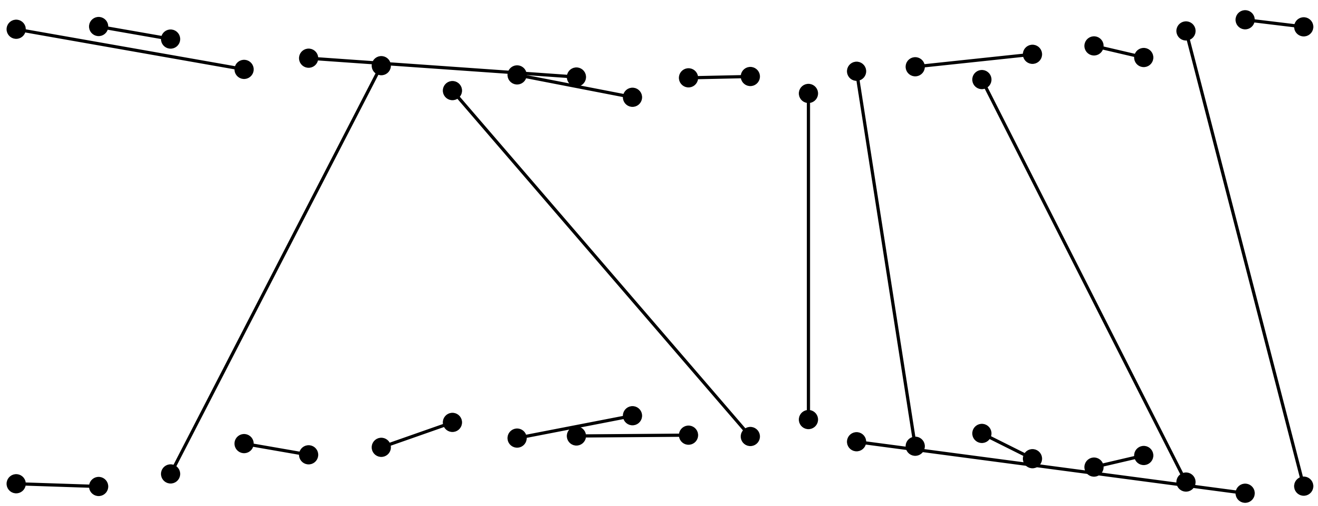
Lattice Paths with States, and Counting Geometric Objects via Production Matrices

(a preliminary report on unproved results)

Günter Rote

Freie Universität Berlin

ongoing joint work with Andrei Asinowski and Alexander Pilz



a non-crossing perfect matching

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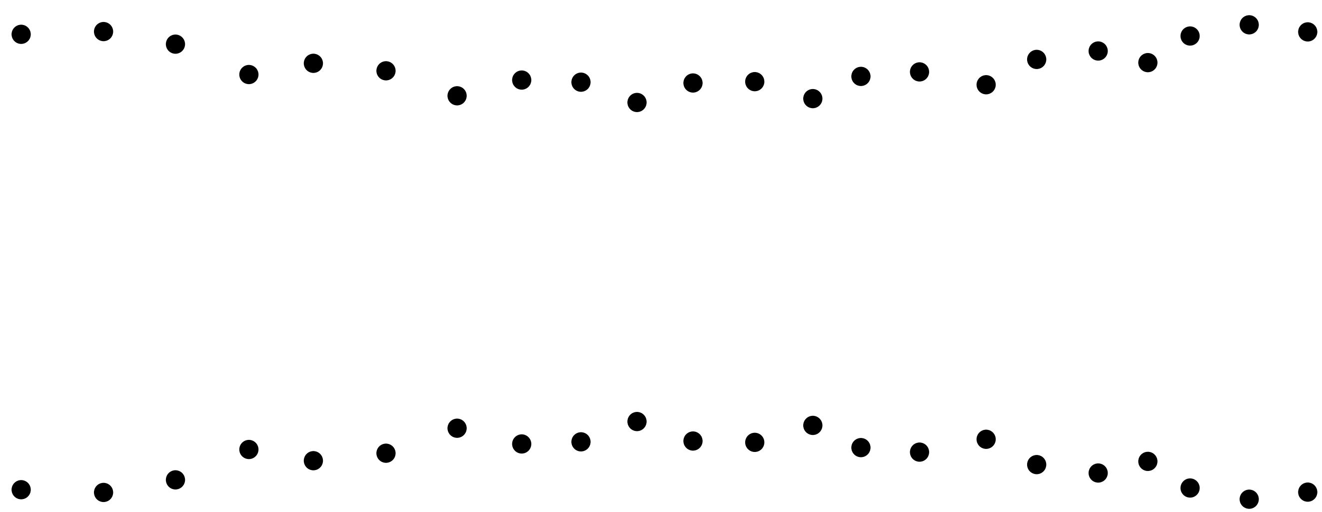

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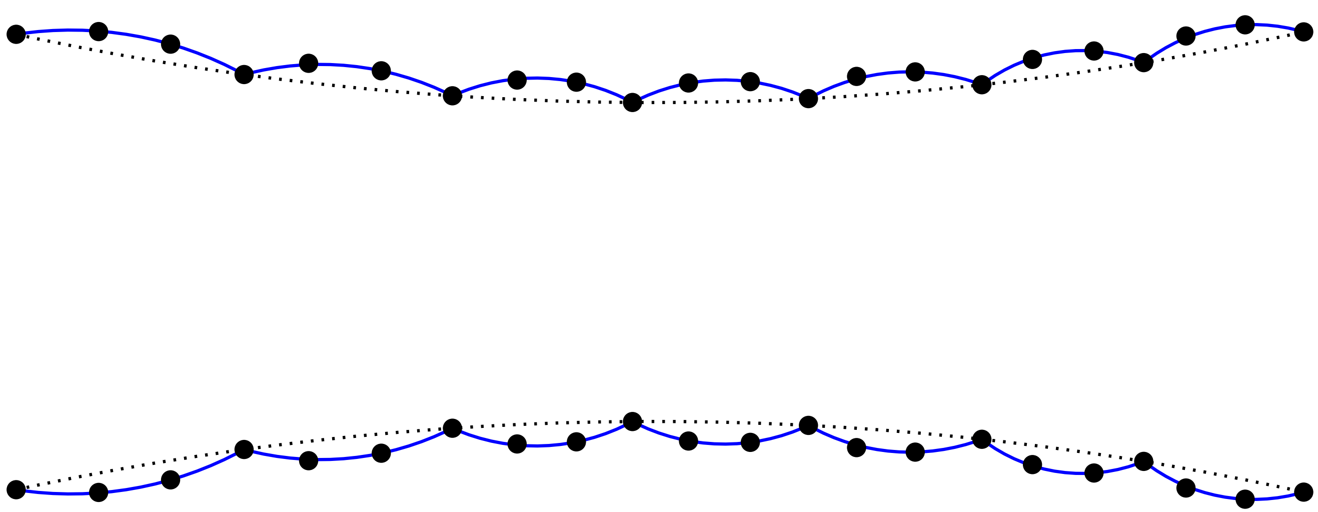

the generalized double zigzag chain

Lattice Paths with States

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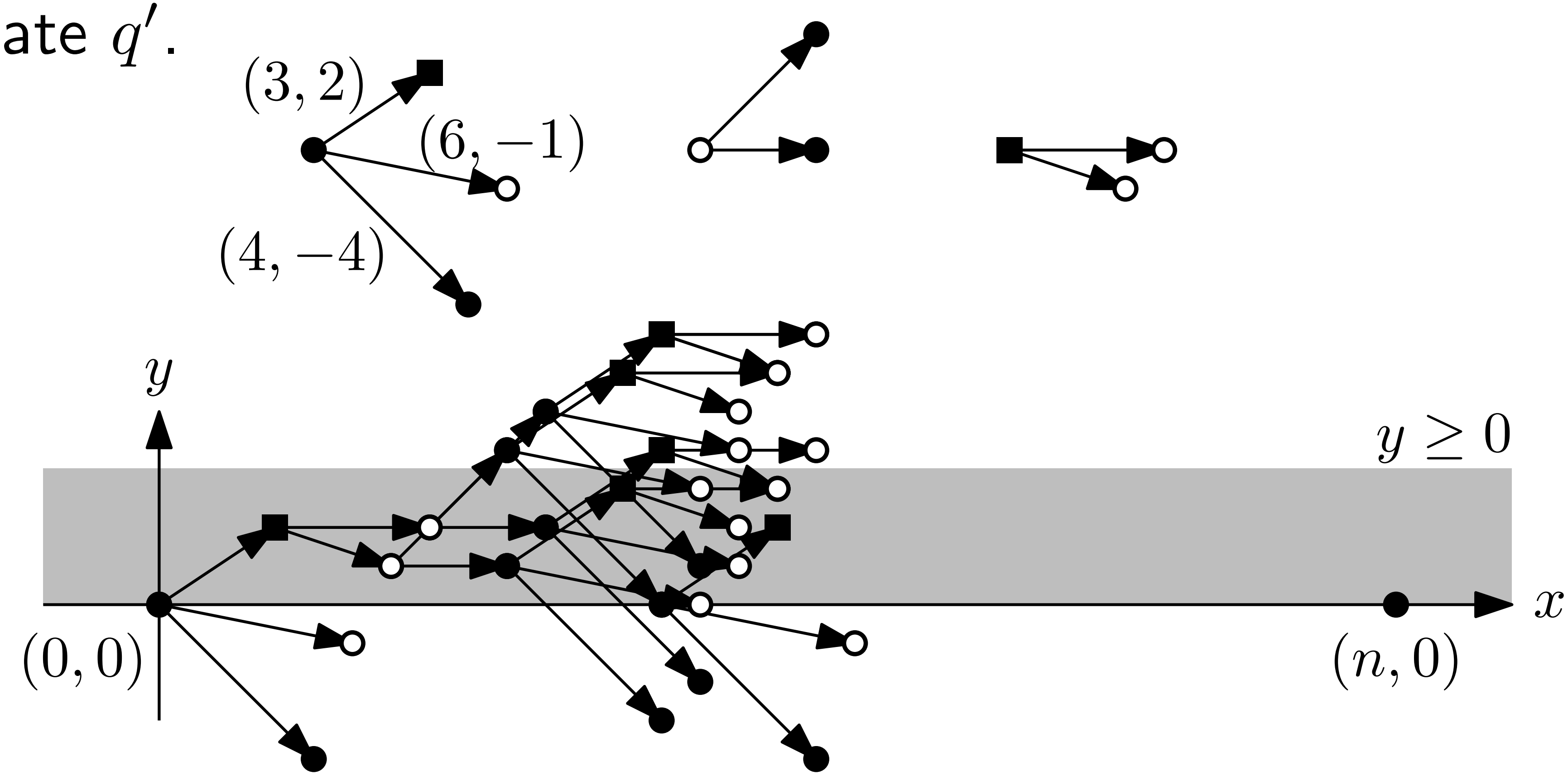
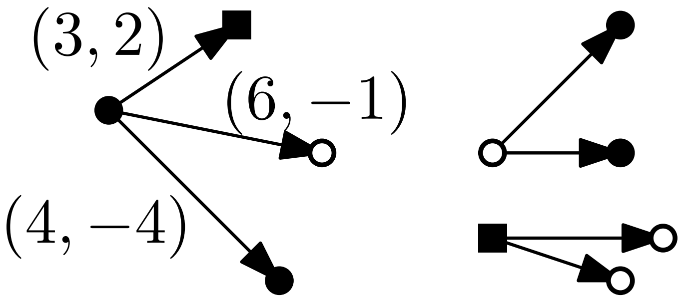
- Finite set of *states* $Q = \{\bullet, \circ, \blacksquare, \square, \triangle, \ldots\}$
- For each $q \in Q$, a set S_q of permissible *steps* $((i, j), q')$:

