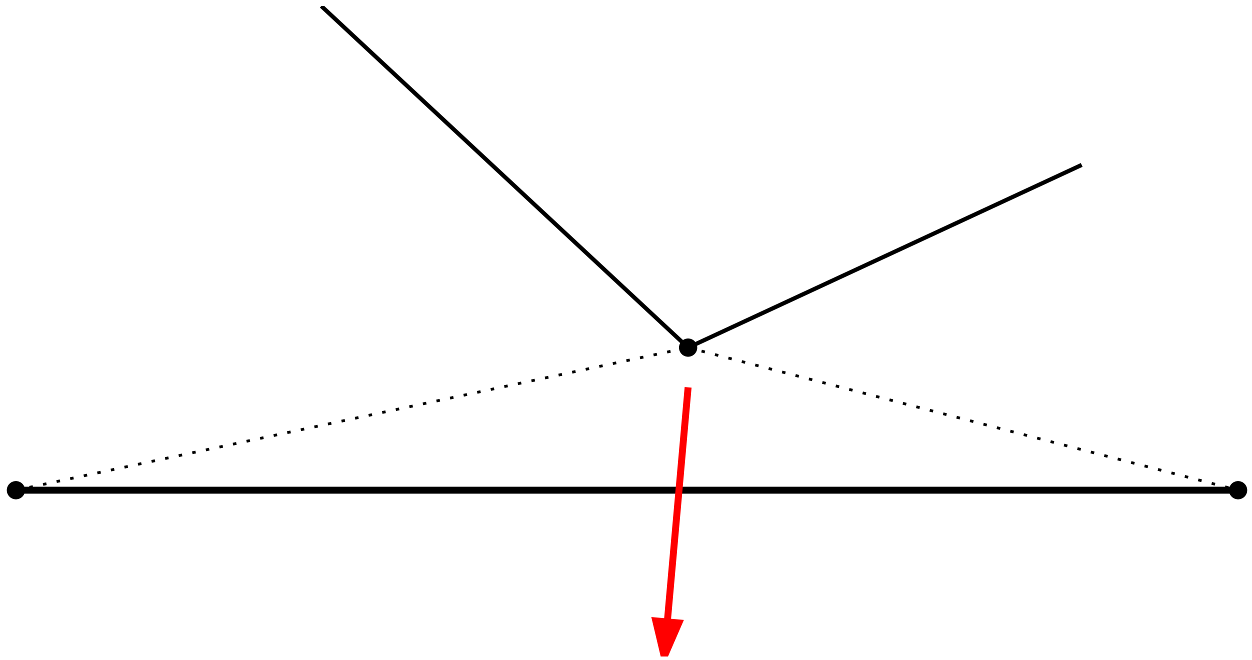


. . . need to show that an expansive motion exists . . .

Every Polygon has an Expansive Motion

Proof I: (Outline)

Existence of an expansive motion

$\Updownarrow$ (duality)

Self-stresses (rigidity)
Self-stresses on planar frameworks

$\Updownarrow$ (Maxwell-Cremona correspondence)

polyhedral terrains

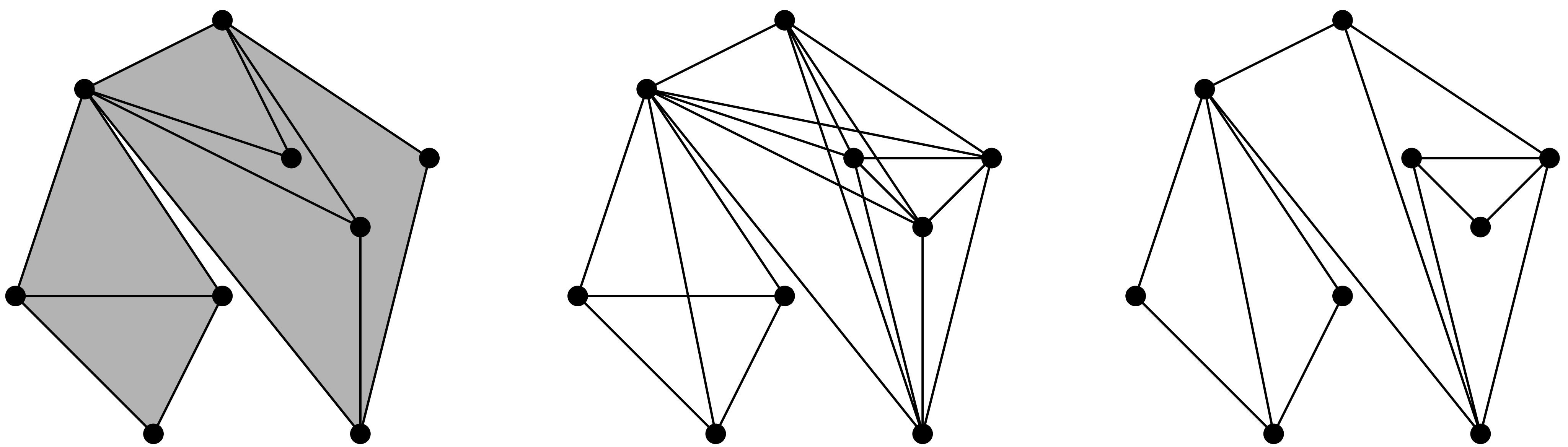
[Connelly, Demaine, Rote 2000]

