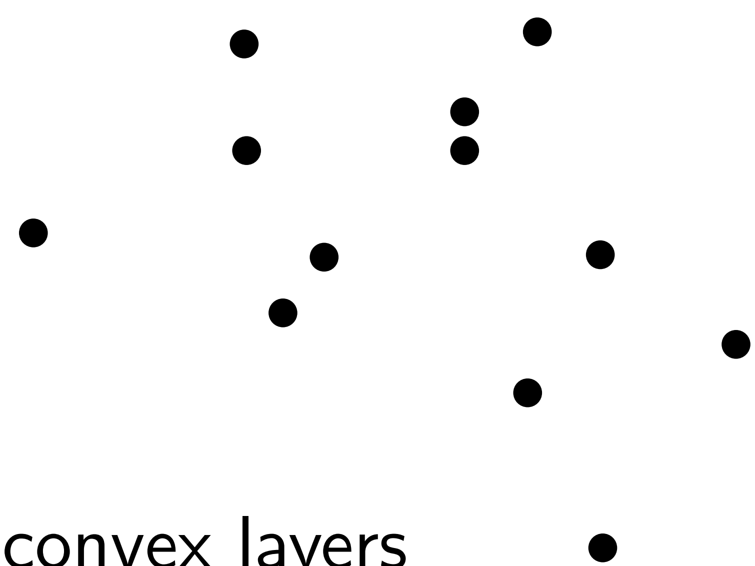
Grid Peeling and the Affine Curvature-Shortening Flow (ACSF)

Günter Rote and Moritz Rüber
Freie Universität Berlin



convex layers
onion layers

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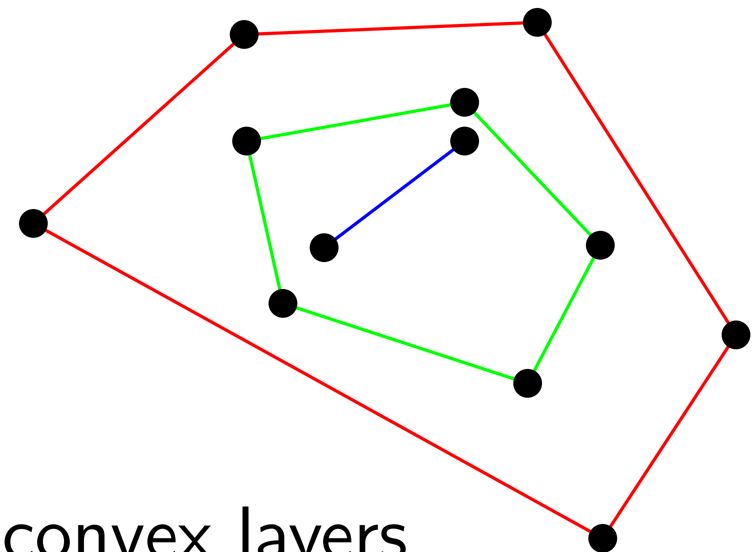

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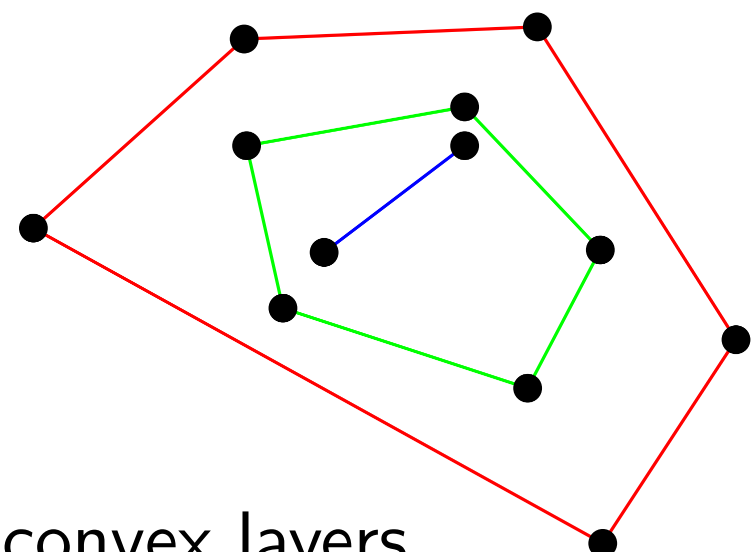

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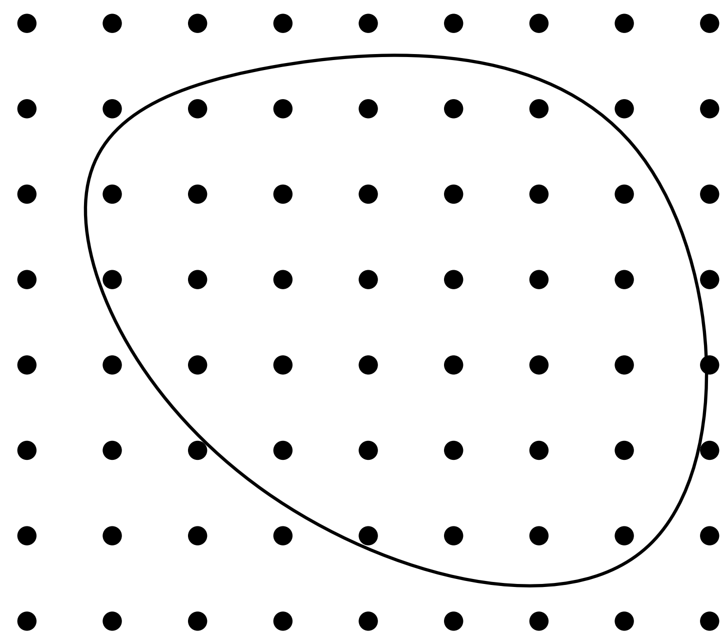

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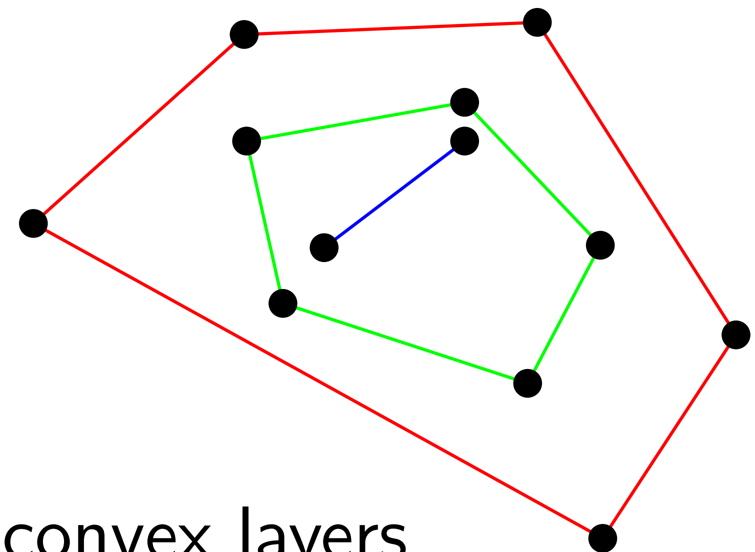

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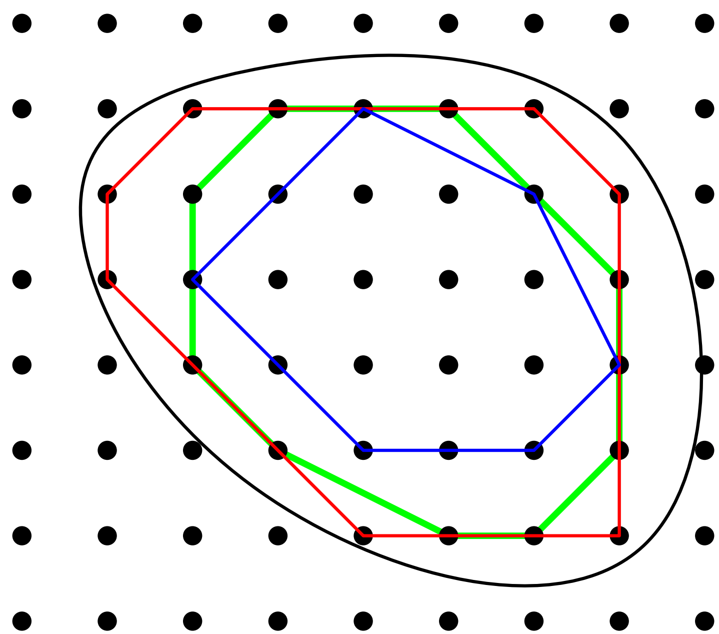
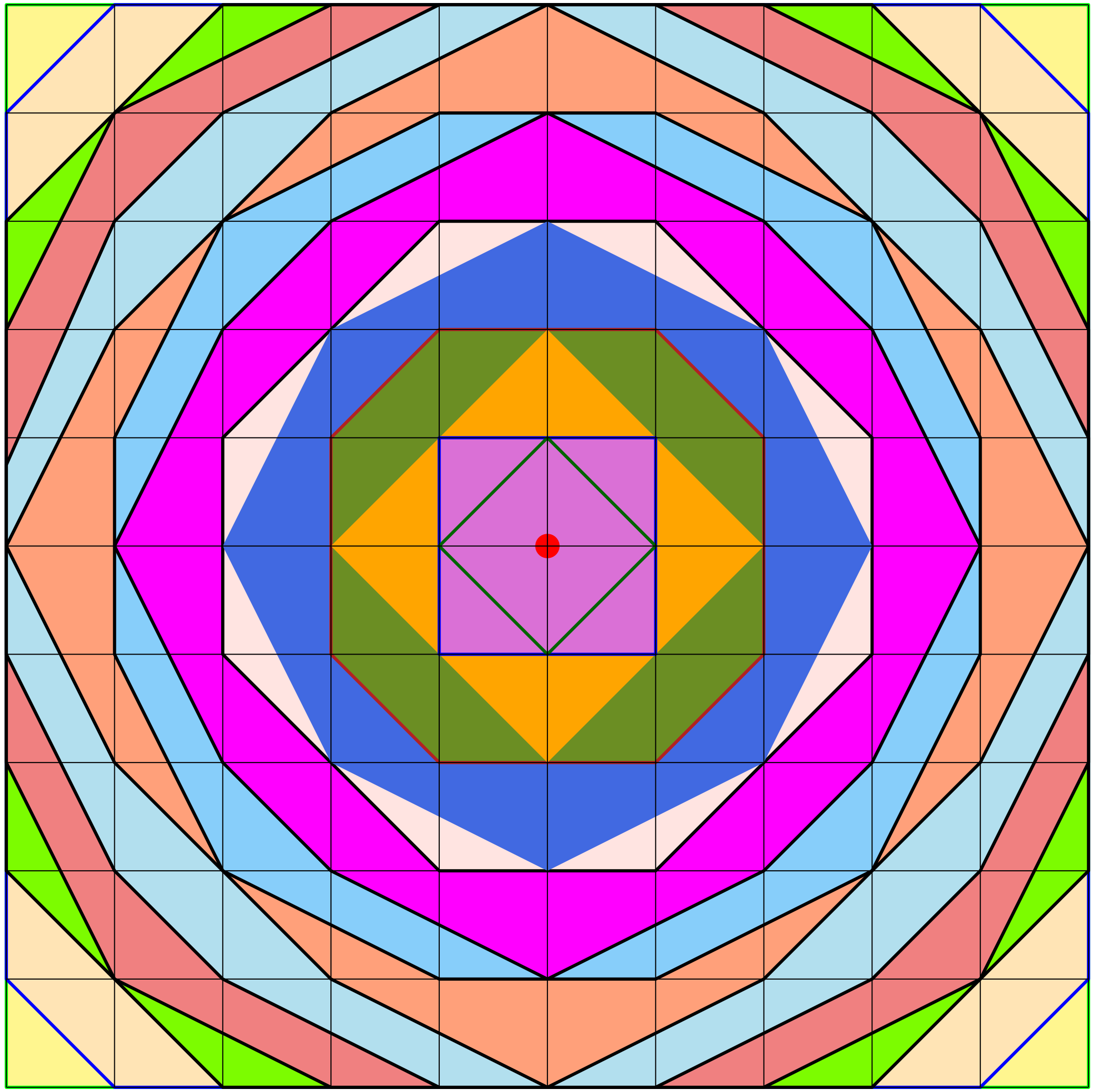


grid peeling

Grid Peeling of the Square

[Sariel Har-Peled and Bernard Lidický 2013]



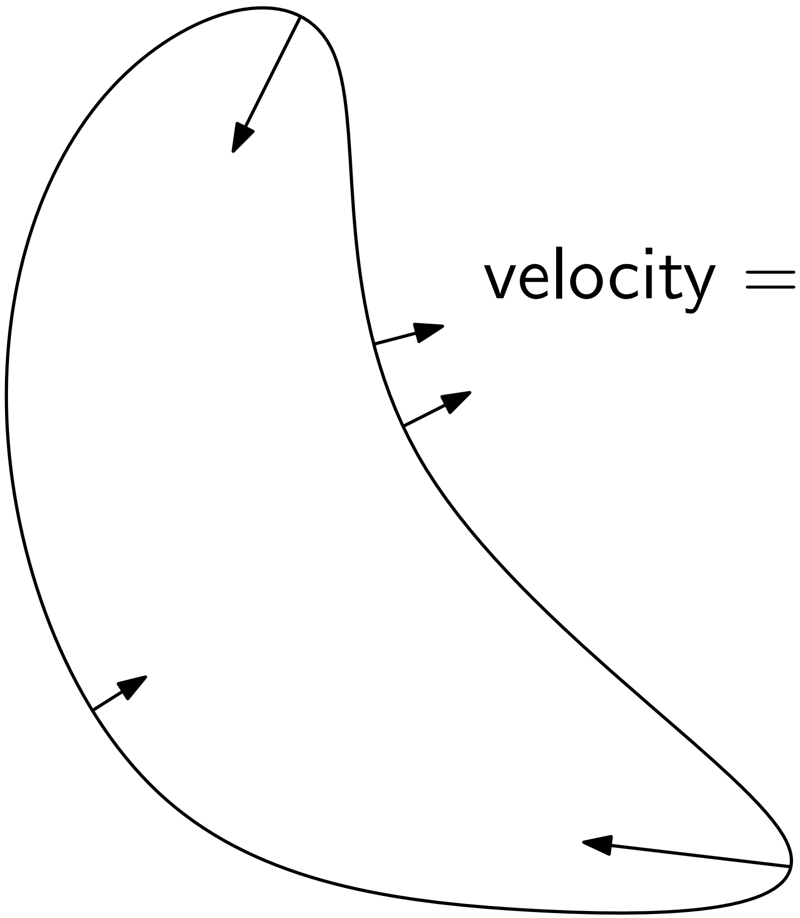
The $n \times n$ grid has $\Theta(n^{4/3})$ convex layers.

Affine Curvature-Shortening Flow (ACSF)

Freie Universität Berlin

[L. Alvarez, F. Guichard, P.-L. Lions, J.-M. Morel:
"Axioms and fundamental equations of image processing" 1993]

[G. Sapiro and A. Tannenbaum:
"Affine invariant scale-space." Int. J. Computer Vision 1993]



velocity $= \kappa^{1/3}$ ($\kappa =$ curvature)

equivariant under area-preserving
affine transformations!

Peeling and the ACSF

Conjecture:

David Eppstein, Sariel Har-Peled, and Gabriel Nivasch. Grid peeling and the affine curve shortening flow. Experimental Mathematics **29** (2020), 306–316

As the grid is more and more refined, grid peeling approaches the ACSF.