  From point (x, y) in state q, can go to $(x + i, y + j)$ in state q'.



Wanted: The number of paths from $(0, 0)$ in state q_0 to $(n, 0)$ in state q_1 that don't go below the x-axis.

Formula for Lattice Paths with States



$$\boxed{(i,j) \mapsto t^i u^j}$$

characteristic matrix

$$A(t,u) = \begin{array}{c|ccc} & \bullet & \circ & \blacksquare \\ \hline \bullet & t^4u^{-4} & t^6u^{-1} & t^3u^2 \\ \circ & t^3+t^3u^3 & 0 & 0 \\ \blacksquare & 0 & t^4+t^3u^{-1} & 0 \end{array}$$

Conjecture: The number of paths from $(0,0)$ in state q_0 to $(n,0)$ in state q_1 that don't go below the x-axis is

$$\sim \text{const} \cdot (1/t^*)^n \cdot n^{-3/2},$$

where

(1) $A(t^*,u^*)$ has largest (Perron-Frobenius) eigenvalue 1.

$$[\implies \det(A(t,u)-I)=0\ ]$$

(2) $u^* > 0$ is chosen such that the value $t^* > 0$ that fulfills (1) is as large as possible. $[\implies \frac{\partial}{\partial u}\det(A(t,u)-I)=0\ ]$

Formula for Lattice Paths with States

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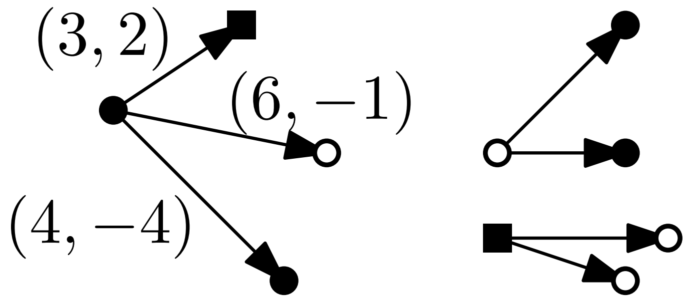

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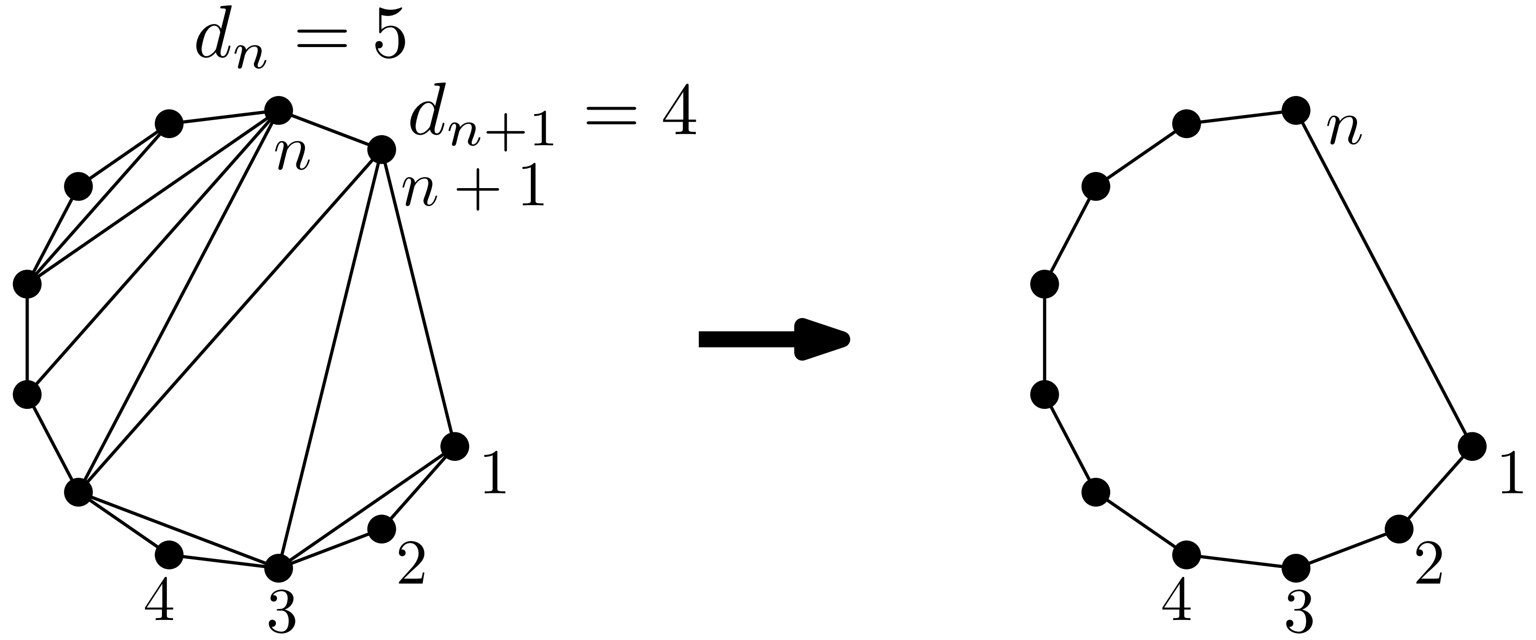
under some obvious *technical conditions*:

- state graph is strongly connected
- no cycles in the lattice paths
- aperiodic
- ...

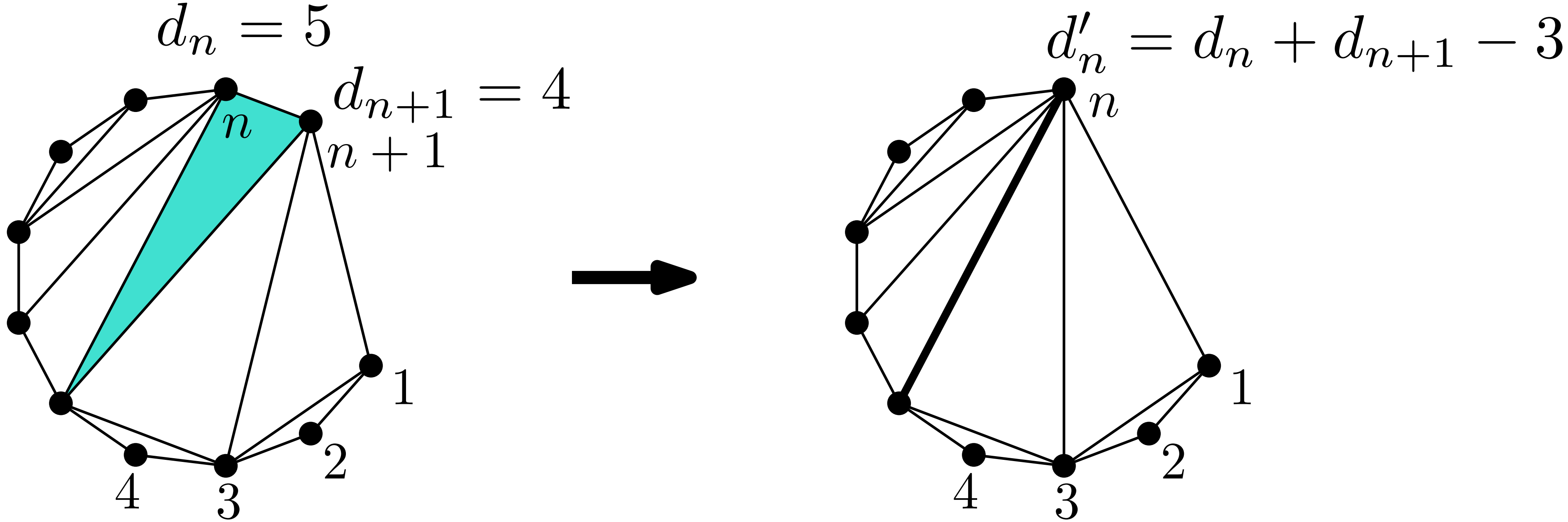
Overview

- Introduction. Point sets with many noncrossing X
- The lattice path formula with states (preview)
- Overview
- Example 1: Triangulations of a convex n-gon
- Production matrices
- Example 2: Noncrossing forests in a convex n-gon
- Example 3: Geometric graphs on the generalized double zigzag chain.
- Proof idea 1. Analytic combinatorics
- Proof idea 2. Random walk