Proof II: via pseudotriangulations and the Pseudotriangulation Polytope

[Streinu 2000] [Rote, Santos, Streinu 2003]

Expansive motions exist

Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]



(There are in general rigid substructures.)

Expansive motions for a chain (or a polygon)

- Add edges to form a pseudotriangulation
- Remove a convex hull edge
- $\rightarrow$ expansive mechanism $\square$

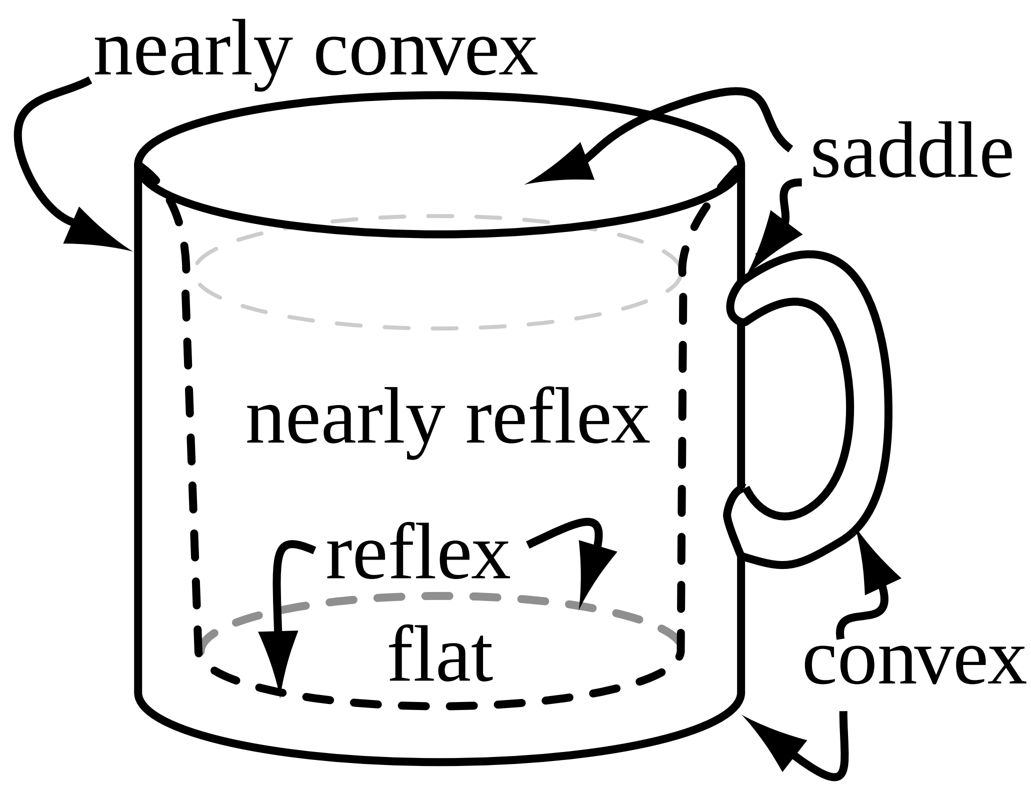
Theorem. Every polygonal arc in the plane can be brought into straight position, without self-overlap.

Every polygon in the plane can be unfolded into convex position.