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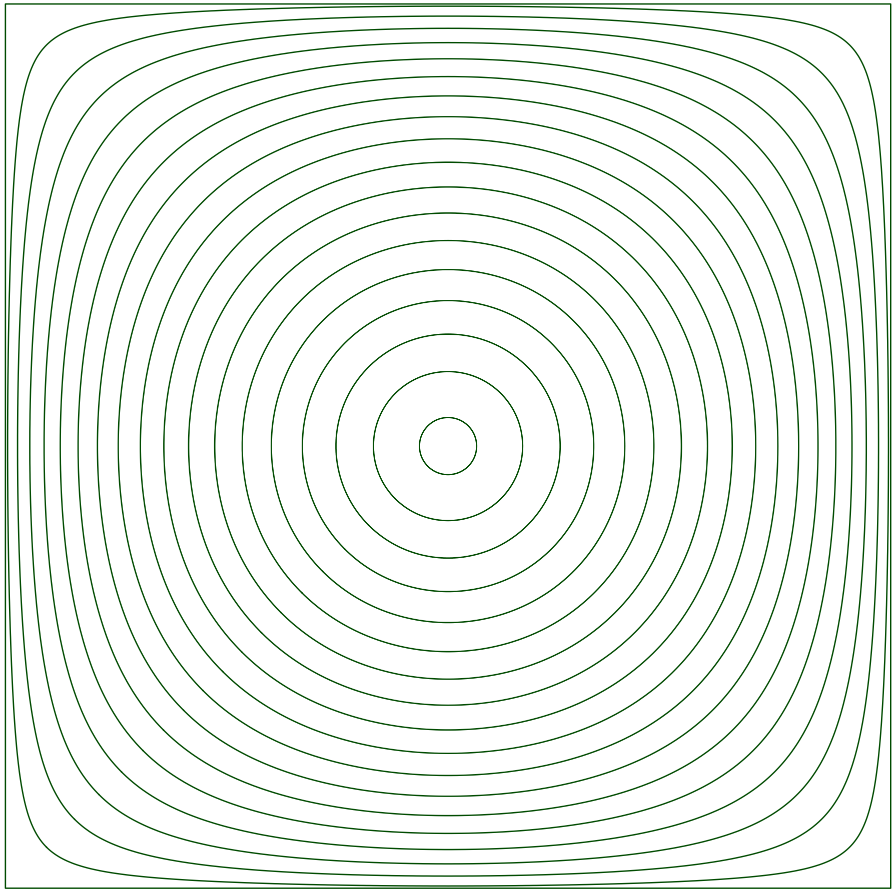

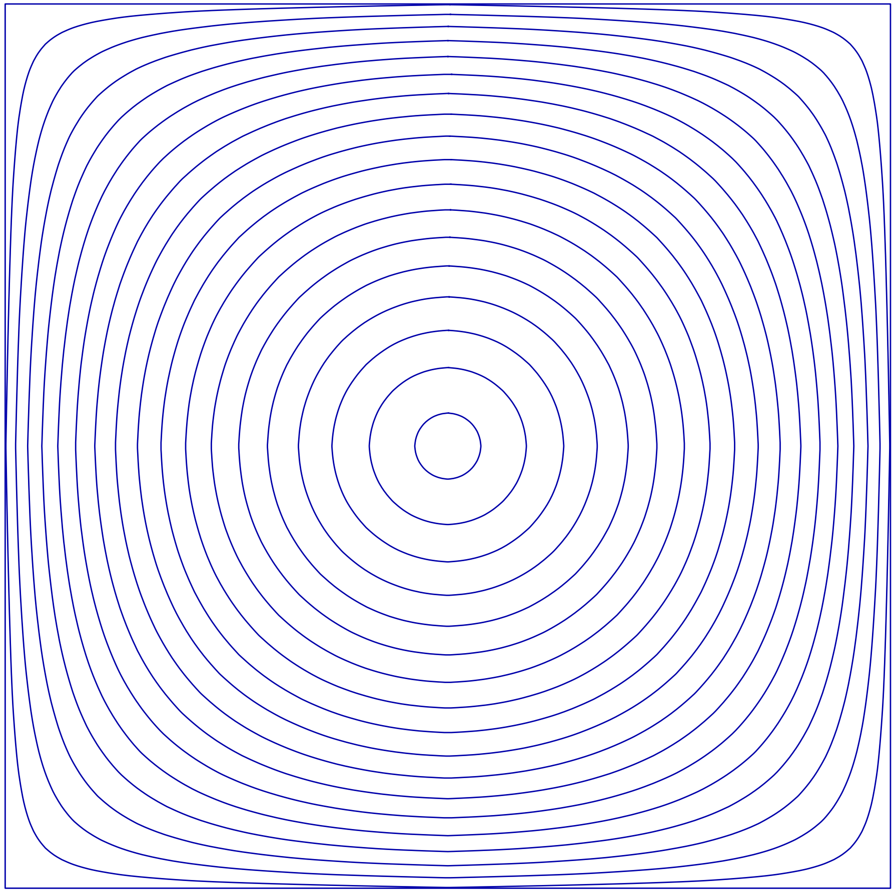

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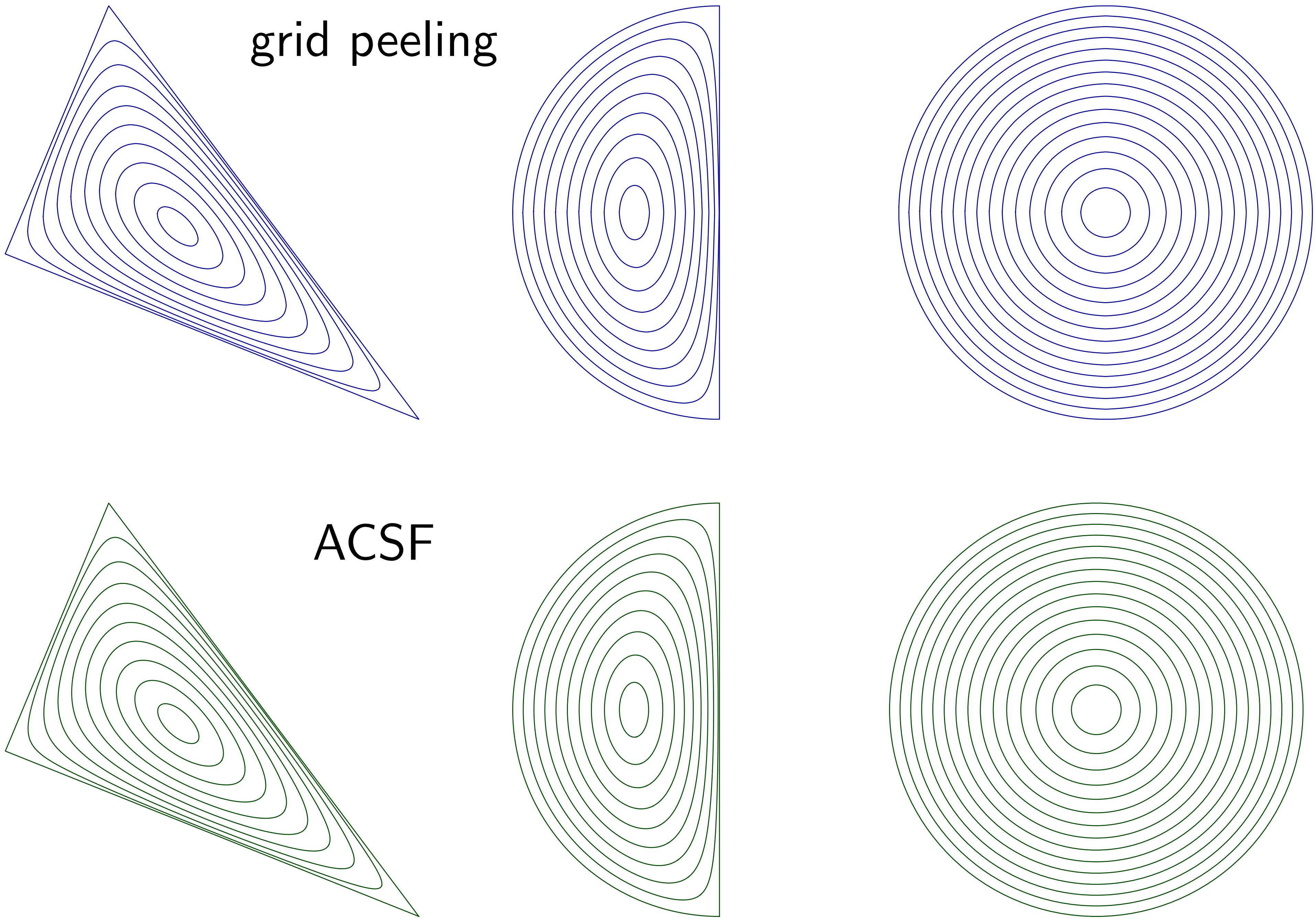

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As the grid is more and more refined, grid peeling approaches the ACSF.

ACSF at time $t \approx$ Grid peeling on $\frac{1}{n}$-grid after $C_g\, t\, n^{4/3}$ steps.

Conjecture: (Moritz Rüber and Günter Rote)

$$C_g = \sqrt[3]{\frac{\pi^2}{2\zeta(3)}} \approx 1.60120980542577$$

Peeling and the ACSF

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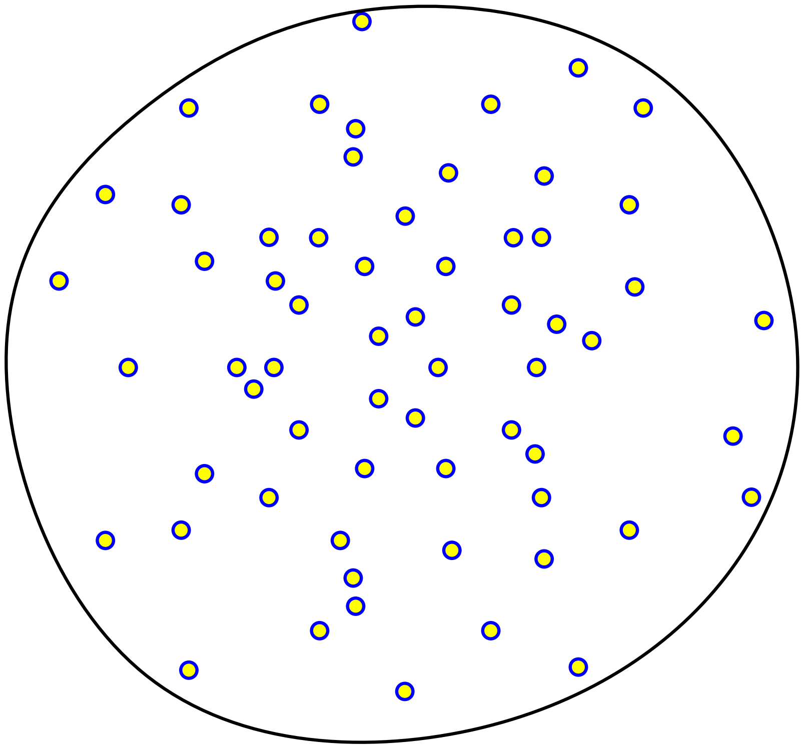
→ Jeff Calder and Charles K Smart. The limit shape of convex hull peeling. Duke Math. J. (2020)

random points



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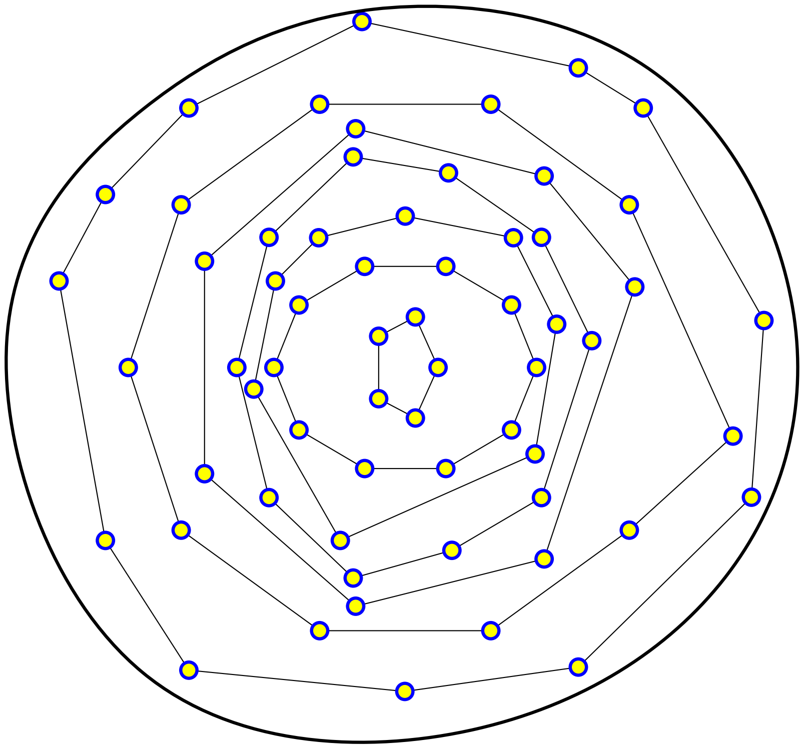
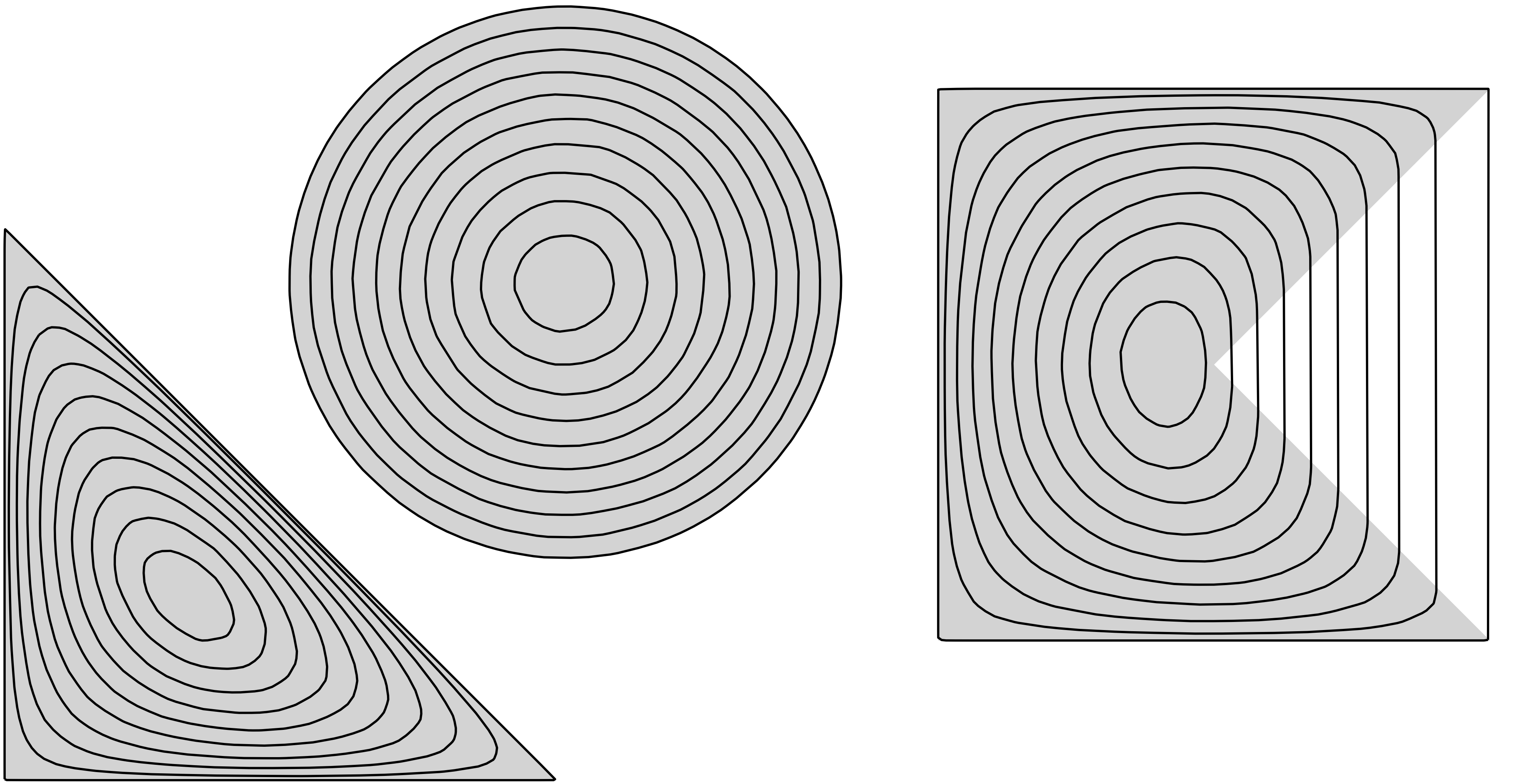