Triangulations of a convex n-gon



Triangulations of a convex n-gon



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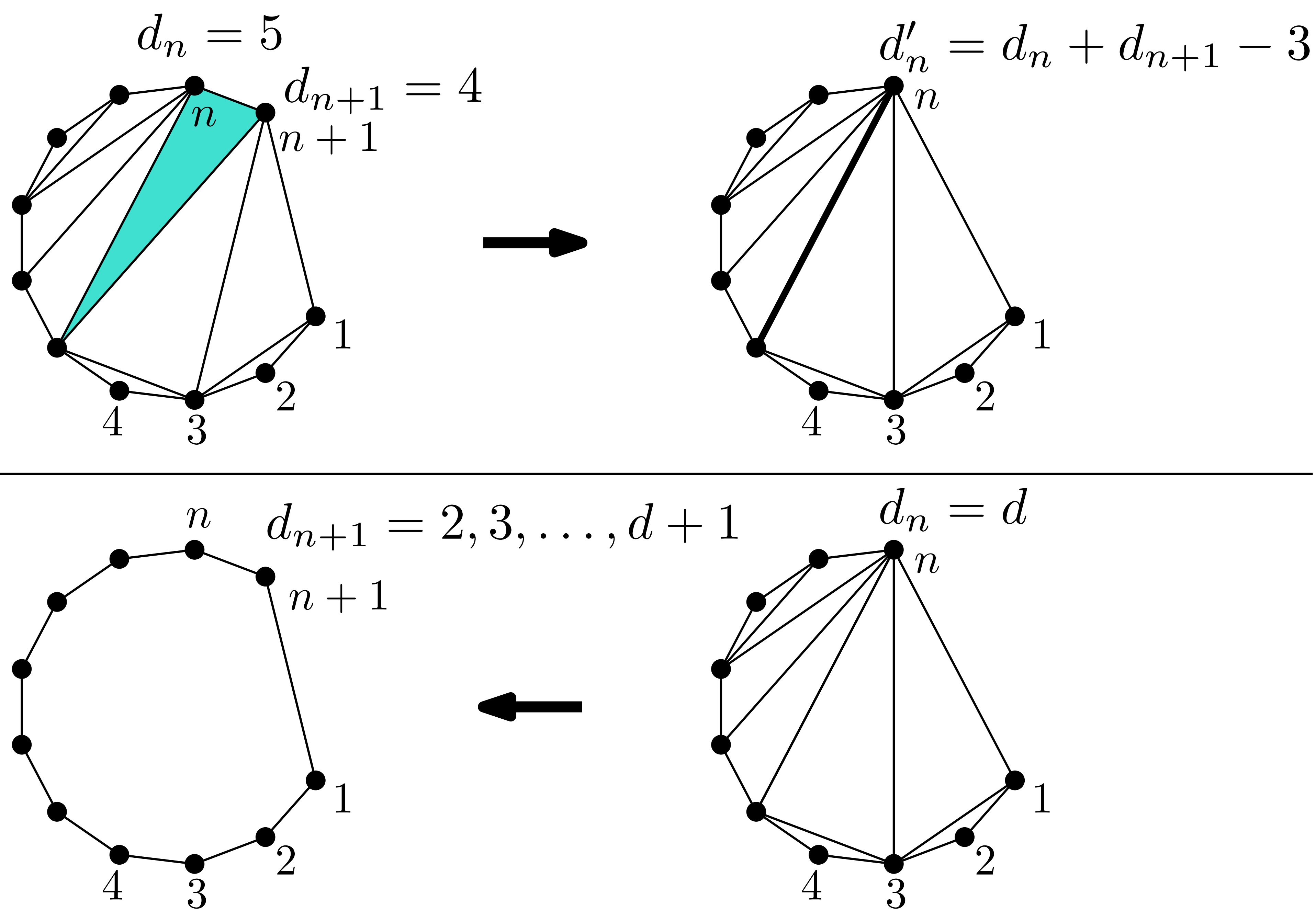

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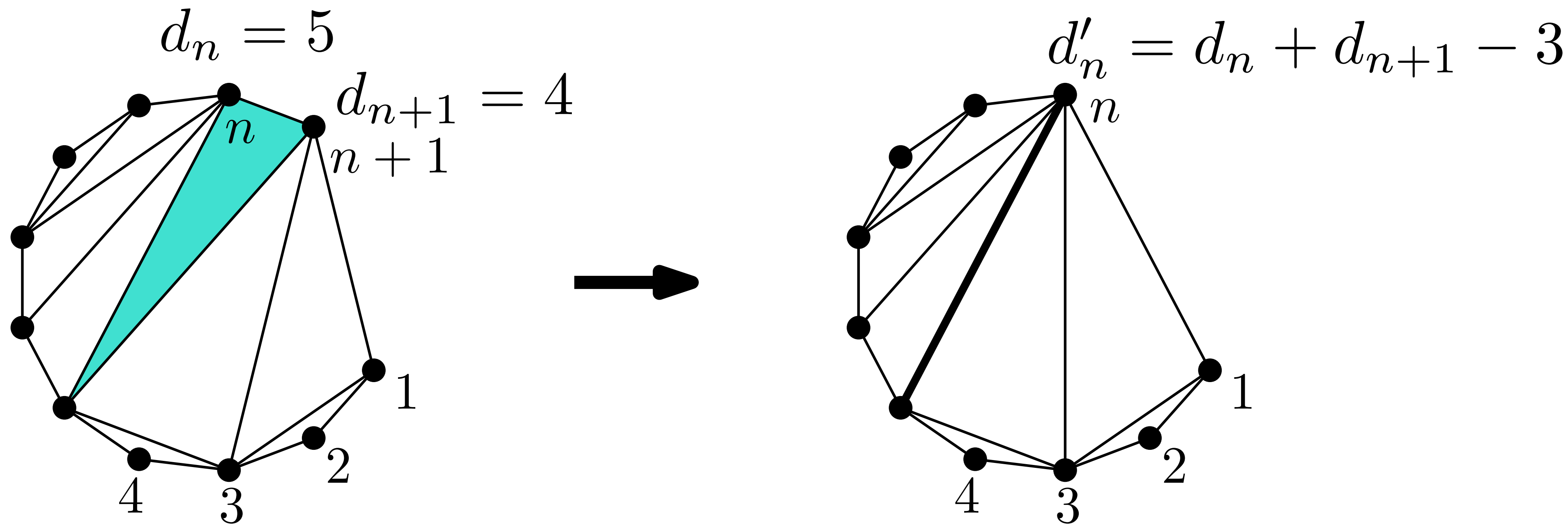
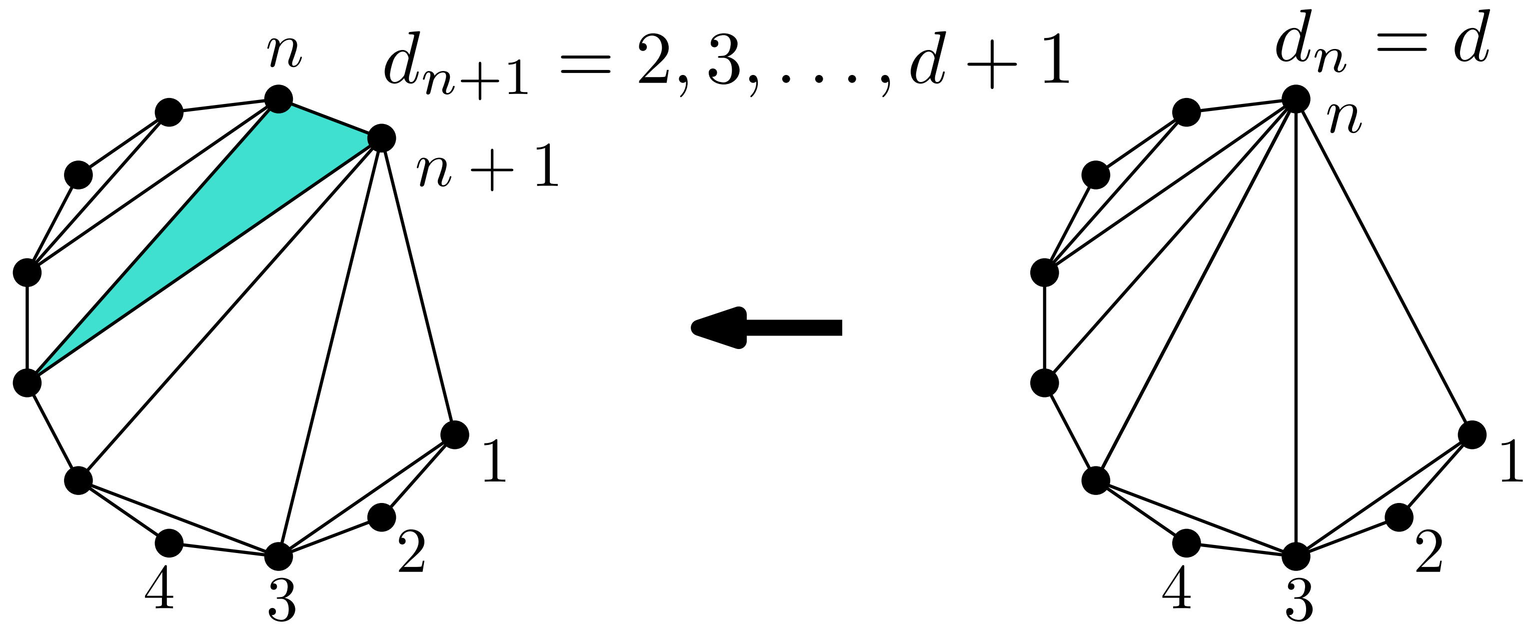