[Connelly, Demaine, Rote 2000], [Streinu 2000]

5. LIFTINGS OF PSEUDOTRIANGULATIONS

Locally convex liftings — the reflex-free hull

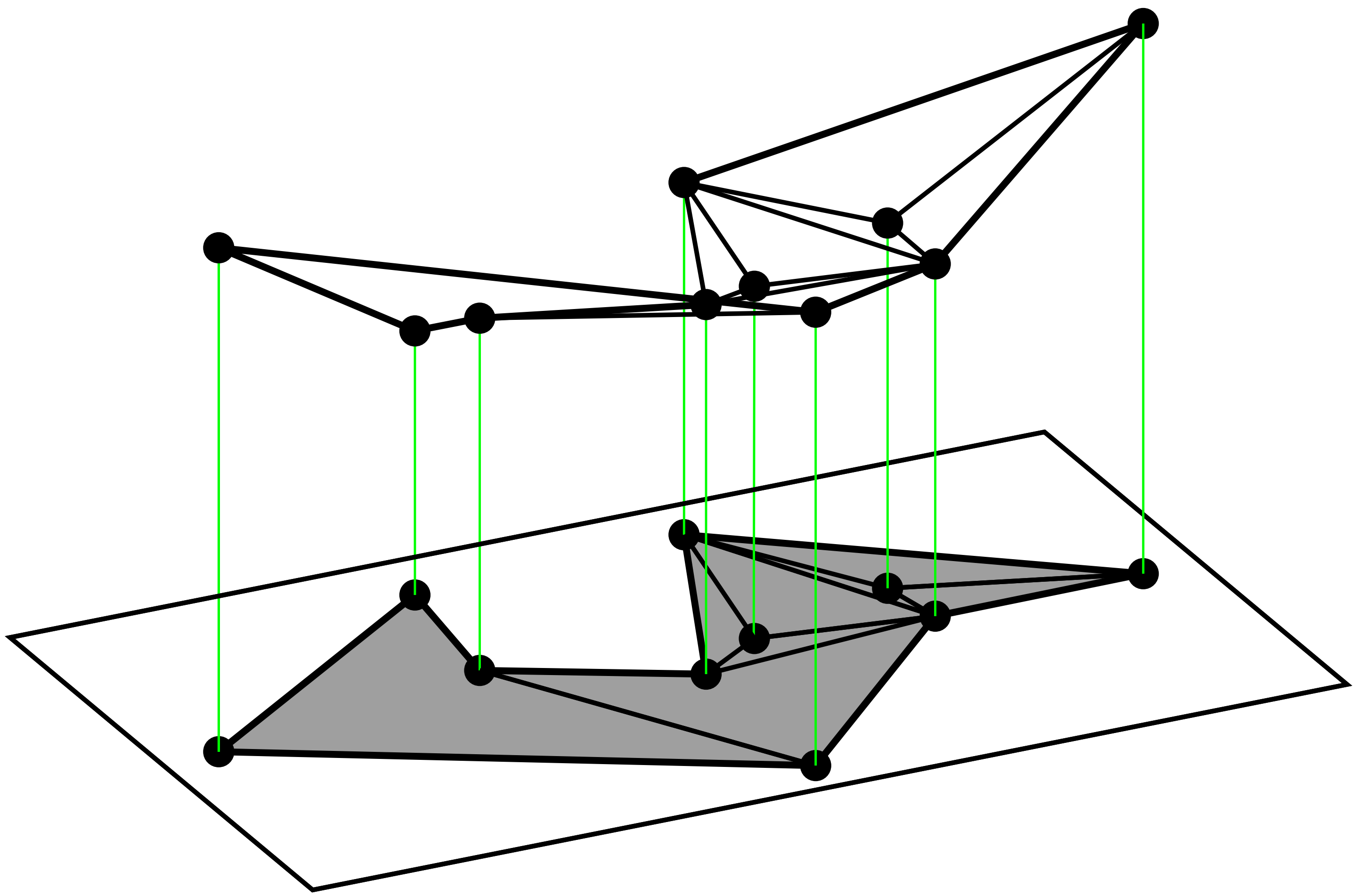


an approach for recognizing pockets in biomolecules

[Ahn, Cheng, Cheong, Snoeyink 2002]

Locally convex surfaces

A function over a polygonal domain P is *locally convex* if it is convex on every segment in P.