Peeling and the ACSF

Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020

10000 random points in the shaded region



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Jeff Calder and Charles K Smart. The limit shape of convex hull peeling. Duke Math. J. **169** (2020)

Theorem:

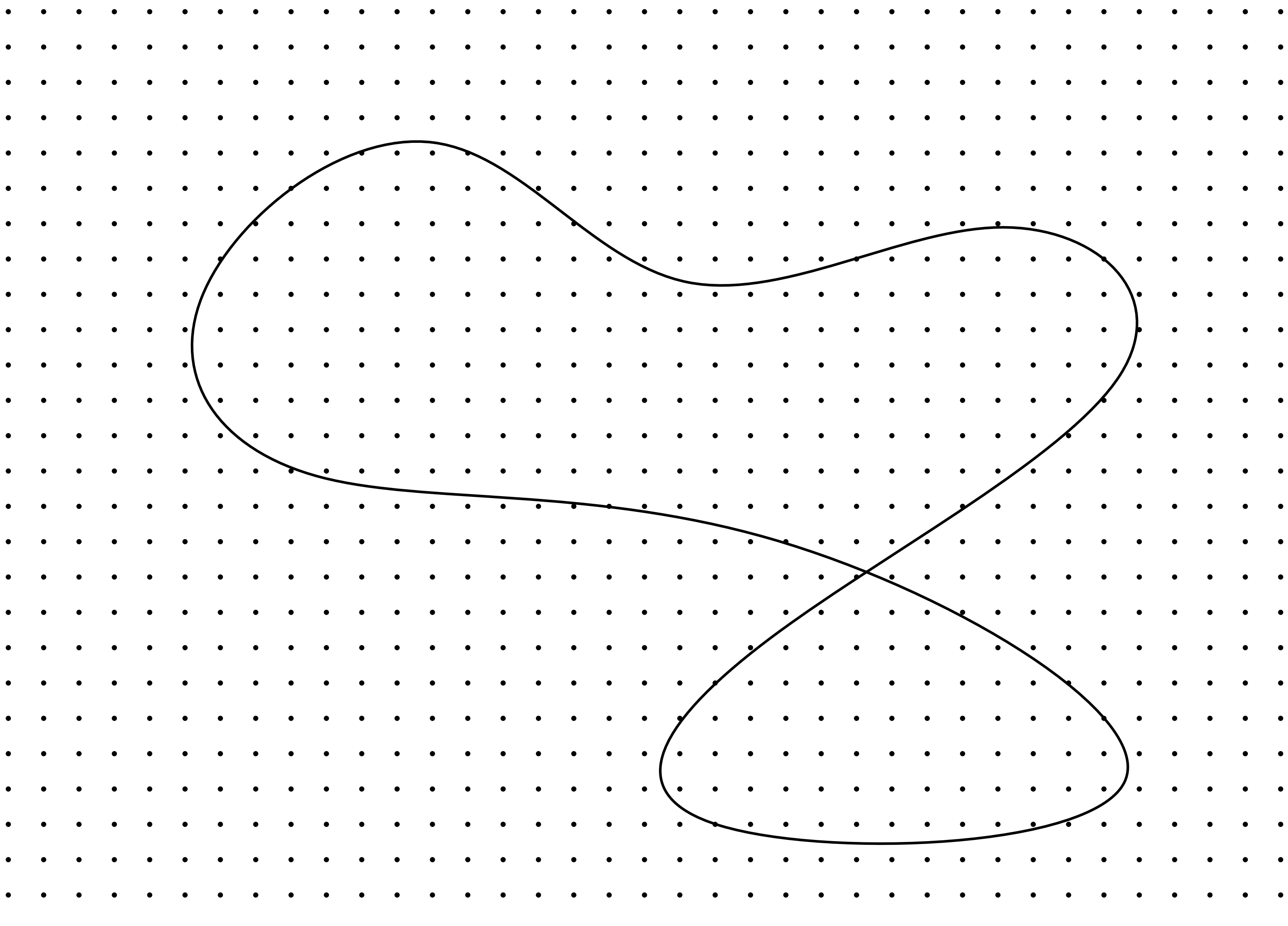
ACSF at time $t \approx$ Peeling on density-n^2 set after $C_r t n^{4/3}$ steps.

$$C_g \approx 1.6, \;\; C_r \approx 1.3$$

- Invariant under affine transformations?

Homotopic peeling

[Sergey Avvakumov and Gabriel Nivasch 2019]



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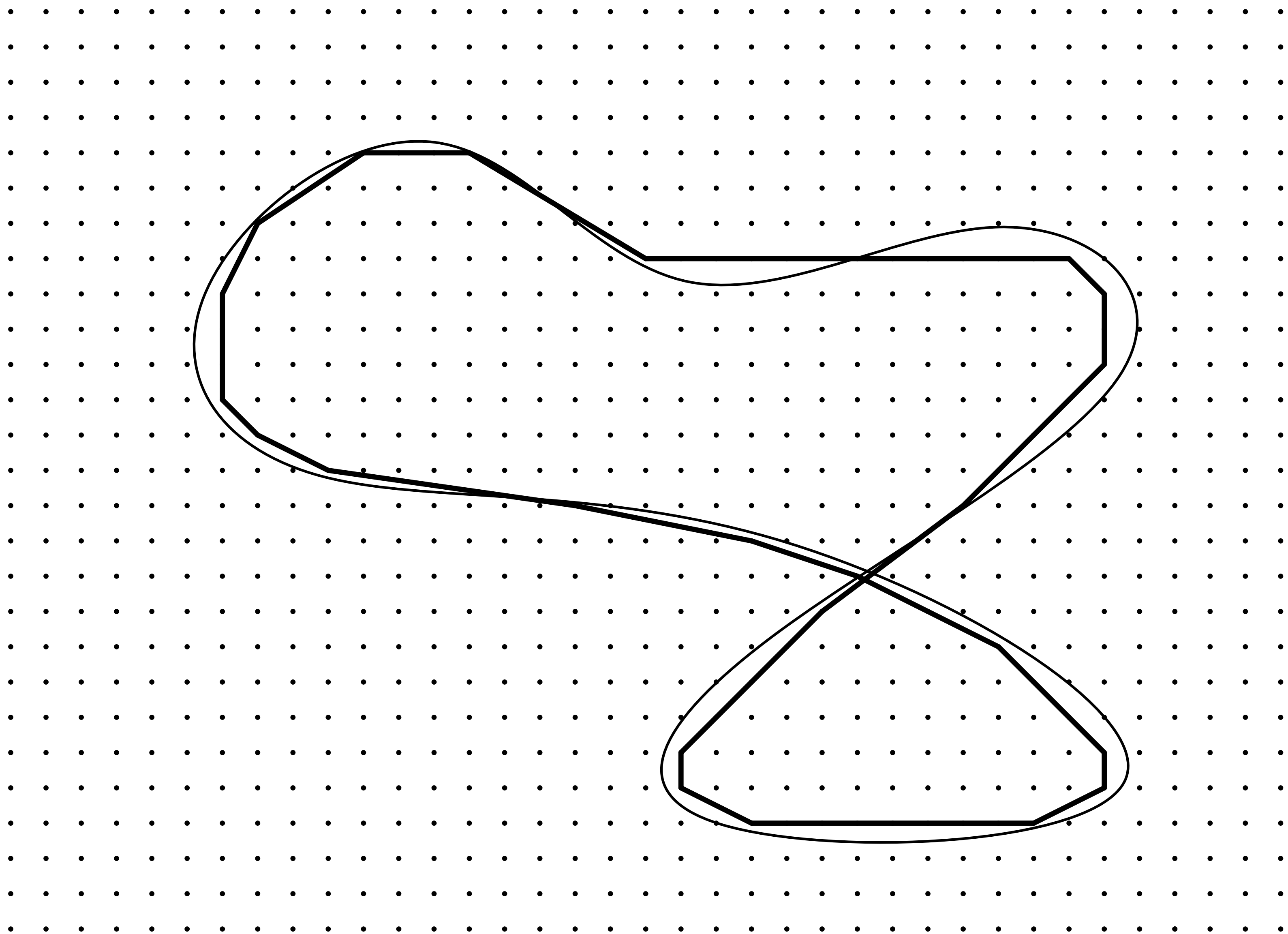

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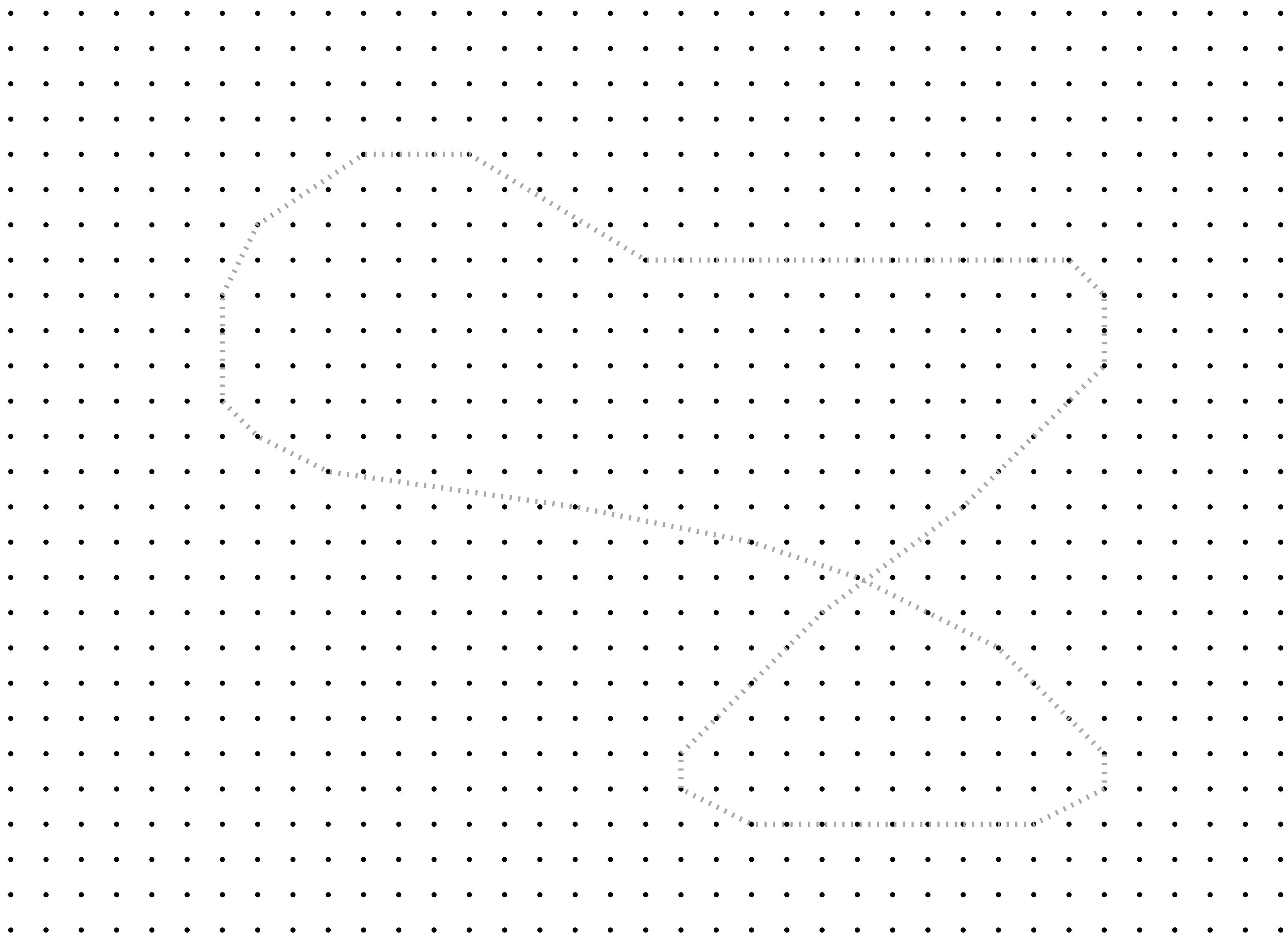

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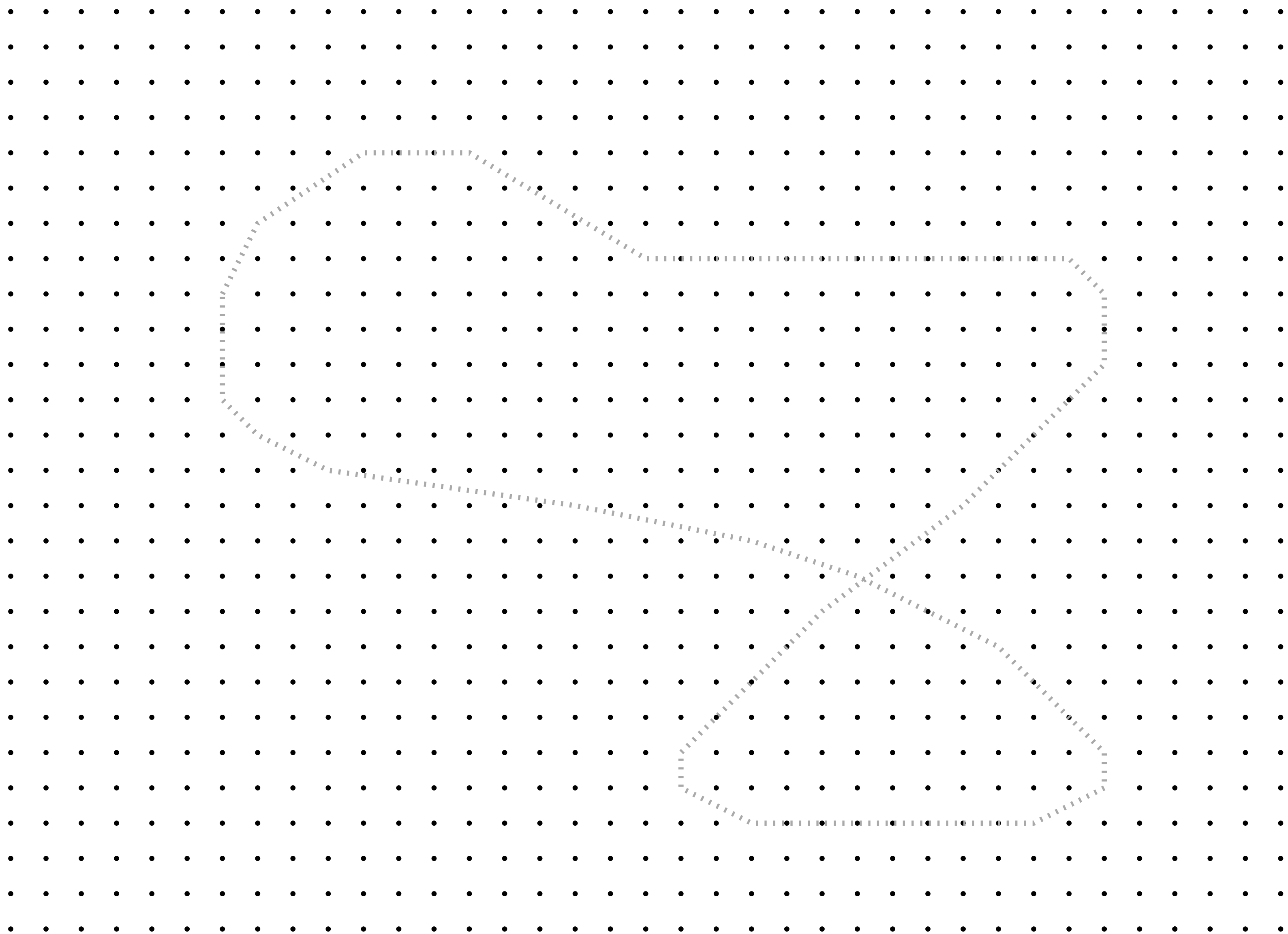