Triangulations of a convex n-gon

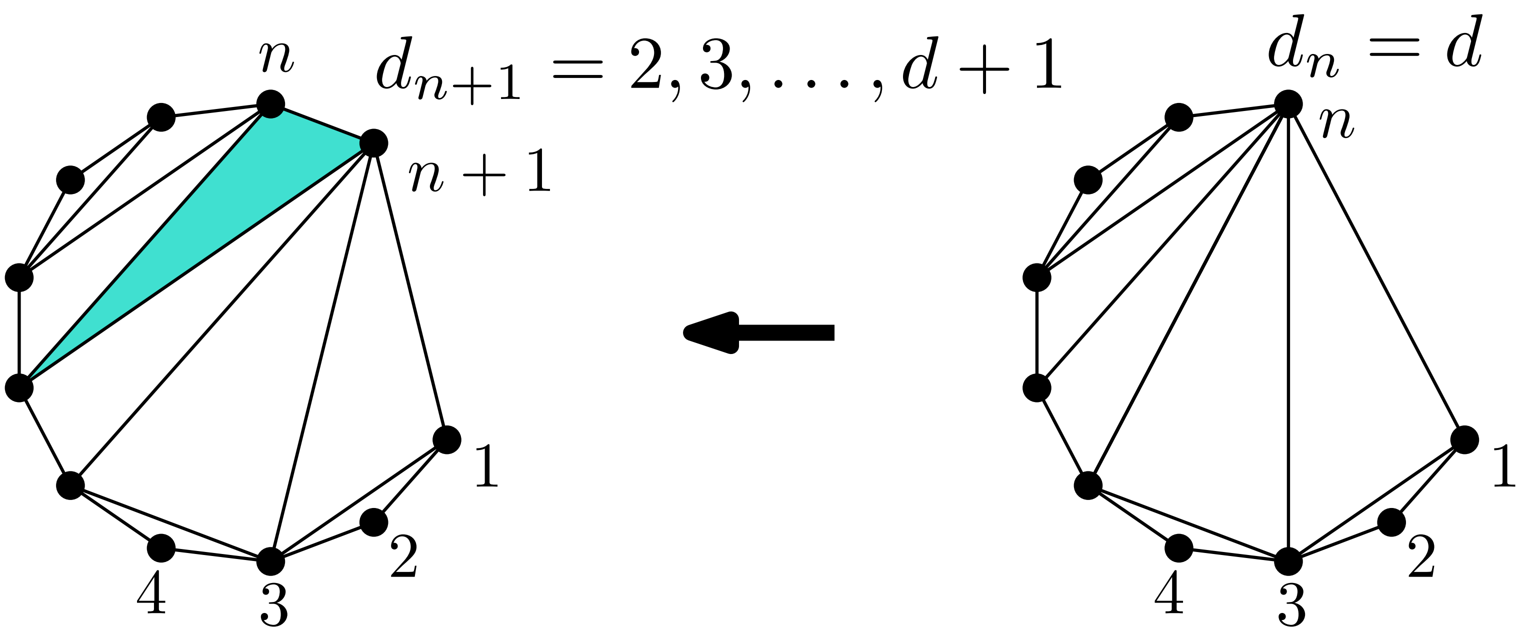
Triangulation of n-gon with last vertex of degree $d_n = d$

$\rightarrow$

Triangulation of $(n+1)$-gon with last vertex of degree

$$d_{n+1} = 2 \text{ or } 3 \text{ or } 4 \text{ or } \ \ldots \ \text{ or } d, \ \text{ or } d+1$$

[Hurtado & Noy 1999]
"tree of triangulations"



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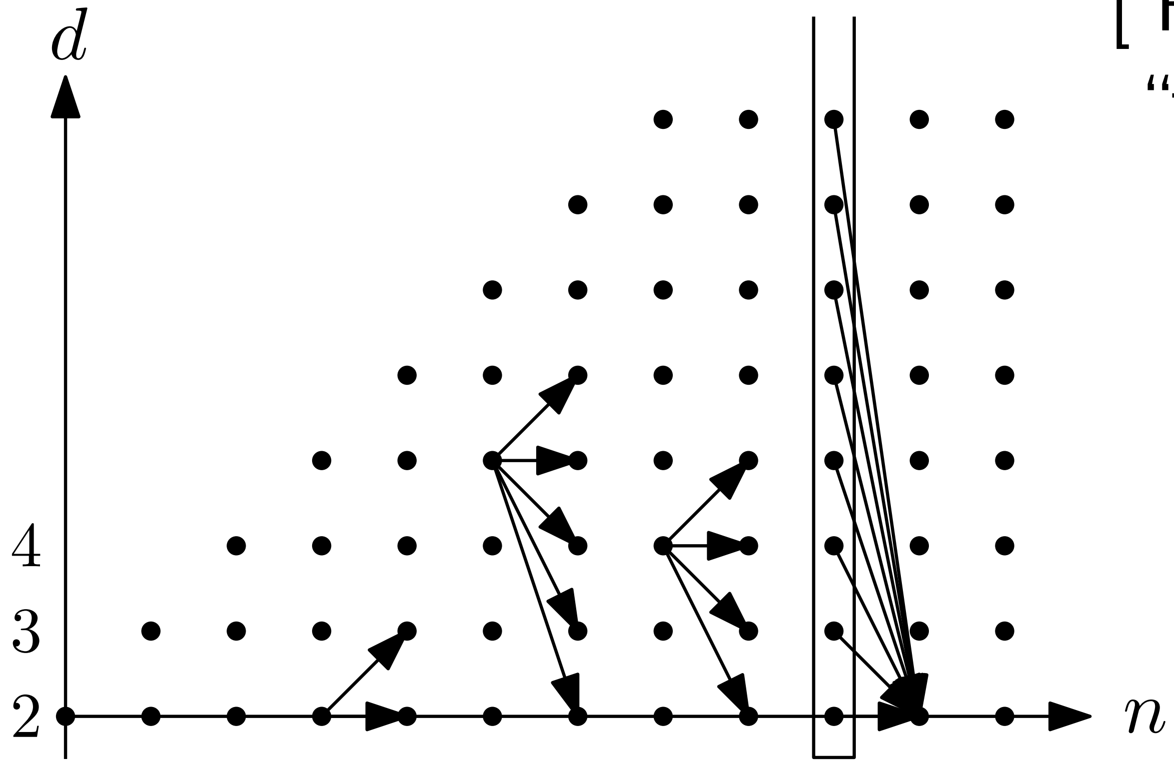
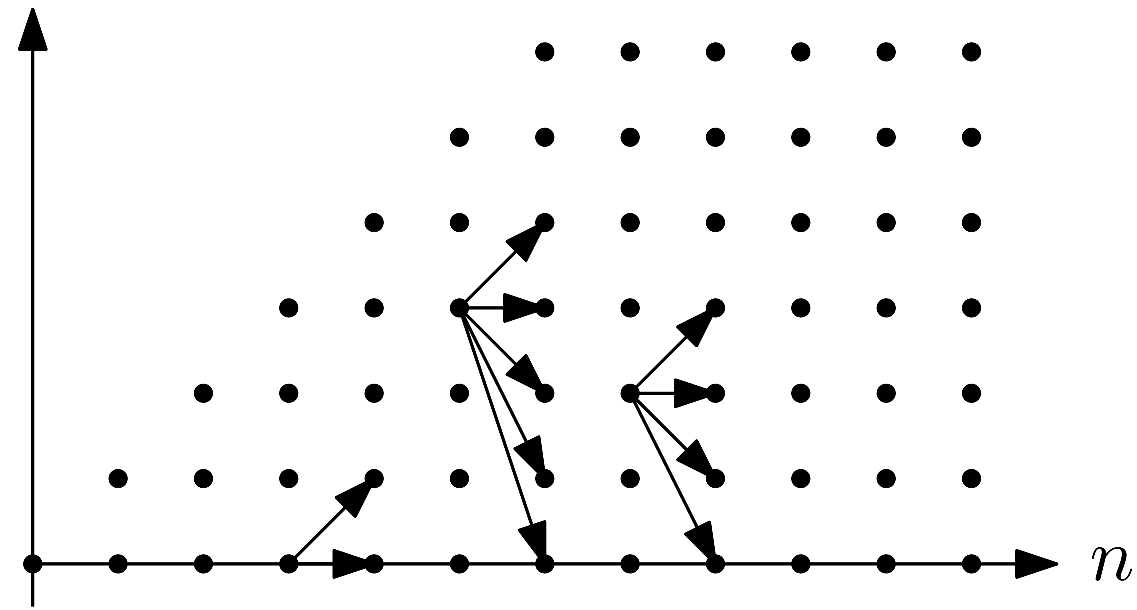


triangulation
$\updownarrow$
lattice path

Production matrices



count paths in a layered graph

The answer is

$$(1 \quad 0 \quad 0 \quad \ldots)\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ 0 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{n}}_{\text{the “production matrix” } P} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Production matrices for enumeration

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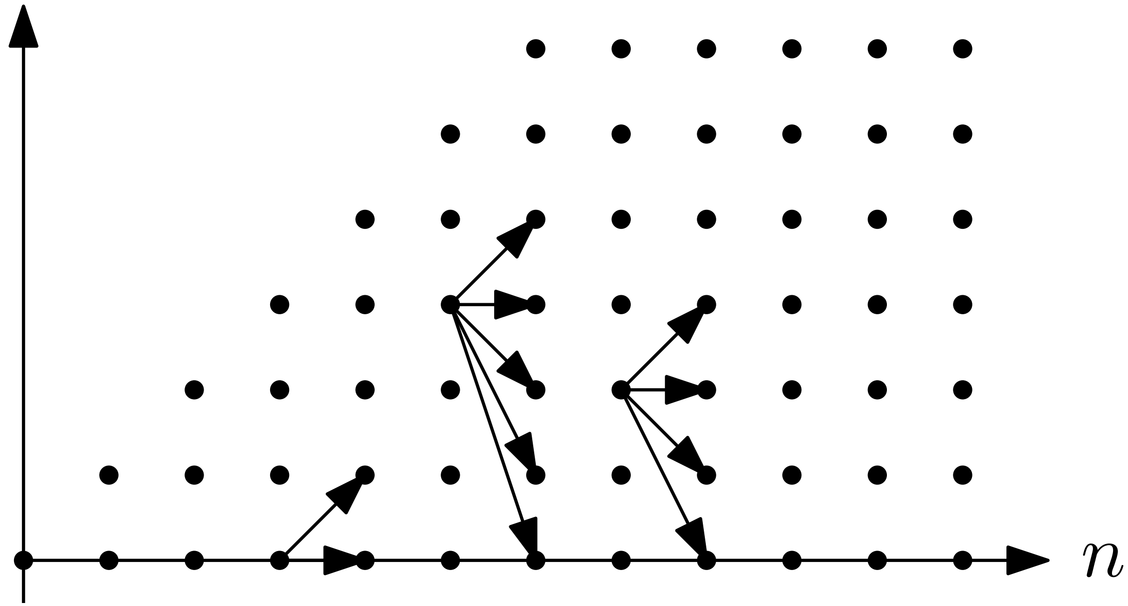
were introduced by Emeric Deutsch, Luca Ferrari, and Simone Rinaldi (2005).