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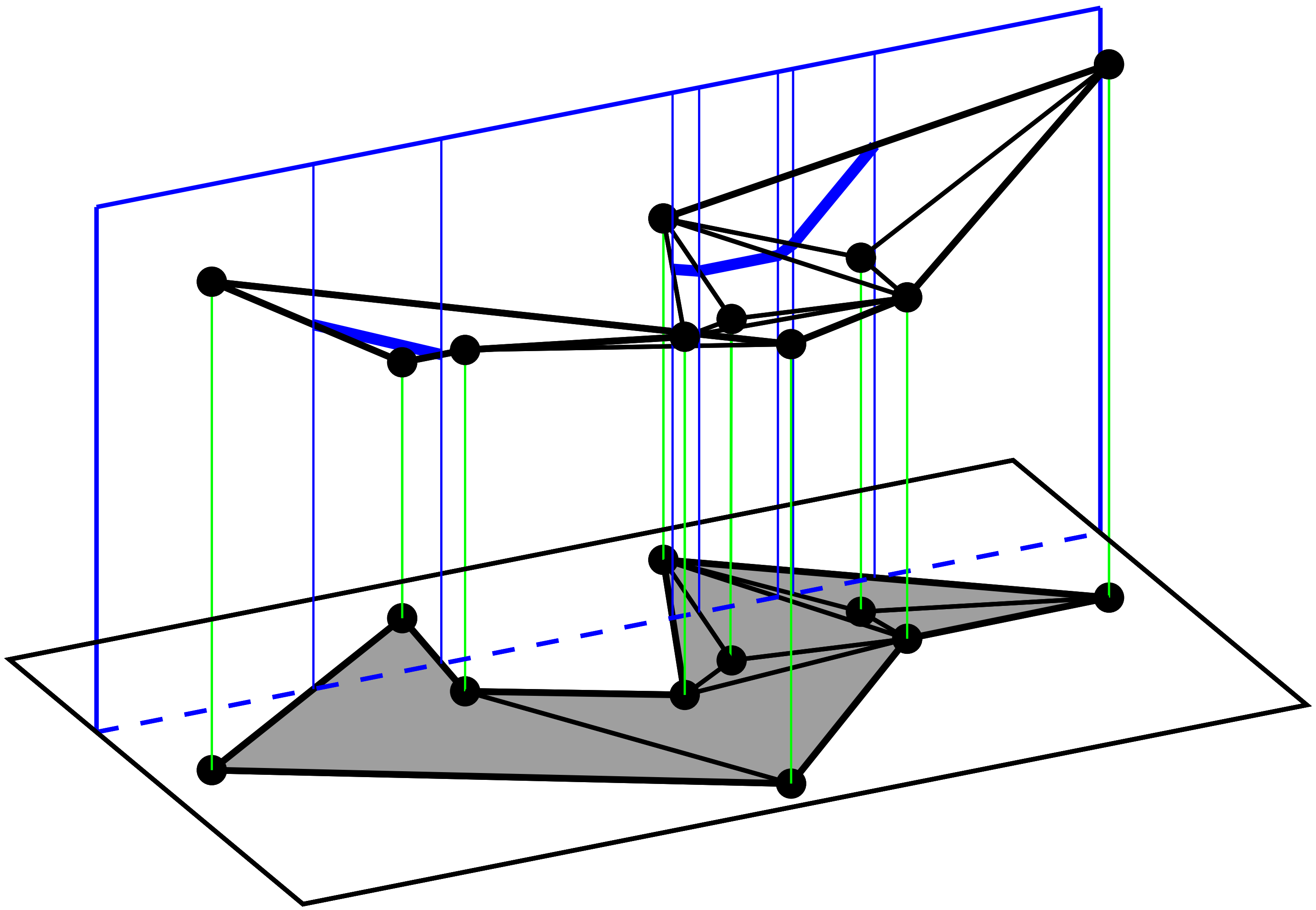

Locally convex functions on a polygon

Given a polygon P and a height value h_i for all vertices plus some additional points p_i in the polygon, find the highest locally convex function $f\colon P \to \mathbb{R}$ with $f(p_i) \le h_i$.

If P is convex, this is the lower convex hull of the three-dimensional point set (p_i, h_i).

In general, the result is a piecewise linear function defined on a pseudotriangulation of (P, S). (Interior vertices may be missing.)

$\rightarrow$ *regular pseudotriangulations*

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

This can be extended to 3-polytopes.

[Aurenhammer, Krasser 2005]

OPEN QUESTIONS

- Pseudotriangulations in 3-space?

  (Rigid graphs are not well-understood in 3-space.)

- How many pseudotriangulations does a point set have?

- Can every pseudotriangulation be (re)drawn on a polynomial-size grid?

INPUT-A NO INPUT