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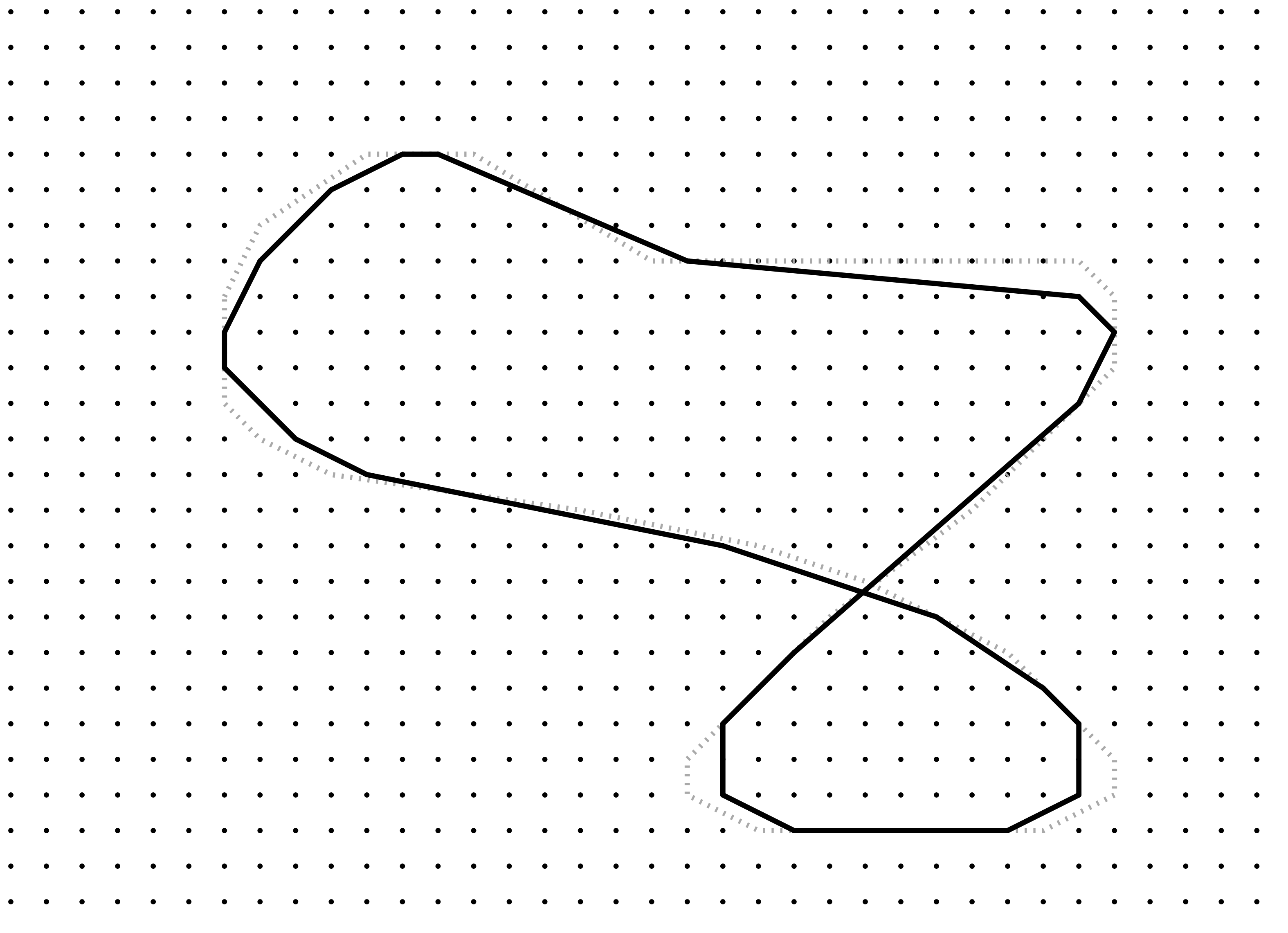

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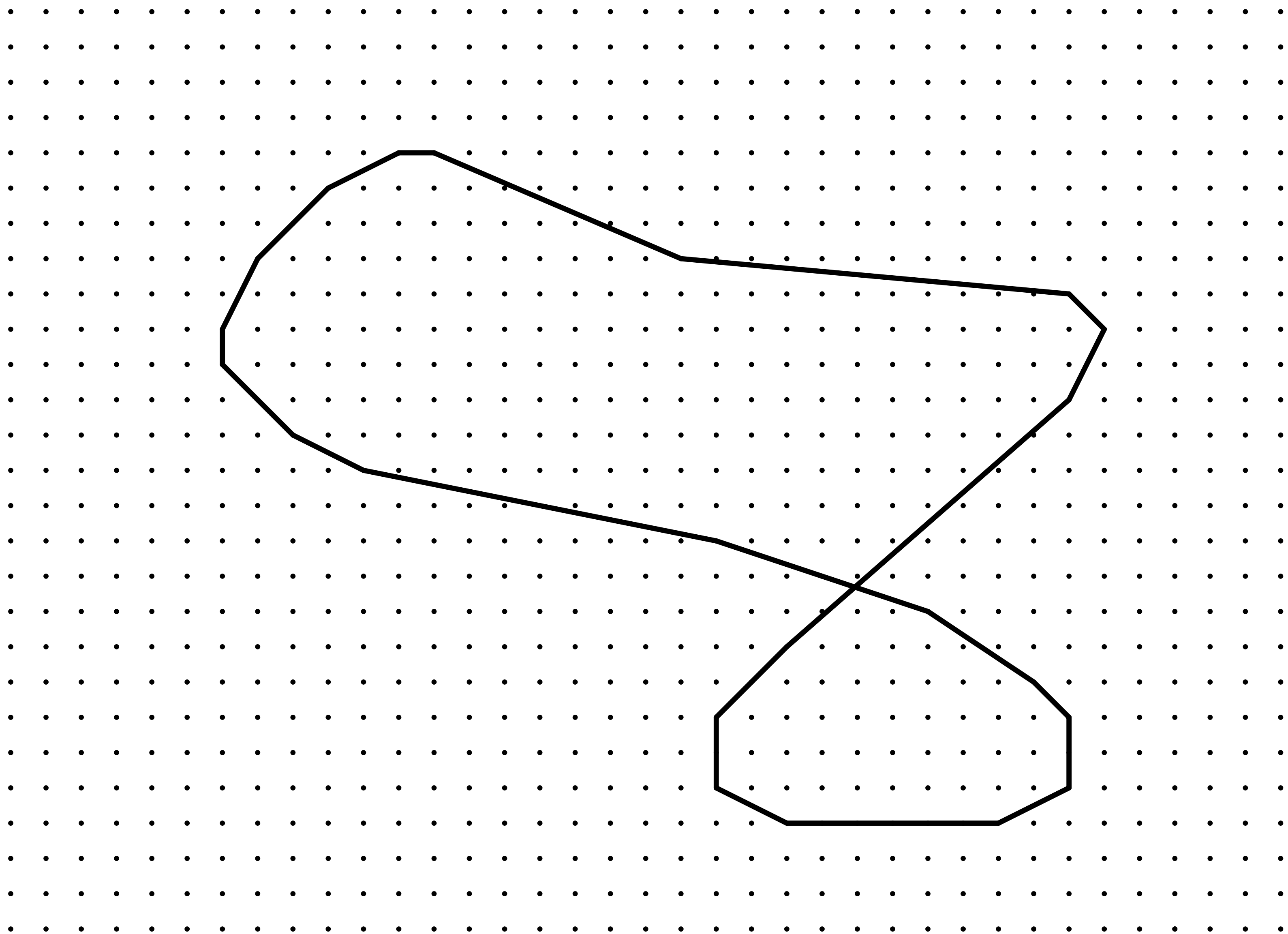

Homotopic peeling

Freie Universität Berlin

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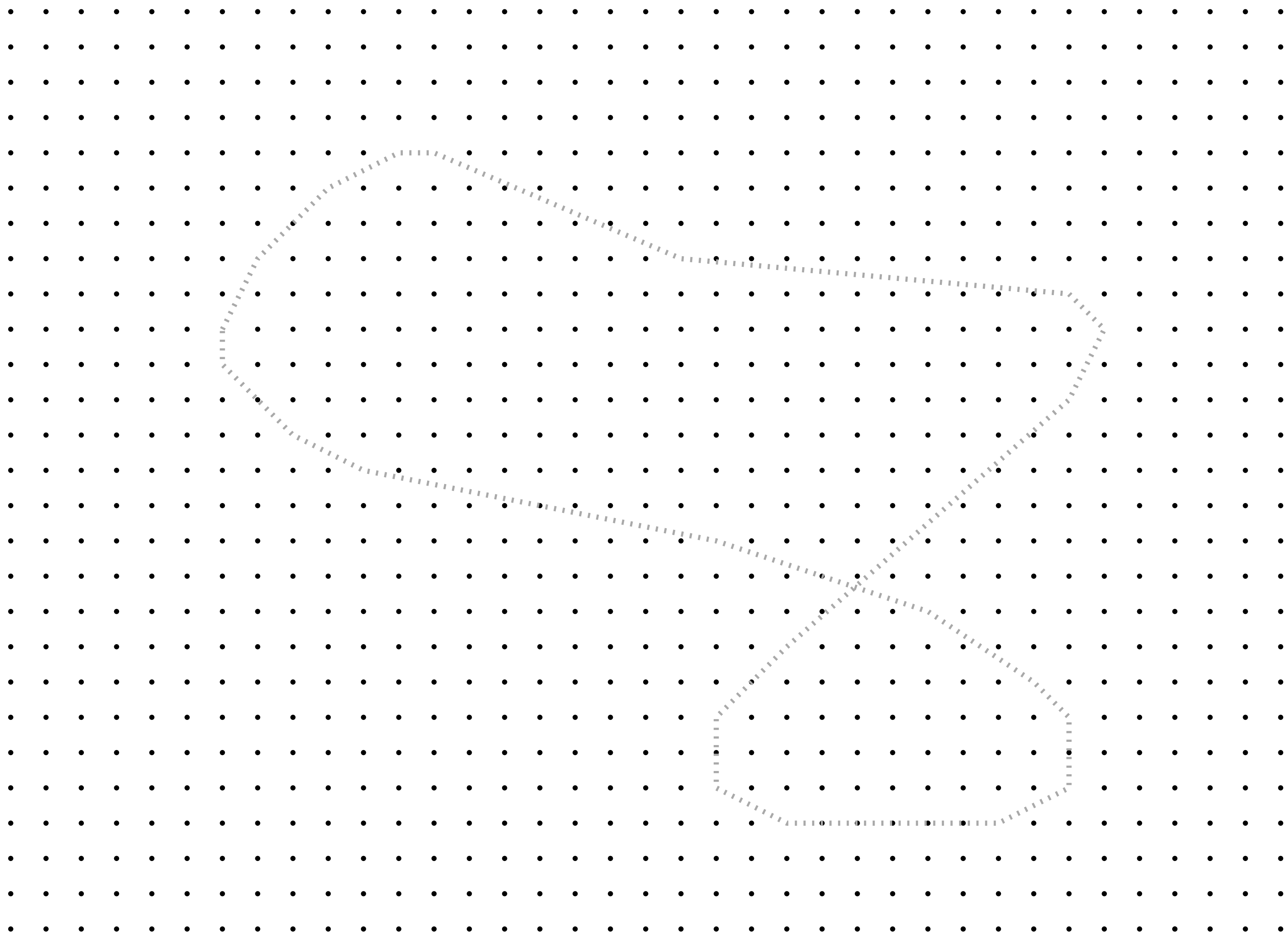

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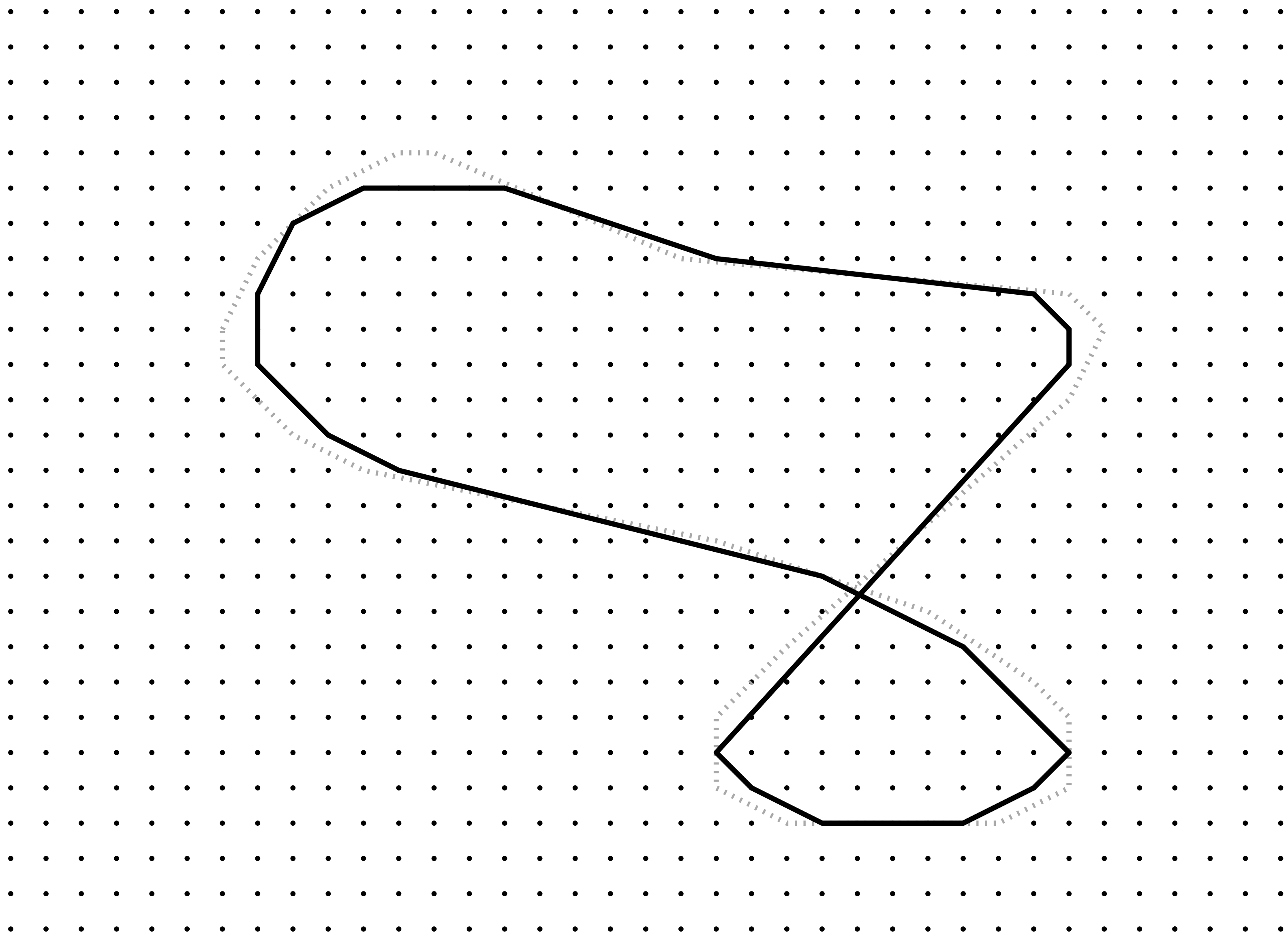