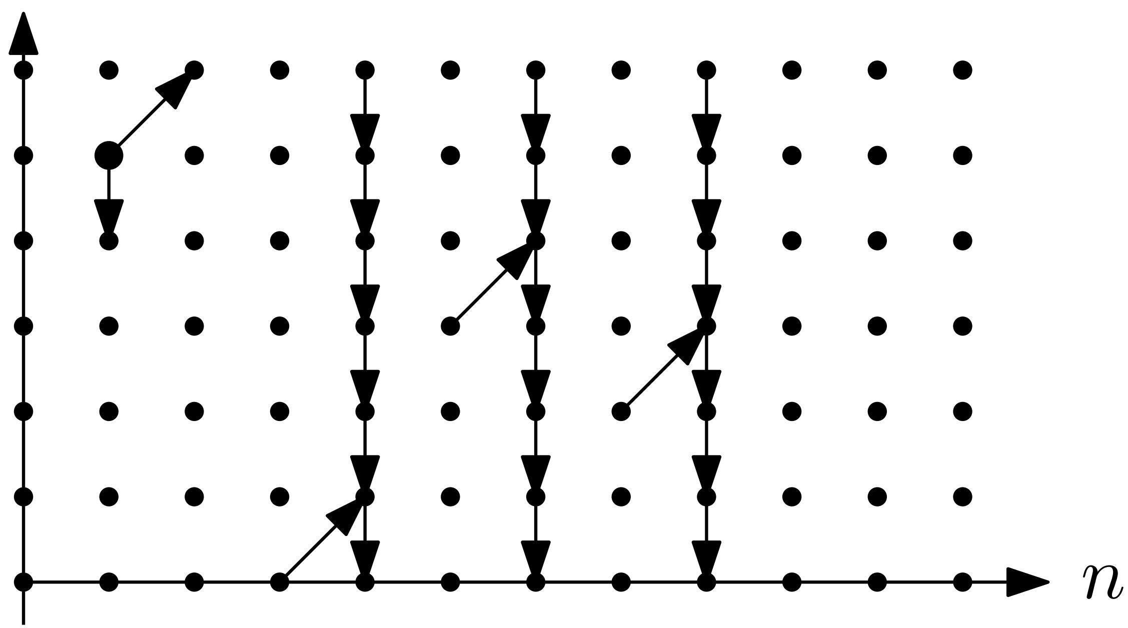
were used for counting noncrossing graphs for points in convex position: Huemer, Seara, Silveira, and Pilz (2016)
Huemer, Pilz, Seara, and Silveira (2017)

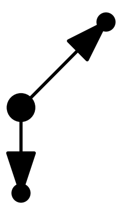
$$\begin{pmatrix} 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 1 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 5 & \dots \\ 1 & 2 & 3 & 4 & \dots \\ 0 & 1 & 2 & 3 & \dots \\ 0 & 0 & 1 & 2 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 4 & 5 & \dots \\ 0 & 1 & 3 & 4 & \dots \\ 0 & 0 & 1 & 3 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

matchings spanning trees forests

Making the degree finite



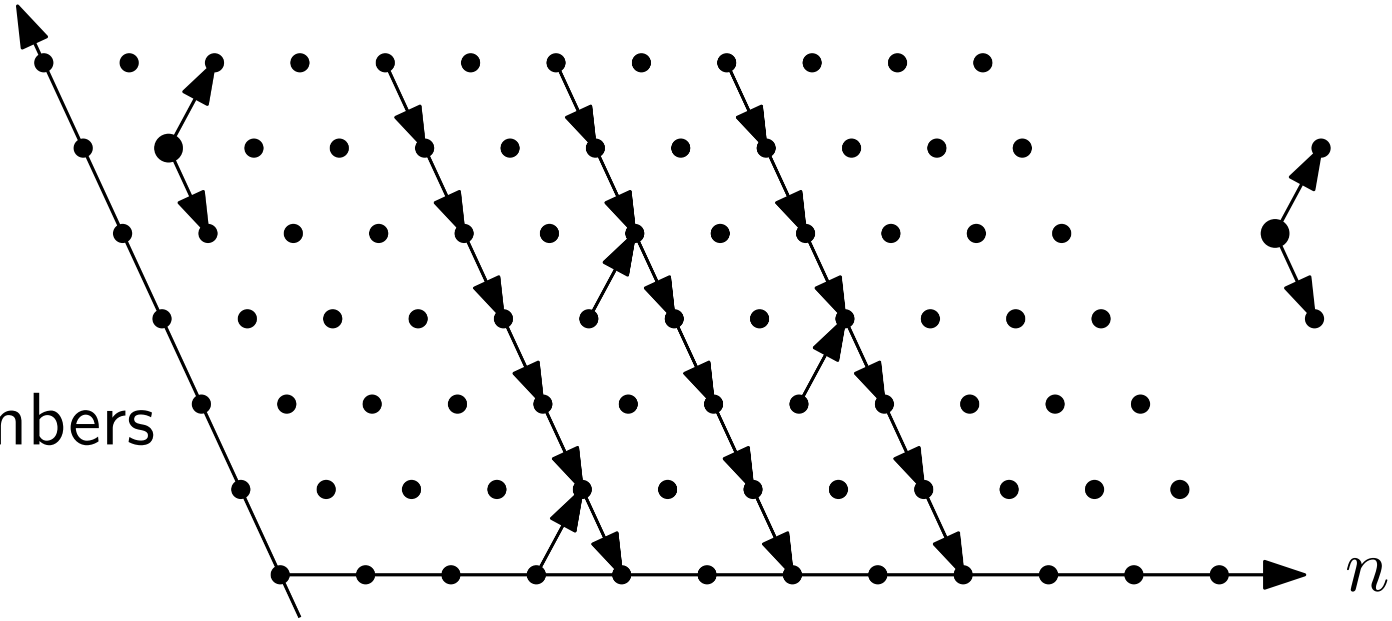


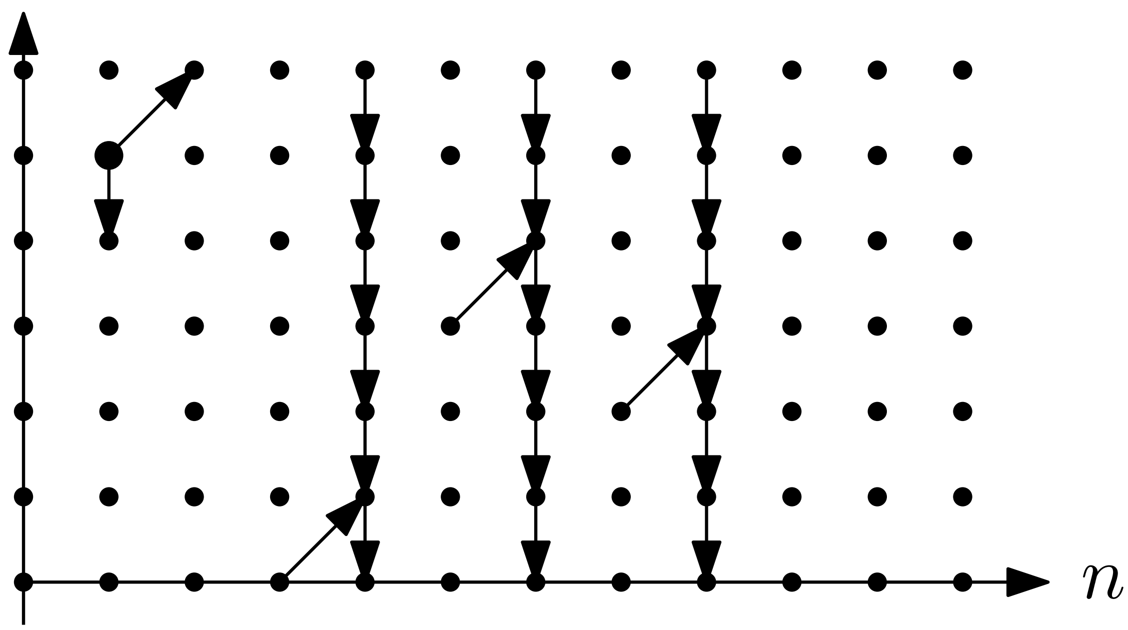


Number of paths is preserved.

Making the degree finite

Shearing
$\rightarrow$ Dyck paths
$\rightarrow$ Catalan numbers





Number of paths
is preserved.