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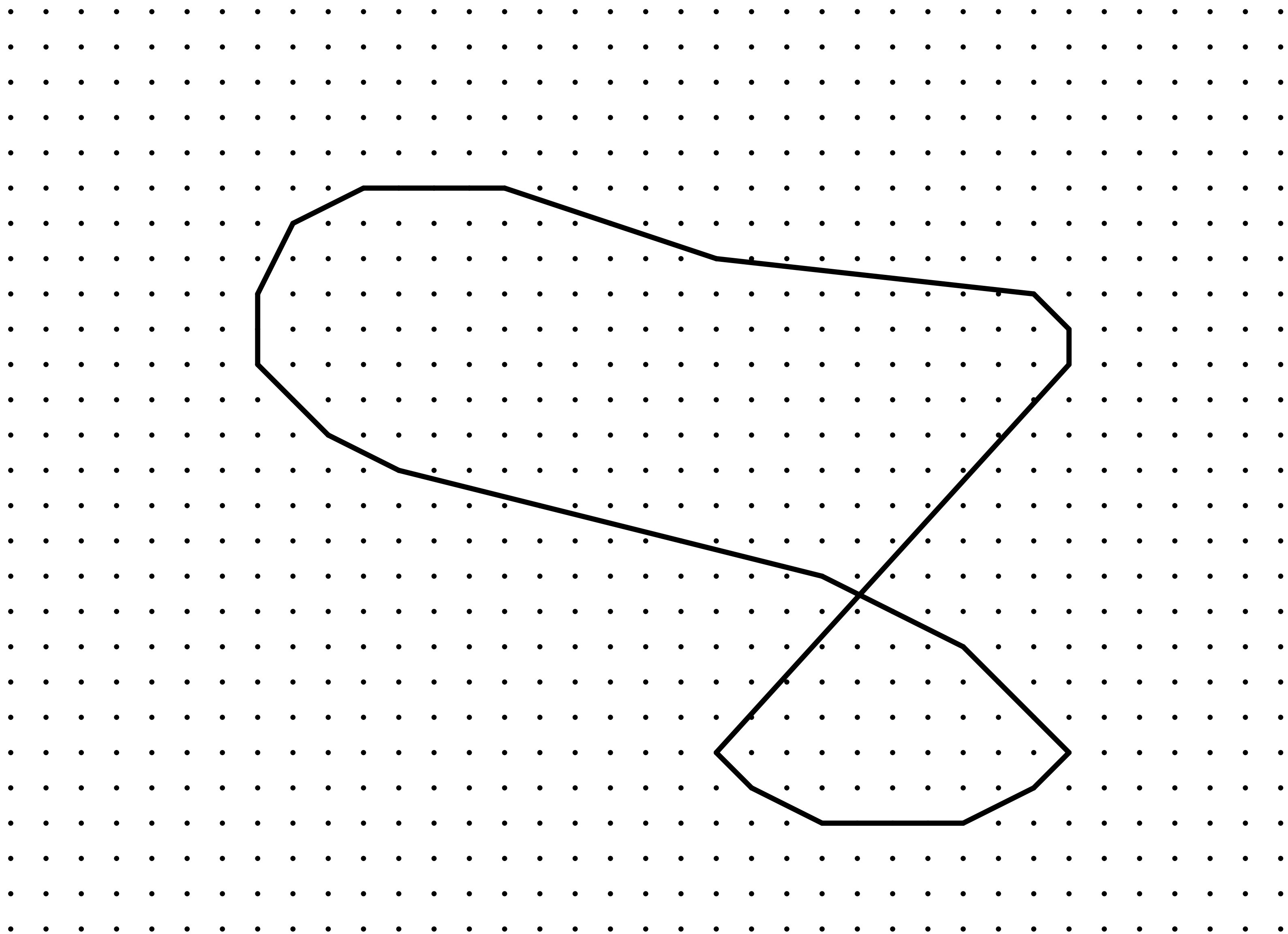

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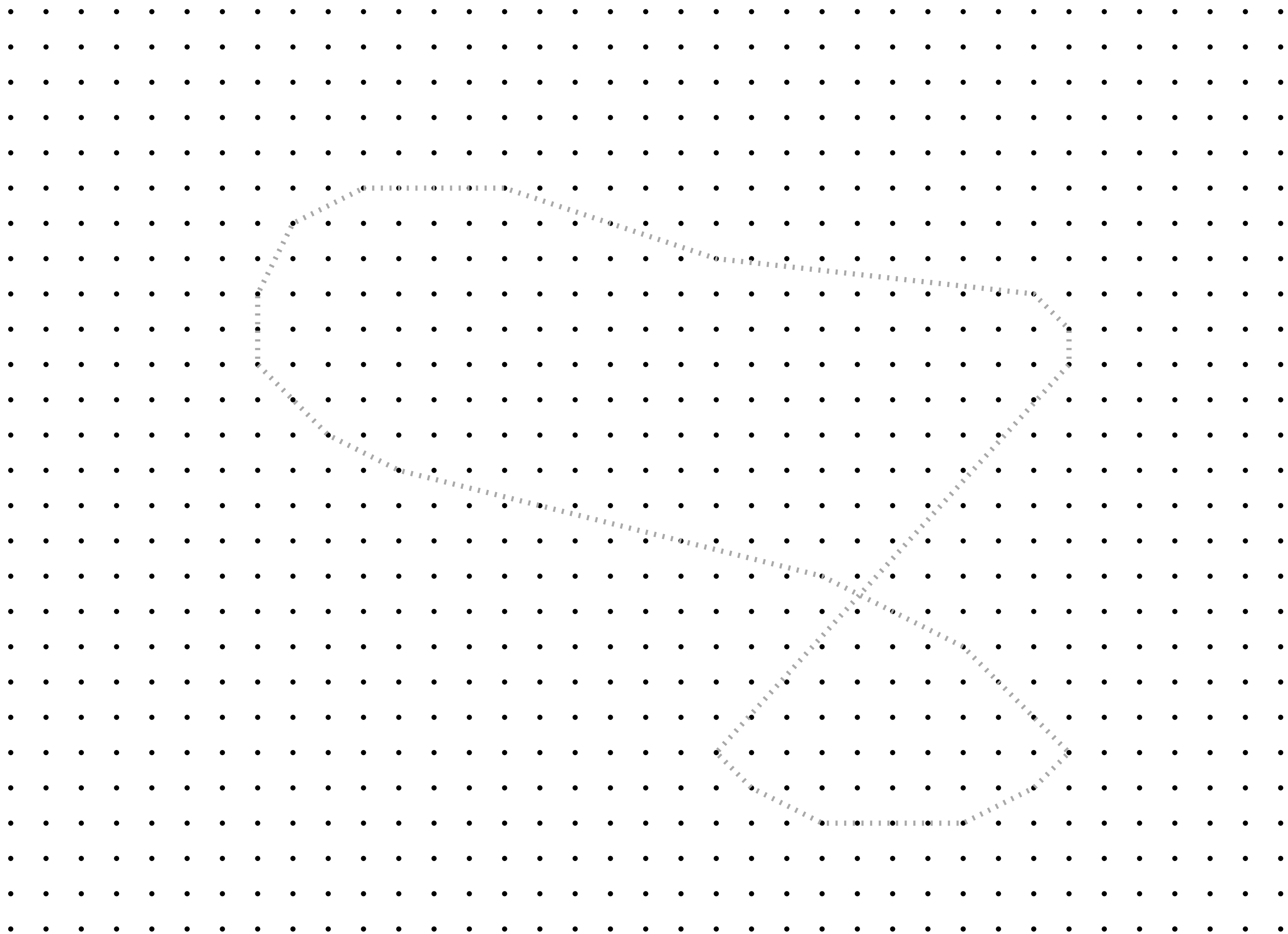

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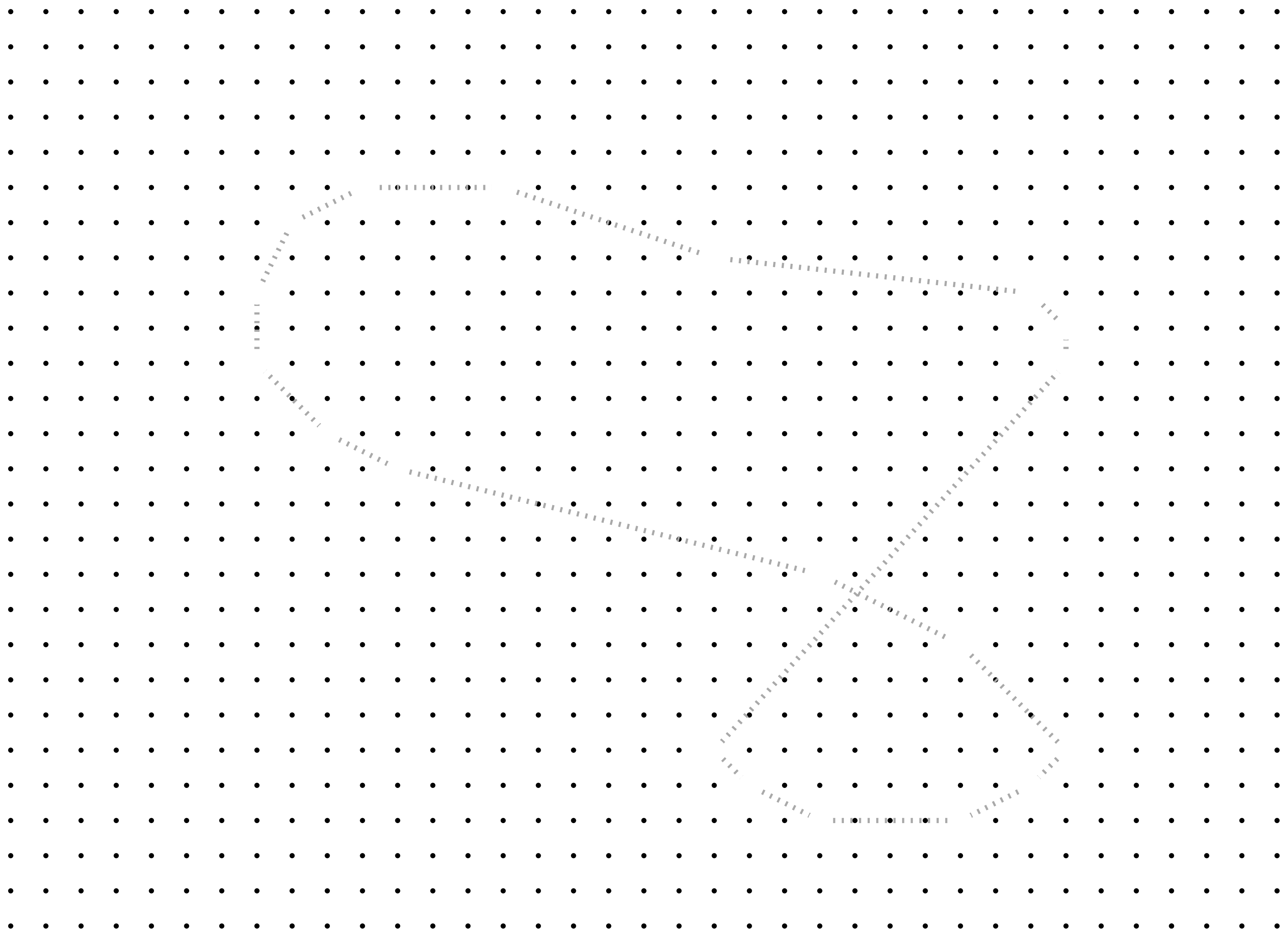

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Freie Universität Berlin

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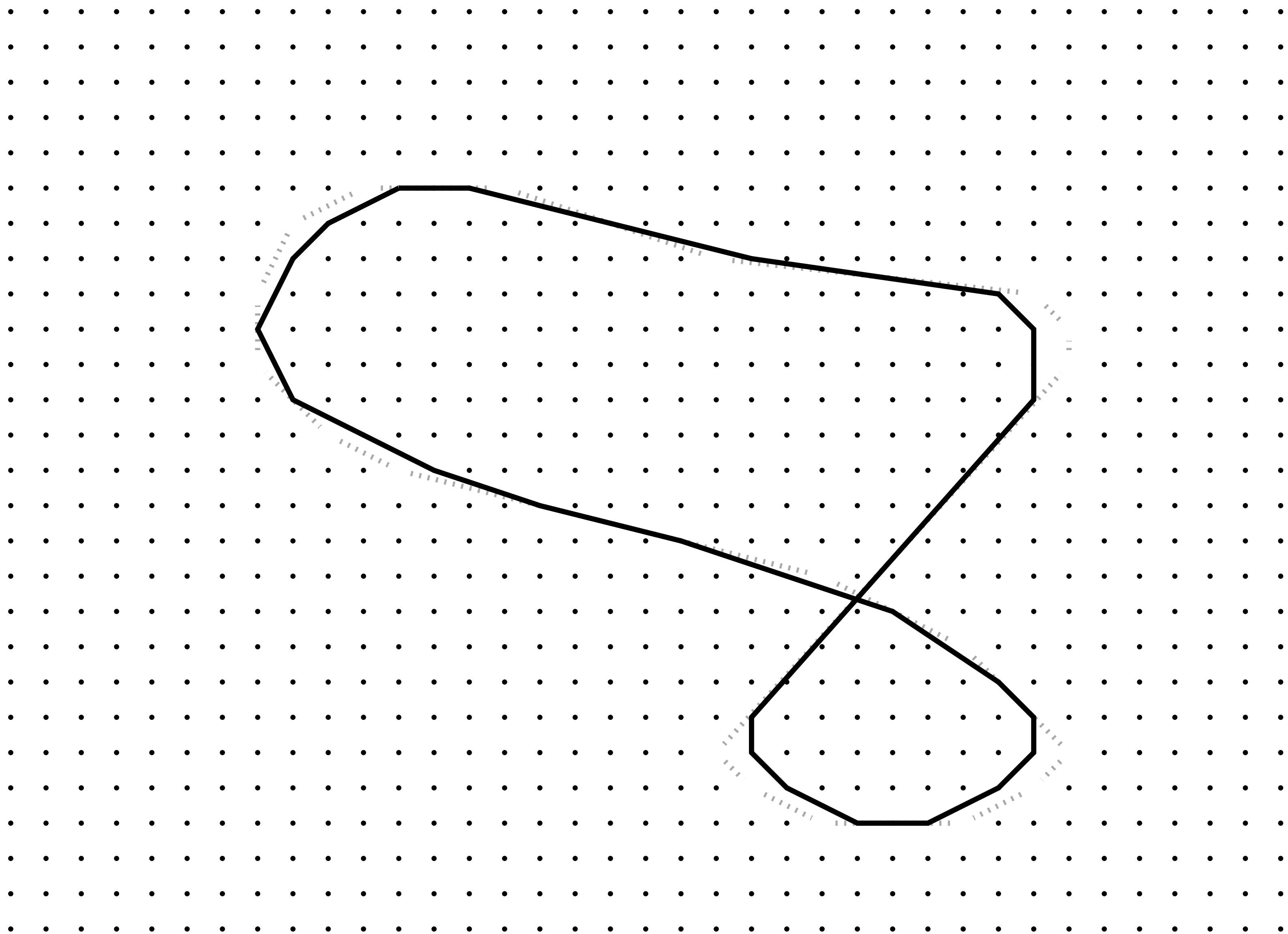