Example 2: Forests

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 4 & 5 & 6 & \dots \\ 0 & 1 & 3 & 4 & 5 & \dots \\ 0 & 0 & 1 & 3 & 4 & \dots \\ 0 & 0 & 0 & 1 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Irregularities at the boundary can be ignored.

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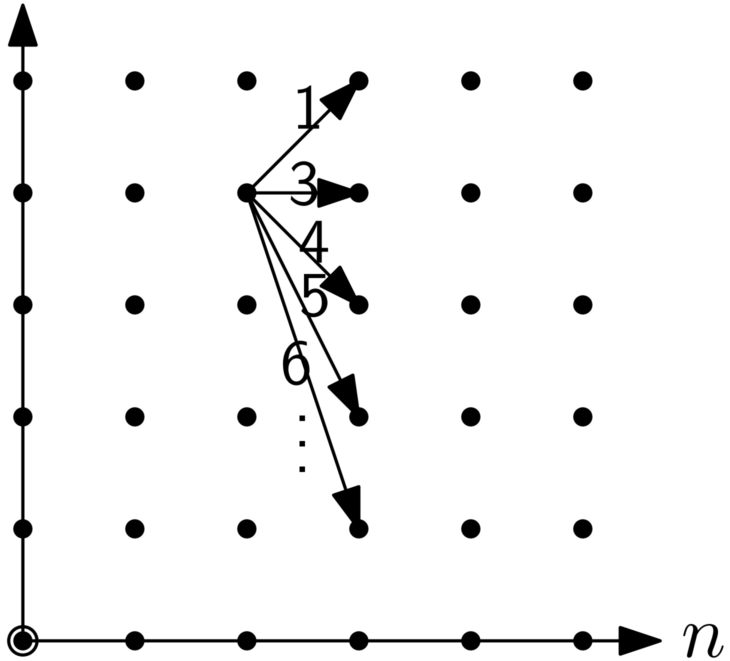

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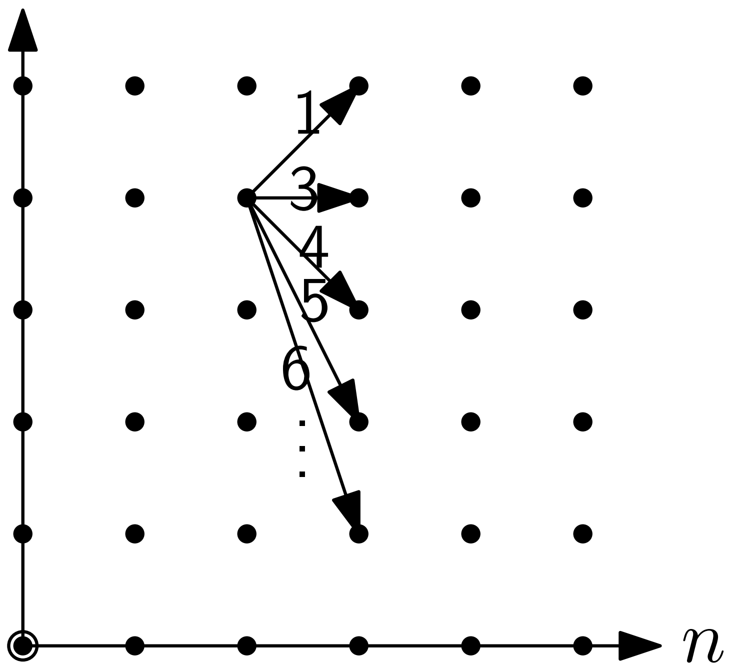

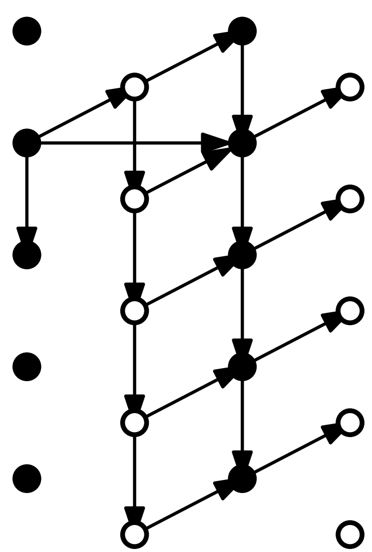

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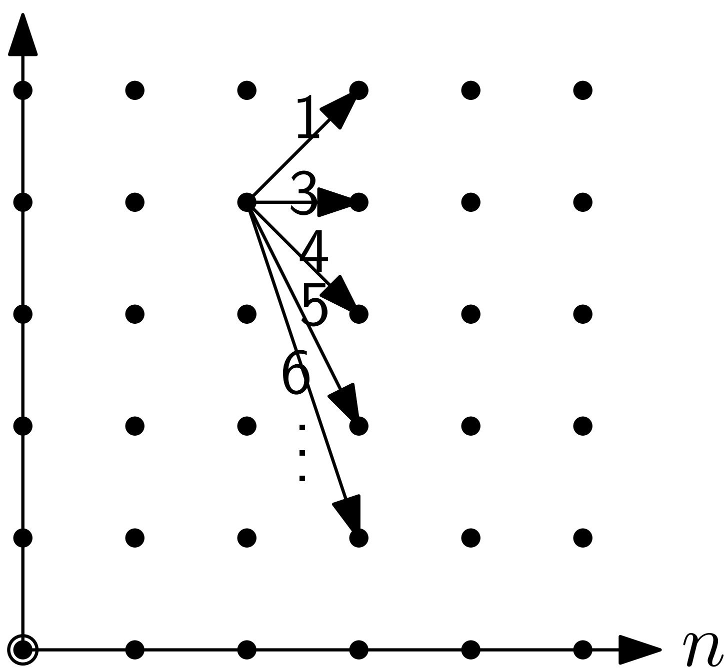

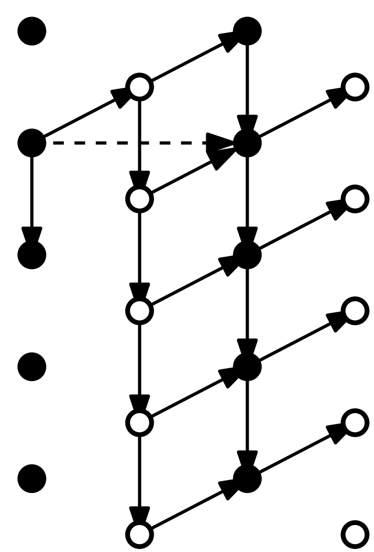

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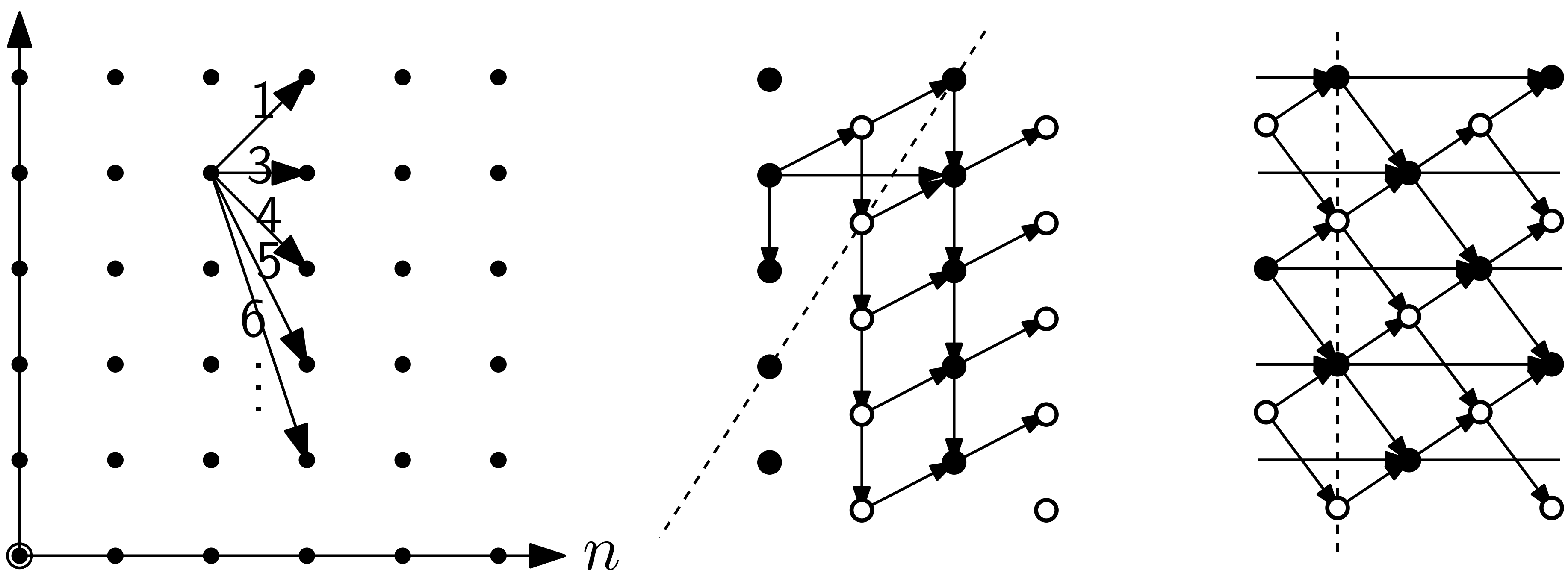
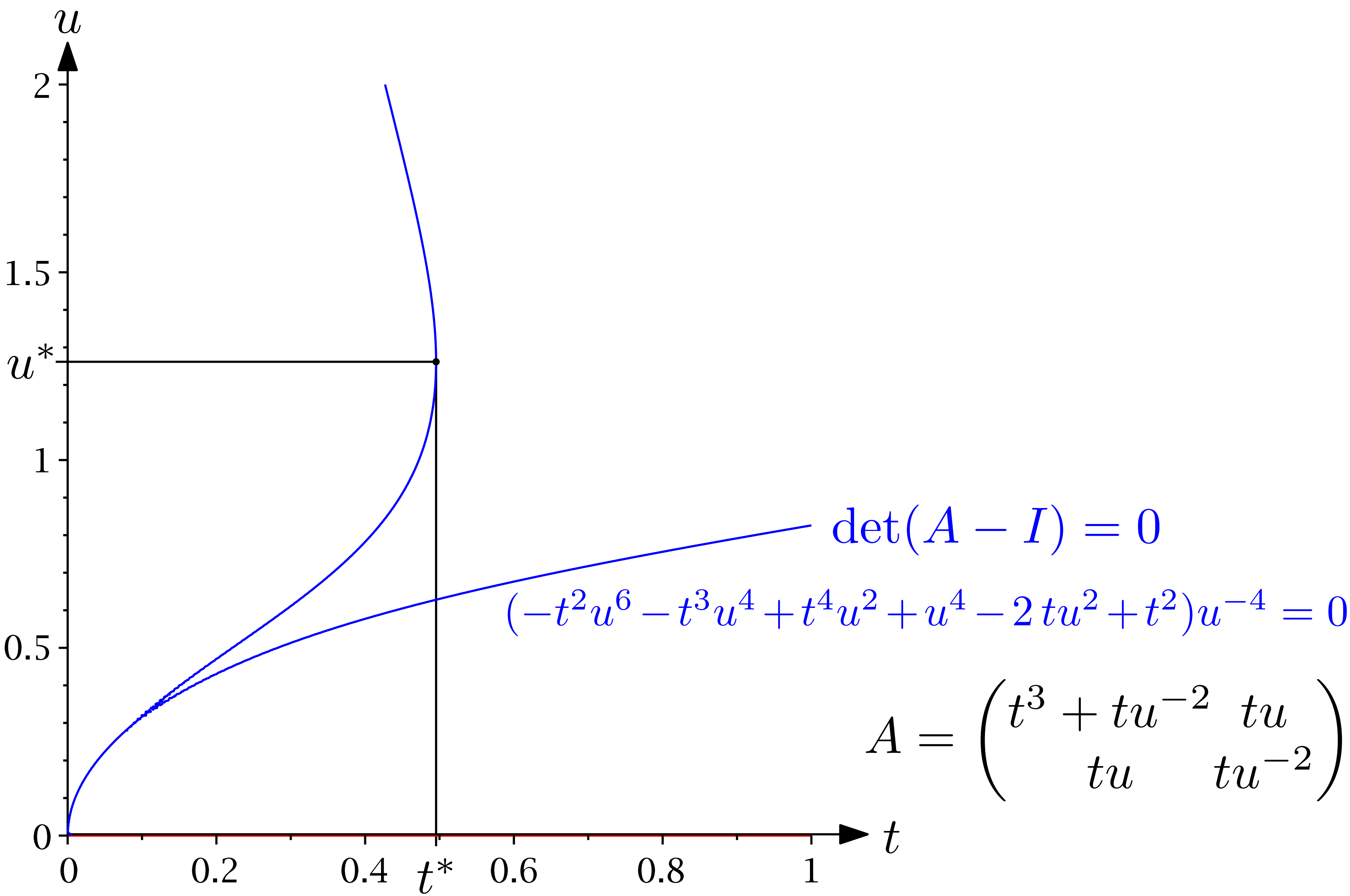
Irregularities at the boundary can be ignored.