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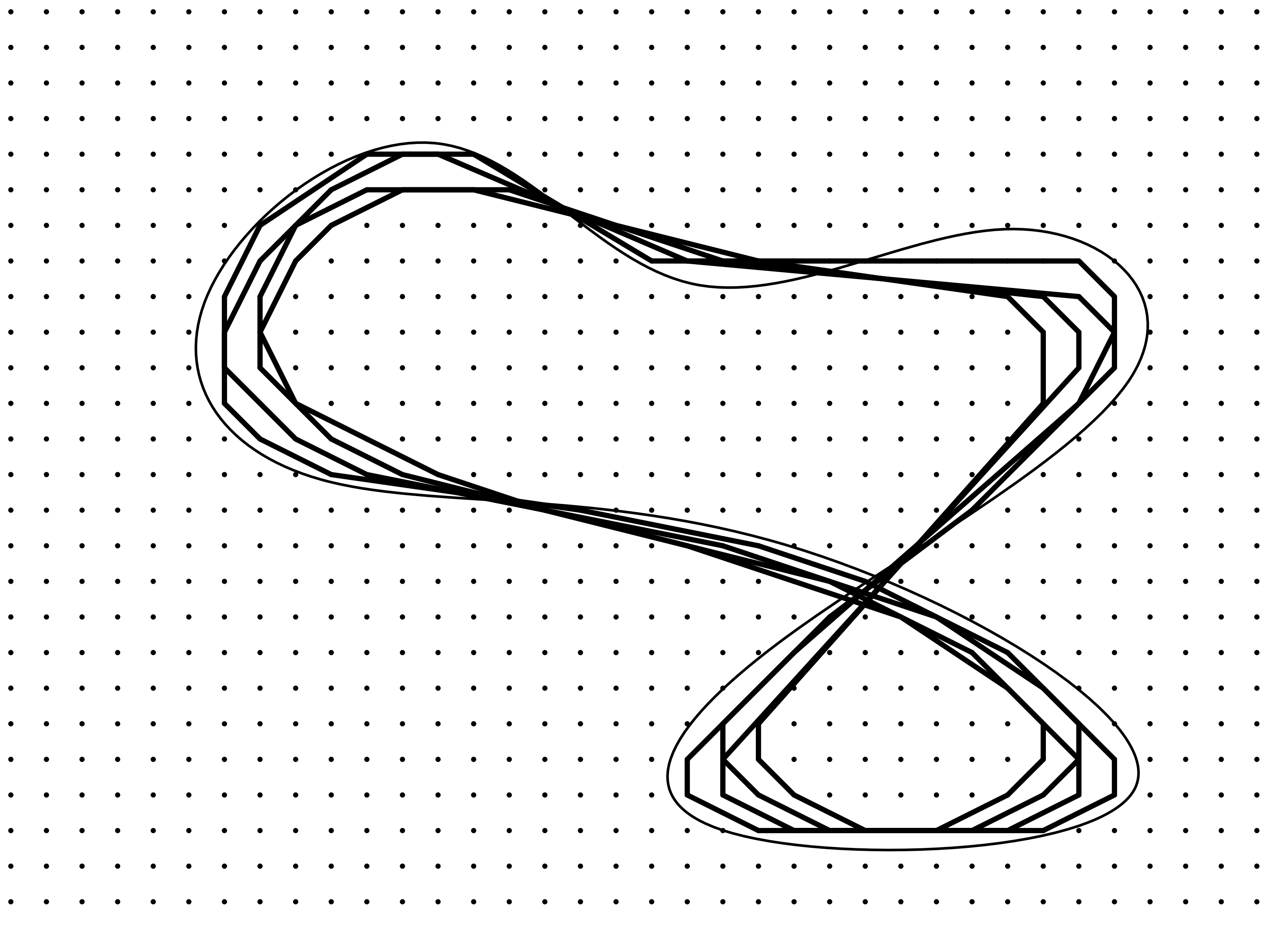
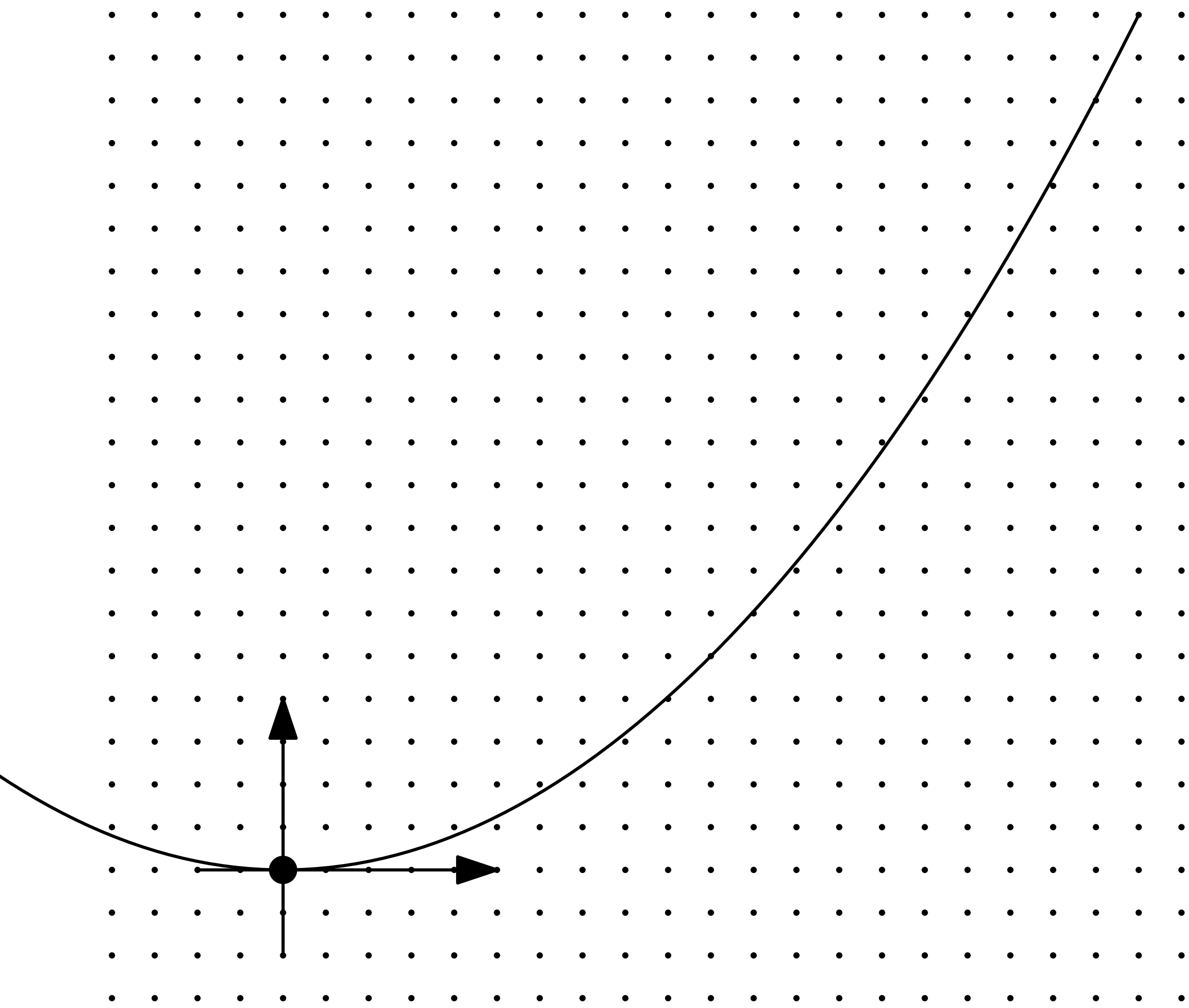
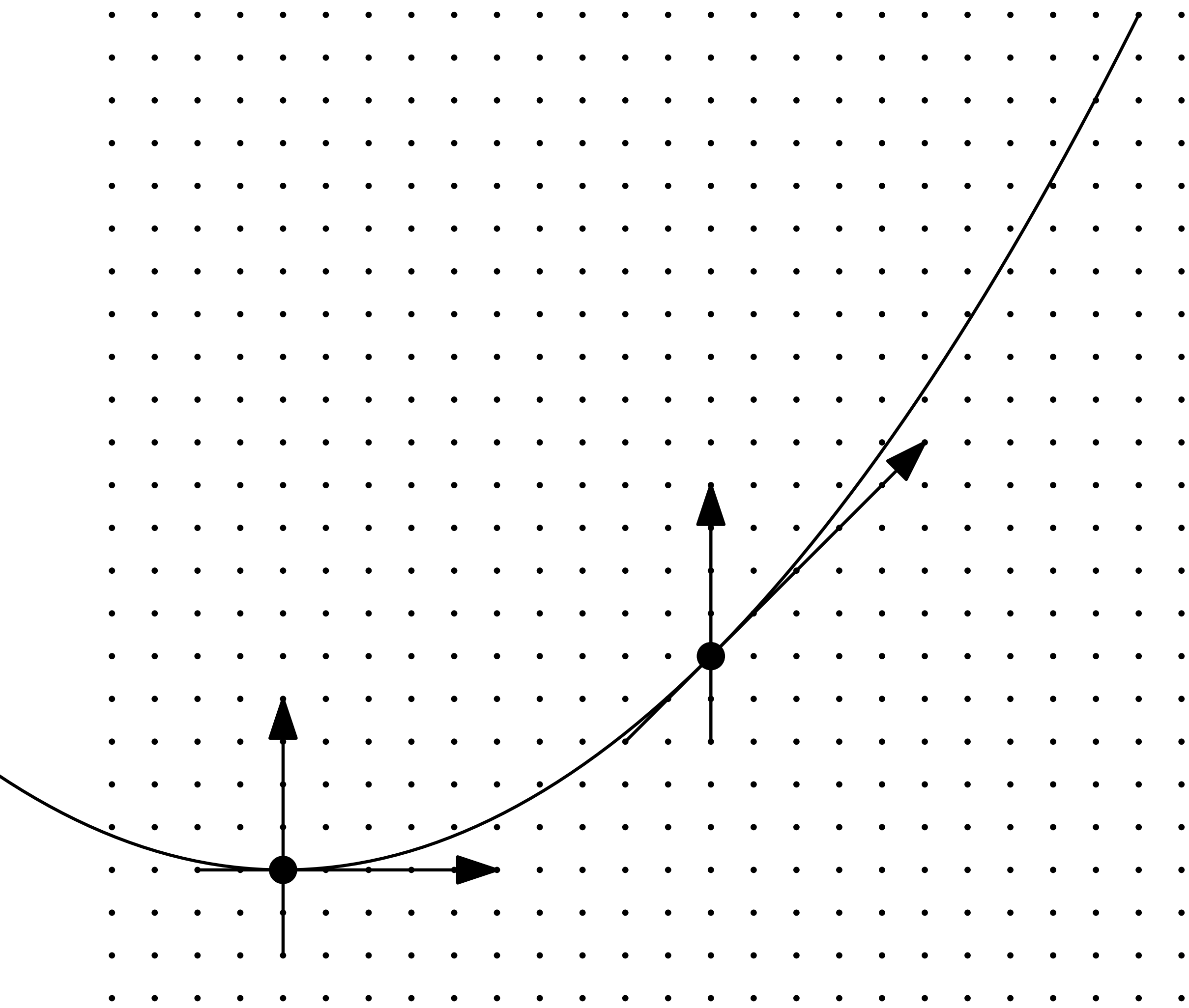


The parabola!



$$y = \frac{1}{20}x^2$$

The parabola!



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affine lattice-preserving
shearing transformations

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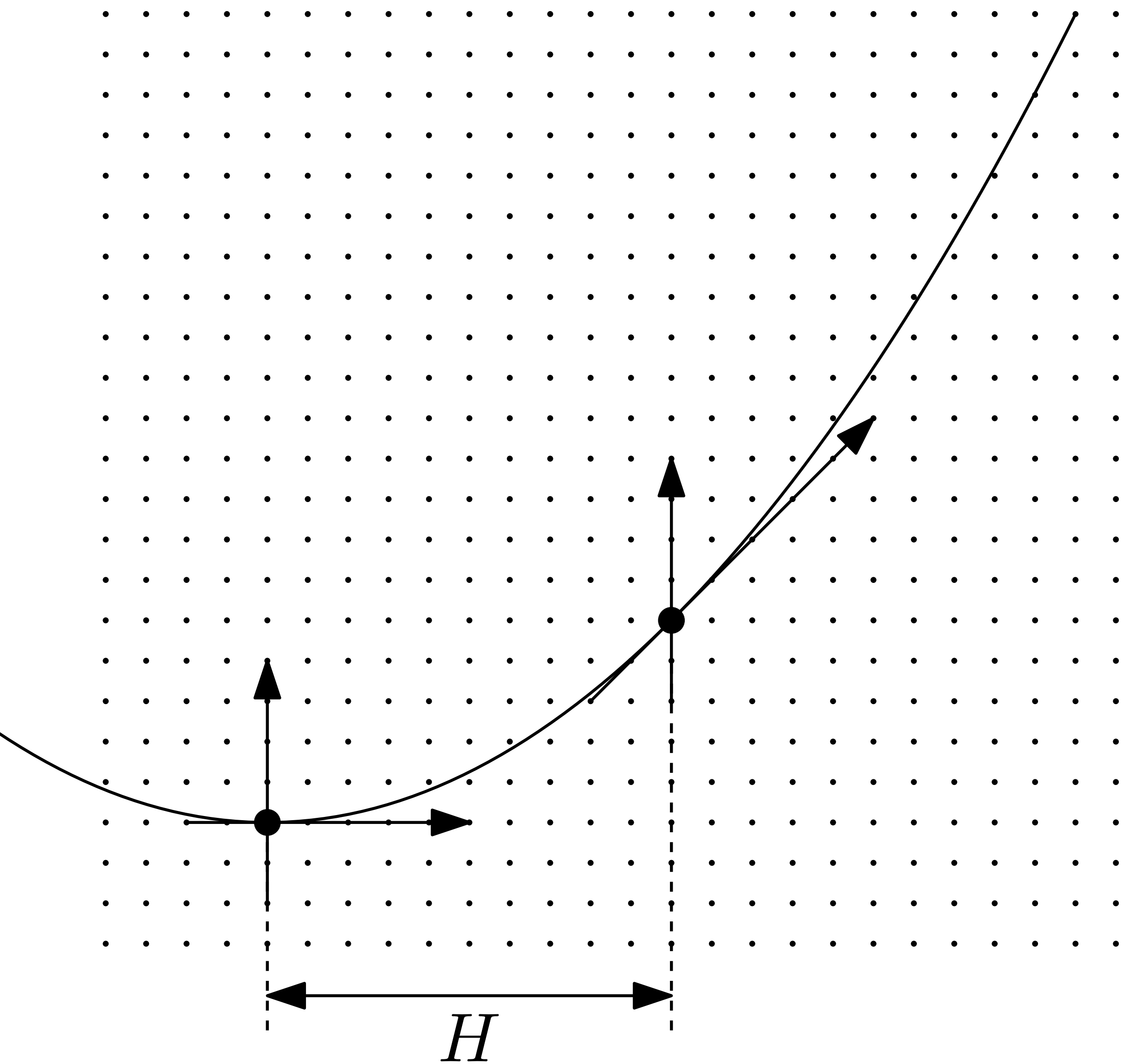