$$A = \left(\begin{array}{c|cc} & \bullet & \circ \\ \hline \bullet & t^3 + tu^{-2} & tu \\ \circ & tu & tu^{-2} \end{array}\right)$$



Solving for t^* and u^*



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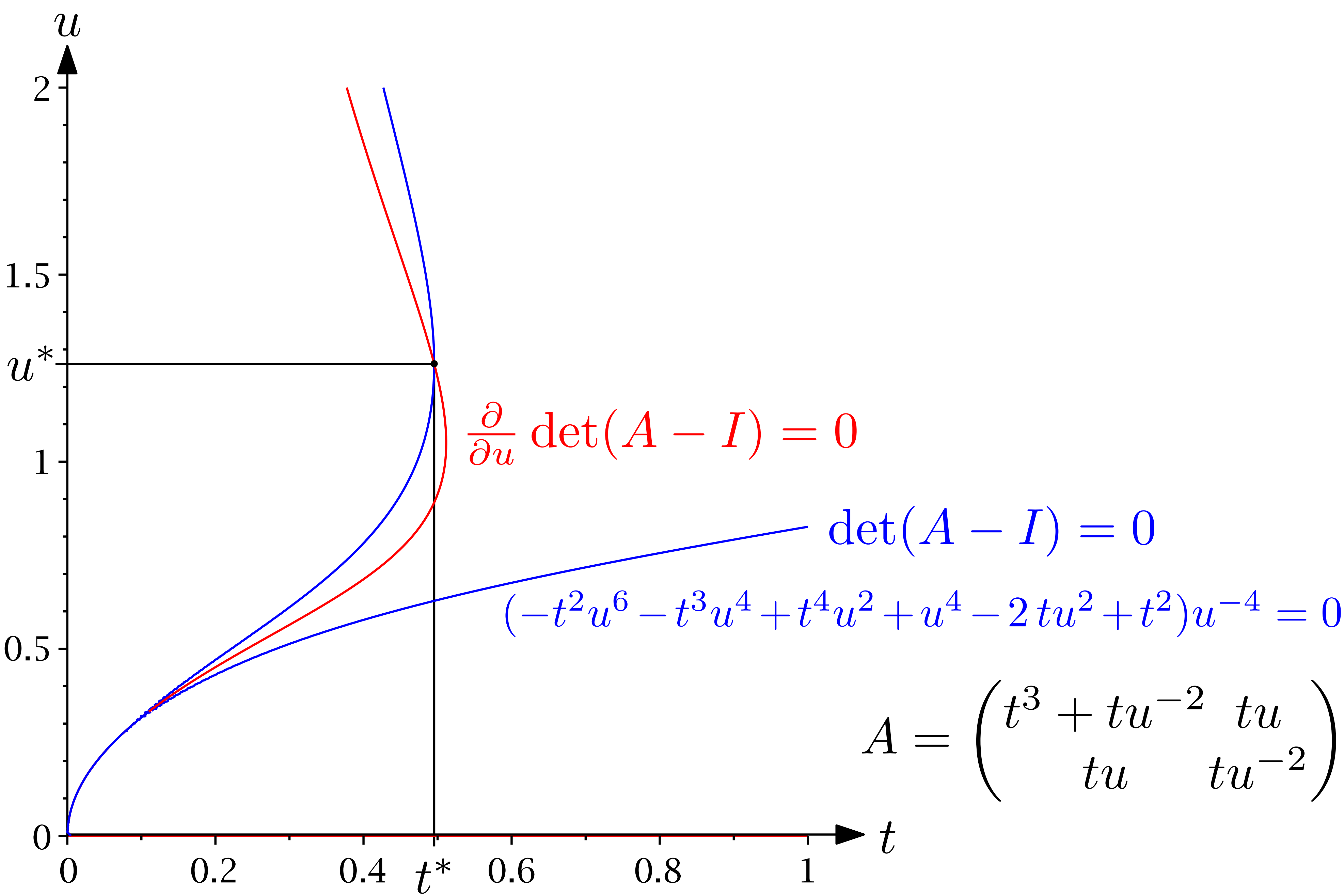
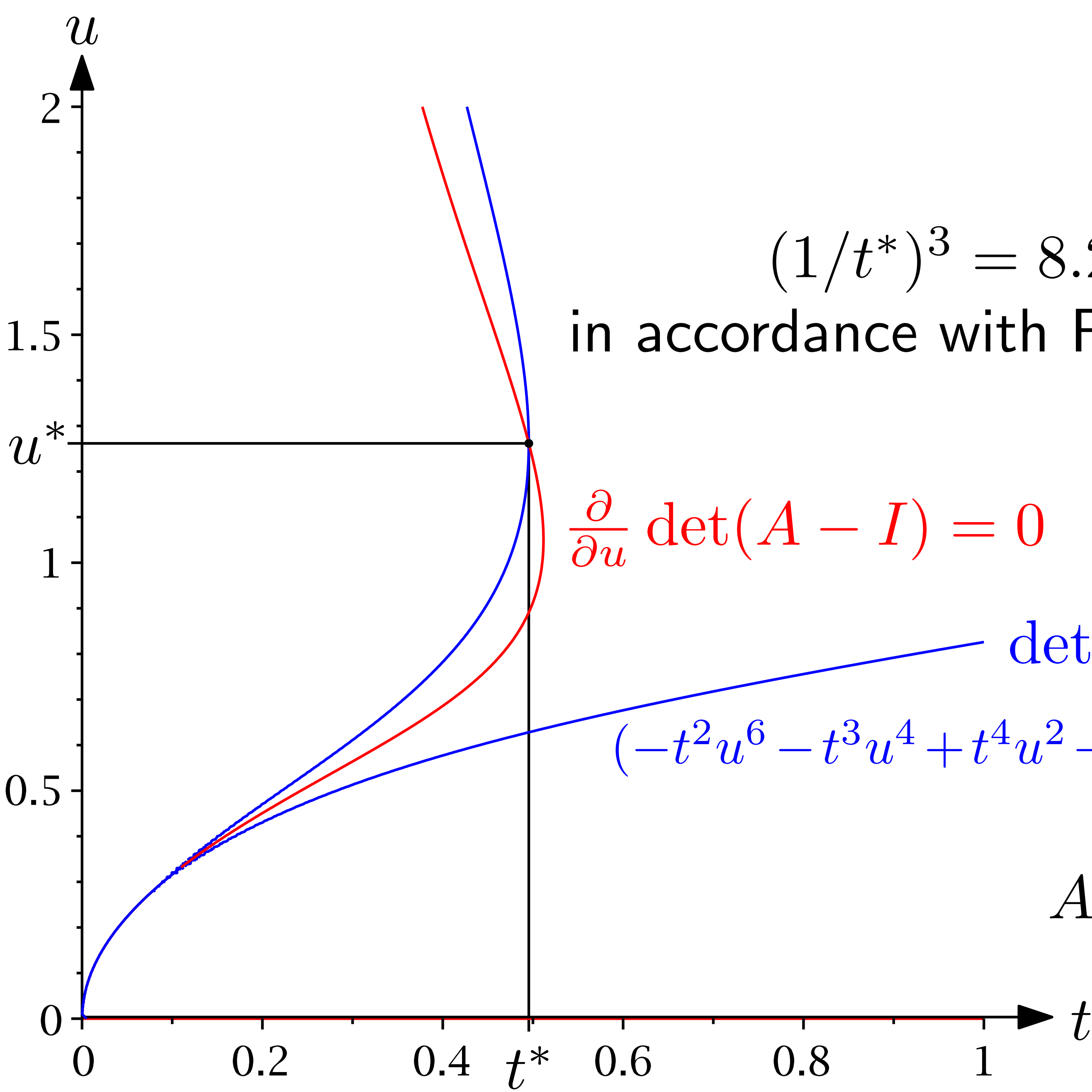