$y = \frac{1}{20}x^2$

affine lattice-preserving shearing transformations

$y = \frac{a_N}{a_D}x^2 + \frac{b_N}{b_D}x + c$

Lemma:
Horizontal period $H = \mathrm{lcm}(a_D, b_D)$ or $H = \mathrm{lcm}(a_D, b_D)/2$

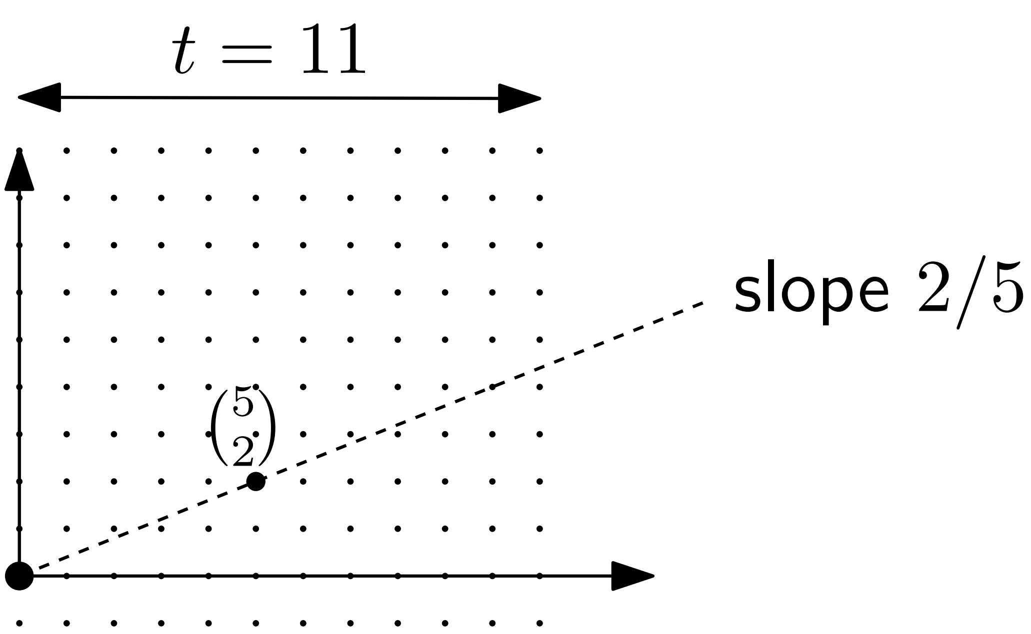
“The grid parabola”

- fixed integer parameter $t \geq 1$
- take all slopes a/b with $0 < b \leq t$
- for each slope a/b, take the longest integer vector

$$\binom{x}{y} = f\binom{b}{a} \quad (f \in \mathbb{Z})$$

with $0 < x \leq t$

Example



"The grid parabola"

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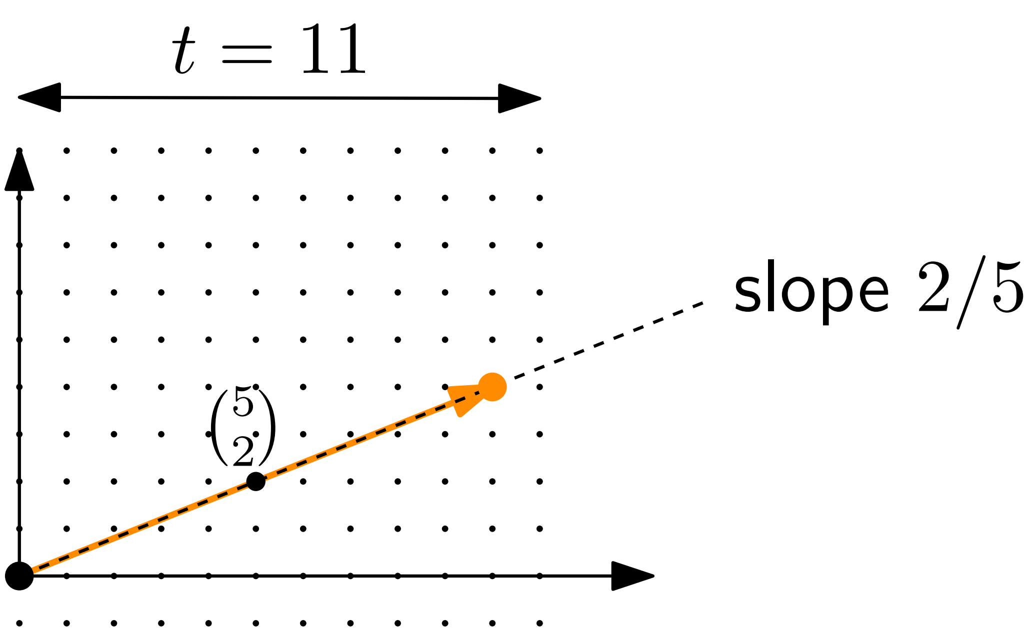

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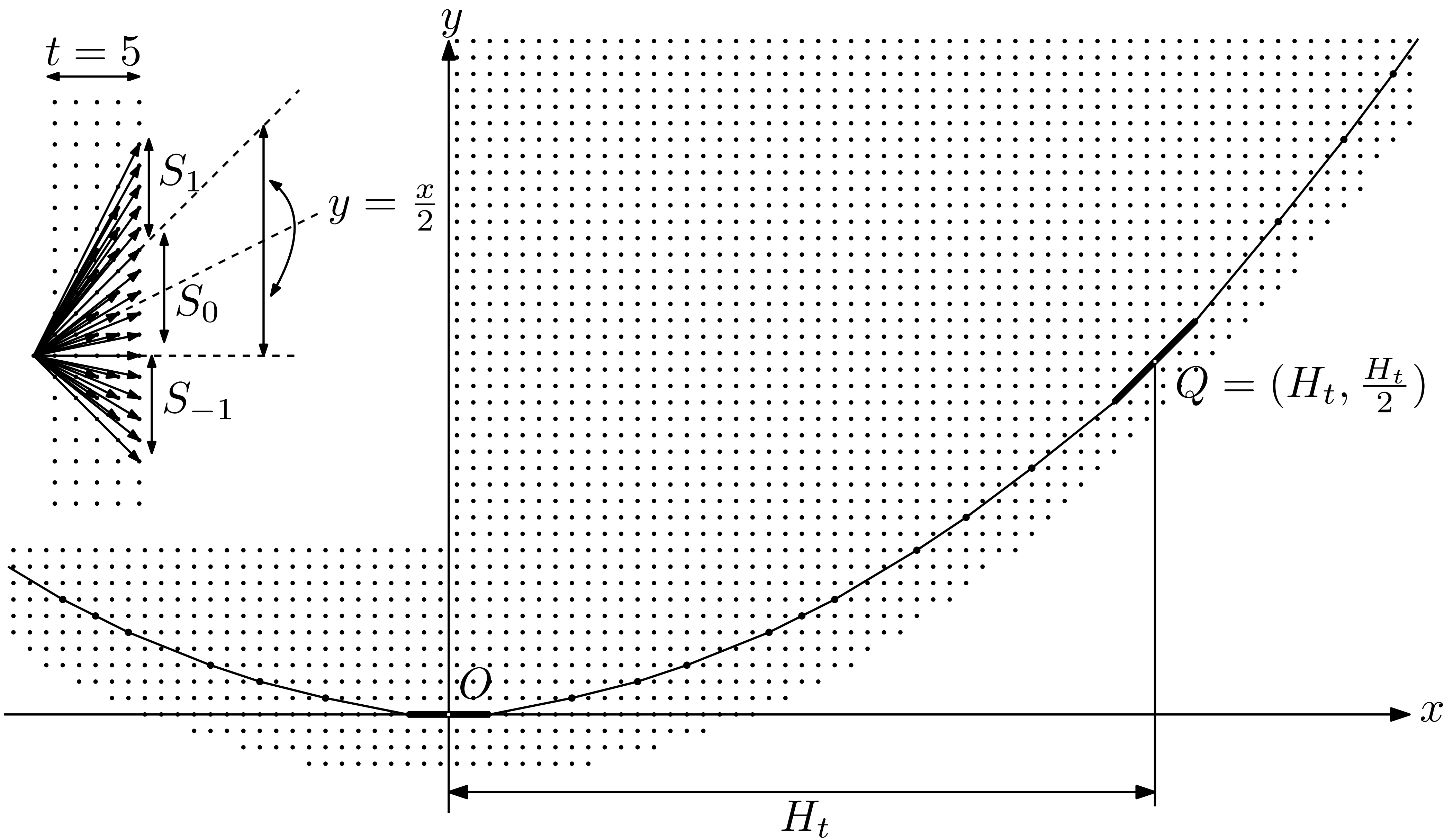

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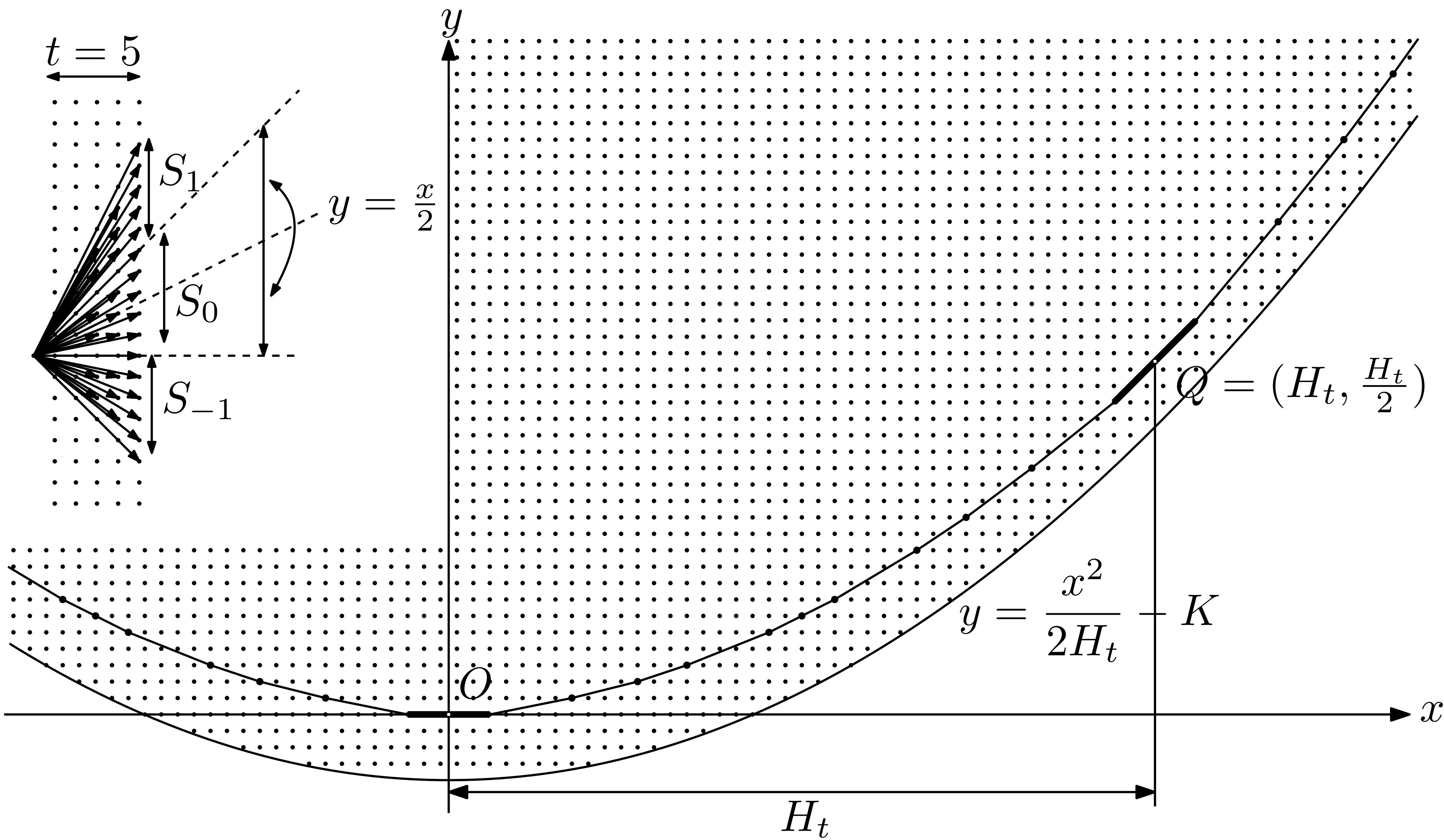