$$A=\begin{pmatrix} t^3+tu^{-2} & tu \\ tu & tu^{-2} \end{pmatrix}$$

Solving for t^* and u^*



$$(1/t^*)^3 = 8.22469257784$$

in accordance with Flajolet & Noy (1999)

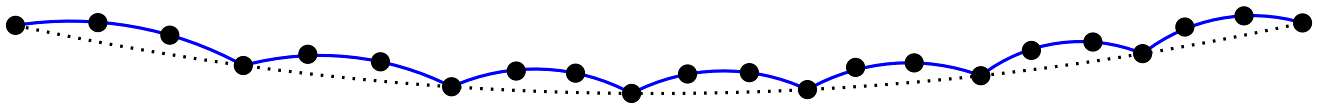
$$\frac{\partial}{\partial u}\det(A-I) = 0$$

$$\det(A-I) = 0$$

$$(-t^2u^6 - t^3u^4 + t^4u^2 + u^4 - 2\,tu^2 + t^2)u^{-4} = 0$$

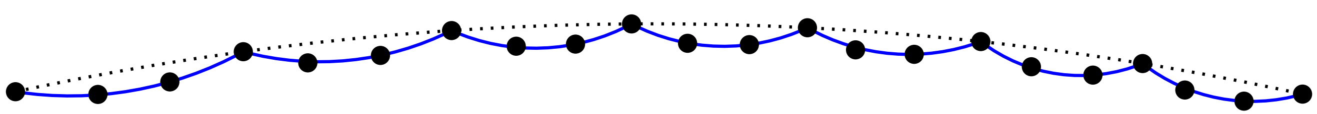
$$A = \begin{pmatrix} t^3 + tu^{-2} & tu \\ tu & tu^{-2} \end{pmatrix}$$

Example 3: Geometric graphs



the generalized double zigzag chain

[Huemer, Pilz, and Silveira 2018]



$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 2 & 2 & 2 & 2 & \dots \\ 0 & 0 & 2 & 2 & 2 & \dots \\ 0 & 0 & 0 & 2 & 2 & \dots \\ 0 & 0 & 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$P = R^3 + SR^2 + S(I+S)R + S(I+S)^2$$

Example 3: Geometric graphs

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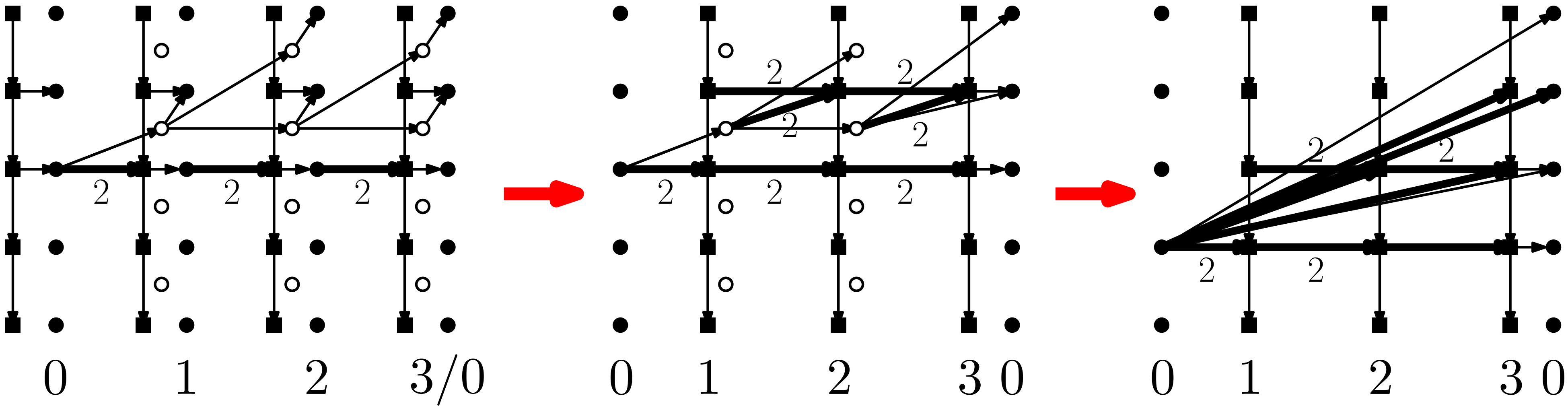

$$A = \begin{array}{cc} & \begin{array}{cccc} \bullet & \blacksquare_1 & \blacksquare_2 & \blacksquare_3 \end{array} \\ \begin{array}{c} \bullet \\ \blacksquare_1 \\ \blacksquare_2 \\ \blacksquare_3 \end{array} & \begin{pmatrix} t(u+2u^2+u^3) & 2 & 2u & 2u+2u^2 \\ 0 & u^{-1} & 2 & 0 \\ 0 & 0 & u^{-1} & 2 \\ t & 0 & 0 & u^{-1} \end{pmatrix} \end{array}$$

Proofs

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Conjecture: The number of paths from $(0,0)$ in state q_0 to $(n,0)$ in state q_1 that don't go below the x-axis is

$$\sim \text{const} \cdot (1/t^*)^n \cdot n^{-3/2},$$

where

(1) $A(t^*, u^*)$ has largest (Perron-Frobenius) eigenvalue 1.
$[\implies \det(A(t,u) - I) = 0\,]$

(2) u^* is chosen such that the value t^* that fulfills (1) is as large as possible. $[\implies \frac{\partial}{\partial u} \det(A(t,u) - I) = 0\,]$

APPROACHES:

A) Analytic Combinatorics, "square-root-type" singularity

B) Probabilistic interpretation, random walk

C) Pedestrian, induction

Proofs

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APPROACHES:

A) Analytic Combinatorics, "square-root-type" singularity

Special case: One state. All steps are of the form $(1, j)$.

[Banderier and Flajolet, 2002]

[$\det(A(t,u) - I) = t \cdot Q(u) - 1 = 0, \;\; Q'(u) = 0$]

Proofs

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(2) u^* is chosen such that the value t^* that fulfills (1) is as large as possible. $[\implies \frac{\partial}{\partial u}\det(A(t,u) - I) = 0\ ]$

(3) Let $\vec{v}$ and $\vec{w}$ be left and right eigenvectors of $A(t,u)$ with eigenvalue 1. Then

$$\vec{v} \cdot \frac{\partial}{\partial u} A(t,u) \cdot \vec{w} = 0.$$

(1)∧(2) ⇔ (1)∧(3). (linear algebra)

(1) $\implies N_{(x,y),q} \le v_q t^{-x} u^{-y}$ by easy induction.

Random walk

Use entries a_{qr} of $A = A(t,u)$ as "weights" for a random walk.

What is the effect of u in $t^i u^j$? Up-jumps ($j > 0$) are favored ($u > 1$) or penalized ($u < 1$) over down-jumps.
The weight of a path from $(0,0)$ to $(n,0)$ is unaffected by u!
Every path weight is multiplied by t^n.

$$A = \begin{pmatrix} 0.71 & 0.25 & 0.05 \\ 0.31 & 0.00 & 0.02 \\ 3.15 & 0.66 & 0.12 \end{pmatrix}$$

Use right eigenvector $\vec{w}$ to rescale into probabilities: $p_{qr} = a_{qr} \frac{w_r}{w_q}$
$\rightarrow$ stochastic matrix

What does $\frac{\partial}{\partial u} A(t,u)$ mean? The expected vertical jump!

Step $(8,5)$: $\frac{\partial}{\partial u} t^8 u^5 = 5t^8 u^4 \implies u \frac{\partial}{\partial u} t^8 u^5 = 5t^8 u^5 = 5a_{qr}$

"No-drift" condition: $$\vec{v} \cdot \left(u \cdot \frac{\partial}{\partial u} A(t,u) \right) \cdot \vec{w} = 0$$

stationary distribution

Local Limit Theorems

Prob[sum of n i.i.d. random variables with mean 0 lies in some small region around 0] $\sim$ const $\cdot\, n^{-1/2}$ [Gnedenko]

Needs to be adapted to sign-restricted case $(y \geq 0)$ and several states.

Local Limit Theorems

Prob[sum of n i.i.d. random variables with mean 0 lies in some small region around 0] $\sim$ const $\cdot\, n^{-1/2}$ [Gnedenko]

Needs to be adapted to sign-restricted case $(y \geq 0)$ and several states.

"Pedestrian" approach. Pioneered for a special case with 2 states in Asinowski and Rote (2018).

- $O((1/t^*)^n)$ by induction.
- $\Omega((1/t^* - \varepsilon)^n)$ for every $\varepsilon > 0$, by induction.

Extensions

- higher dimensions
- jumps $(i, j) \in \mathbb{R}^2$