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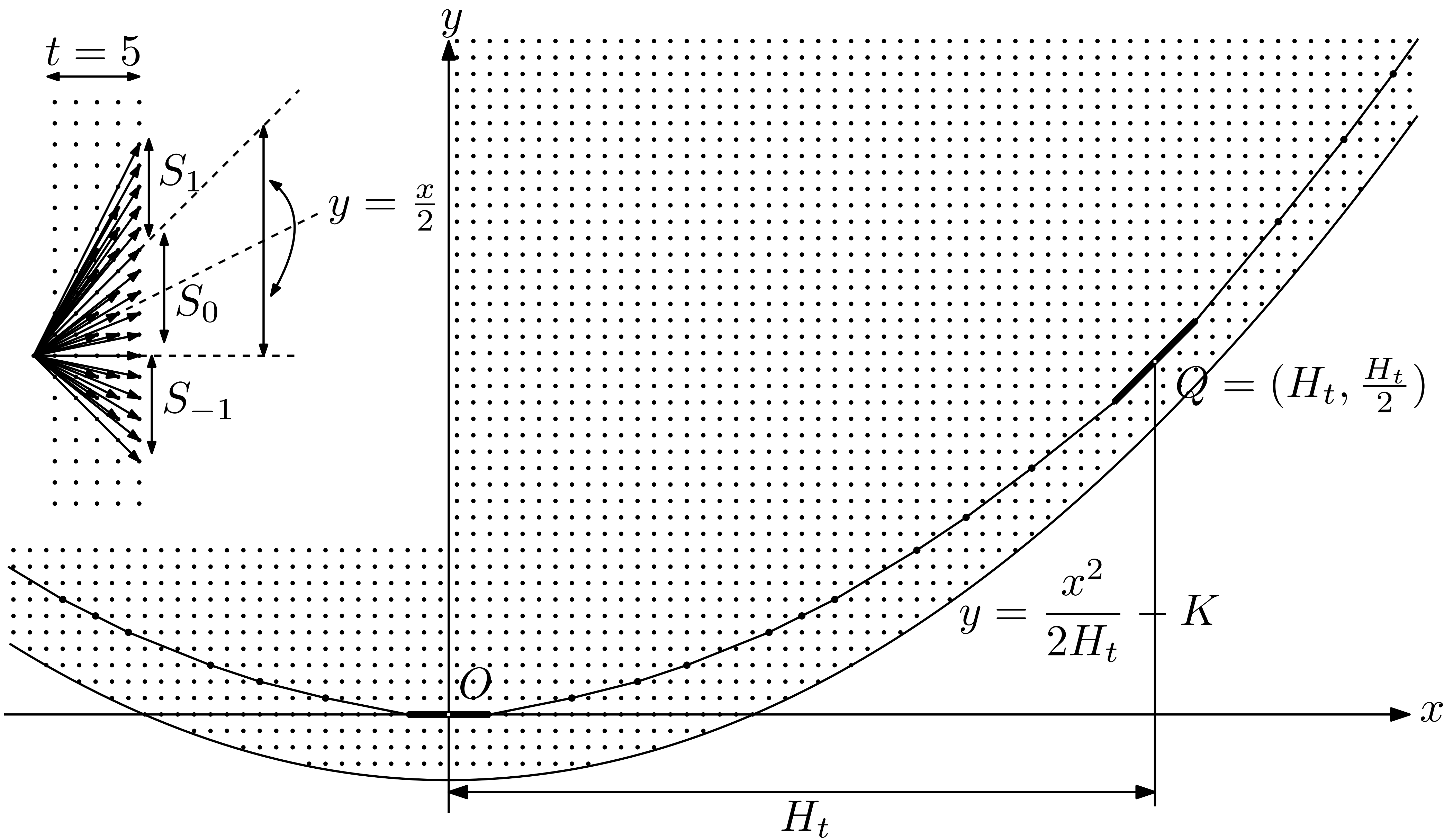
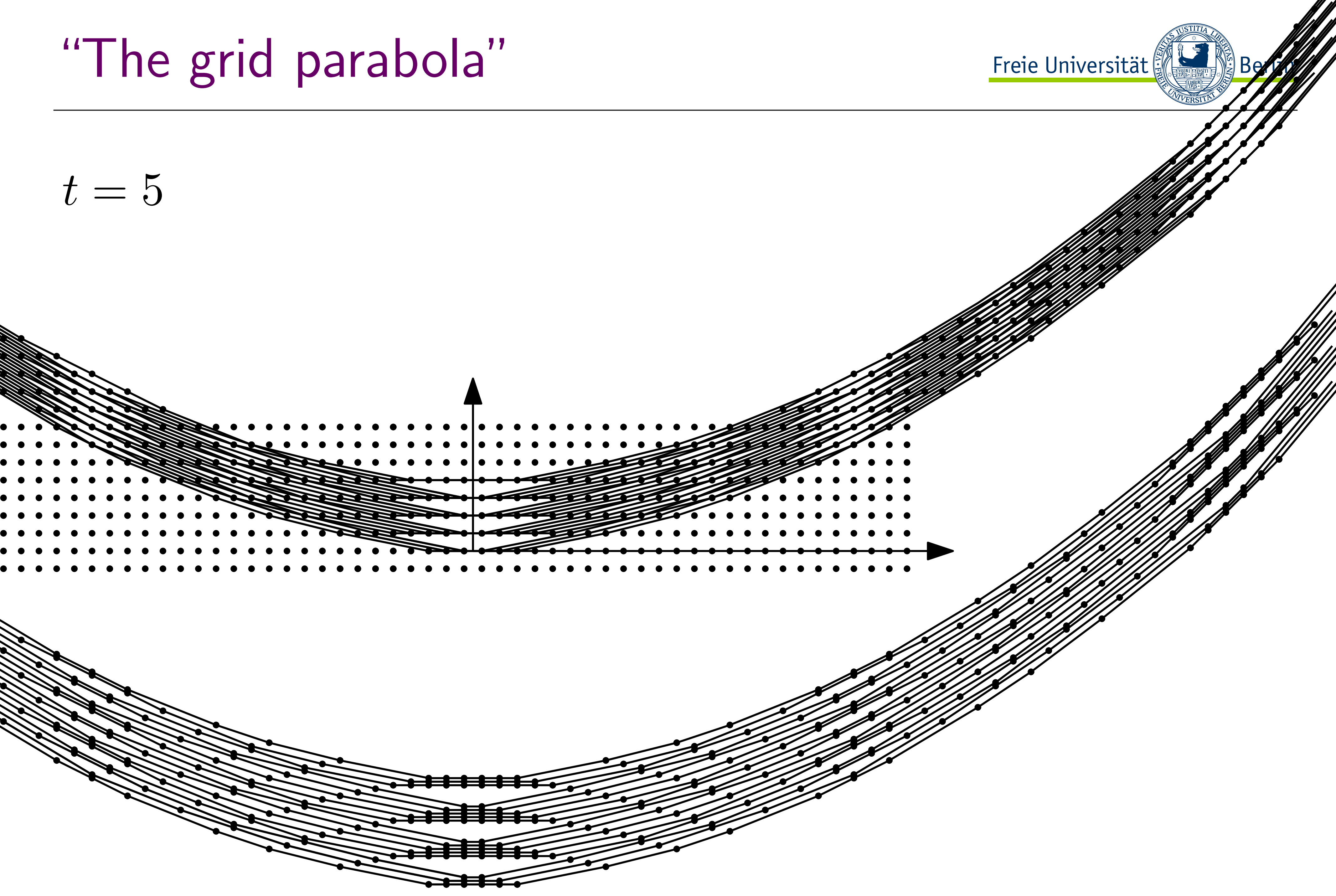


$$H_1, H_2, H_3, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$$

“The grid parabola”

$t = 5$



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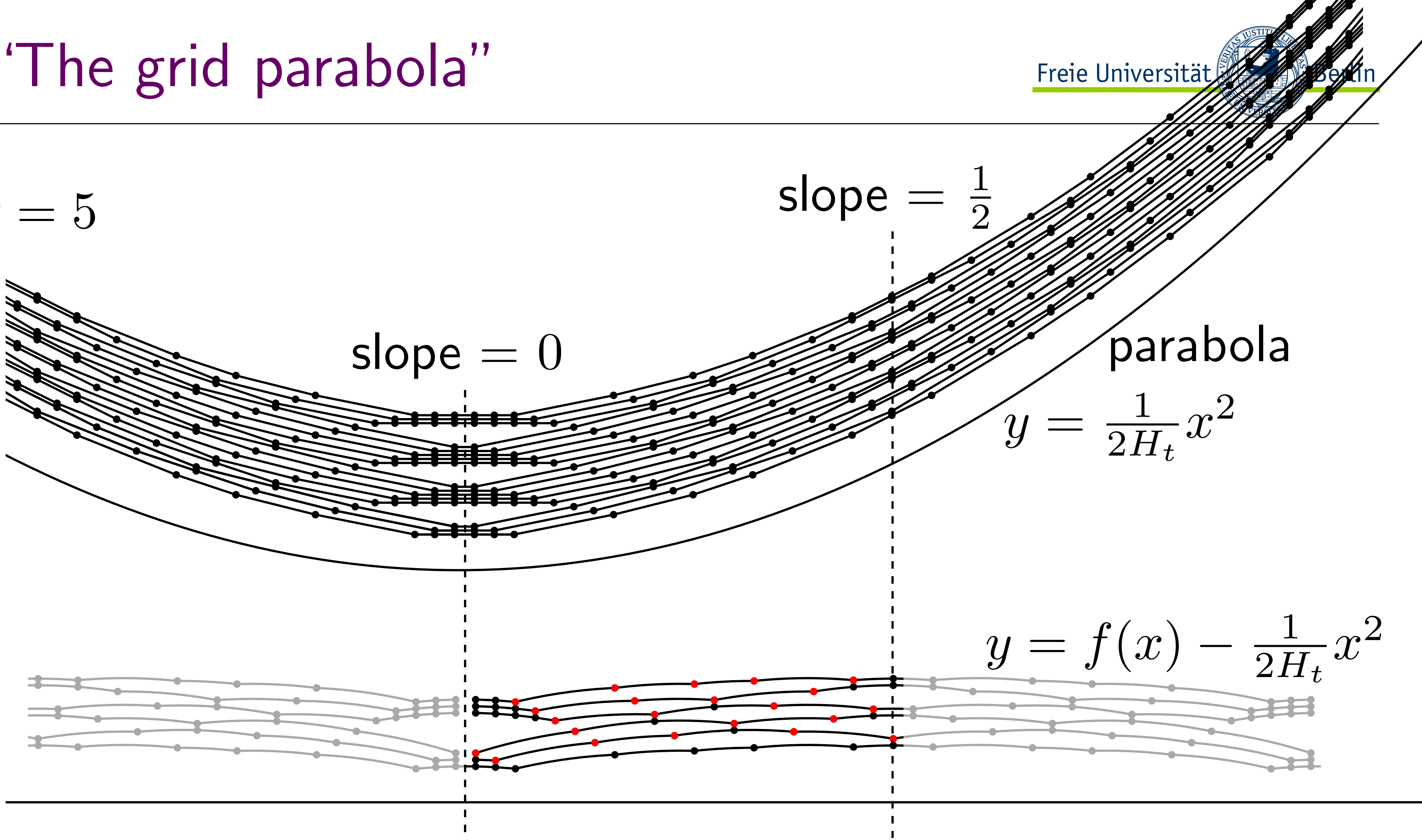

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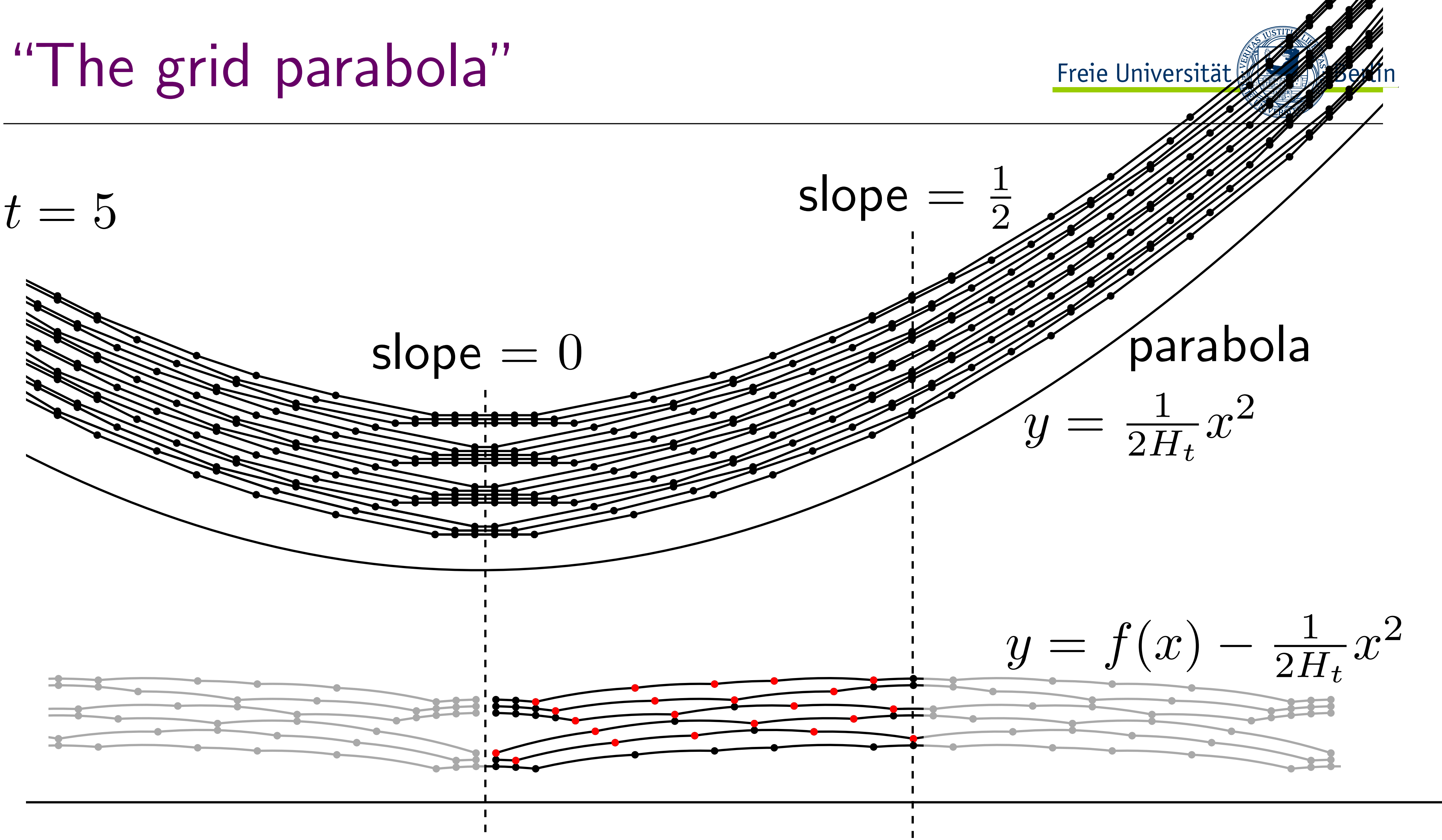
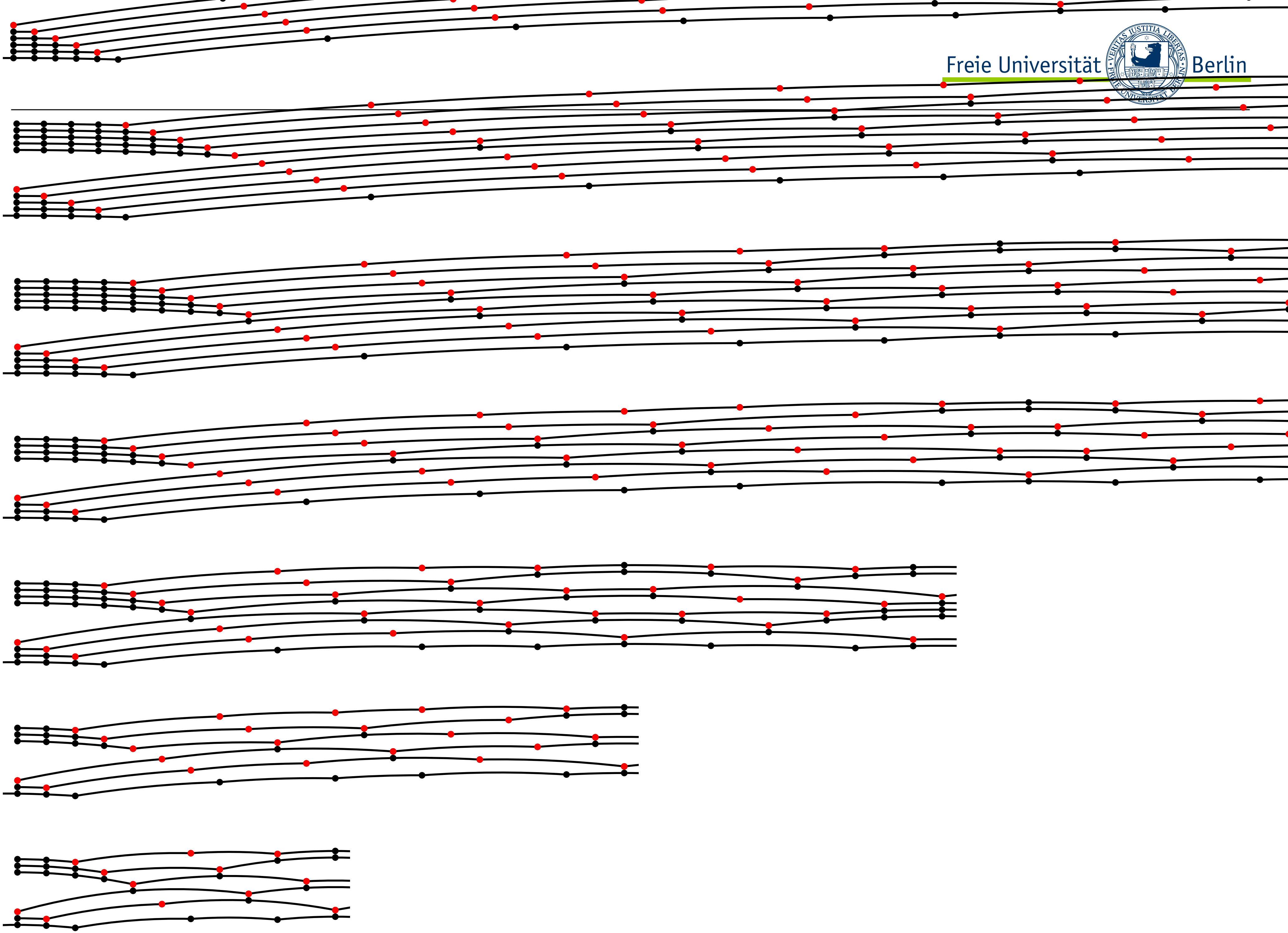
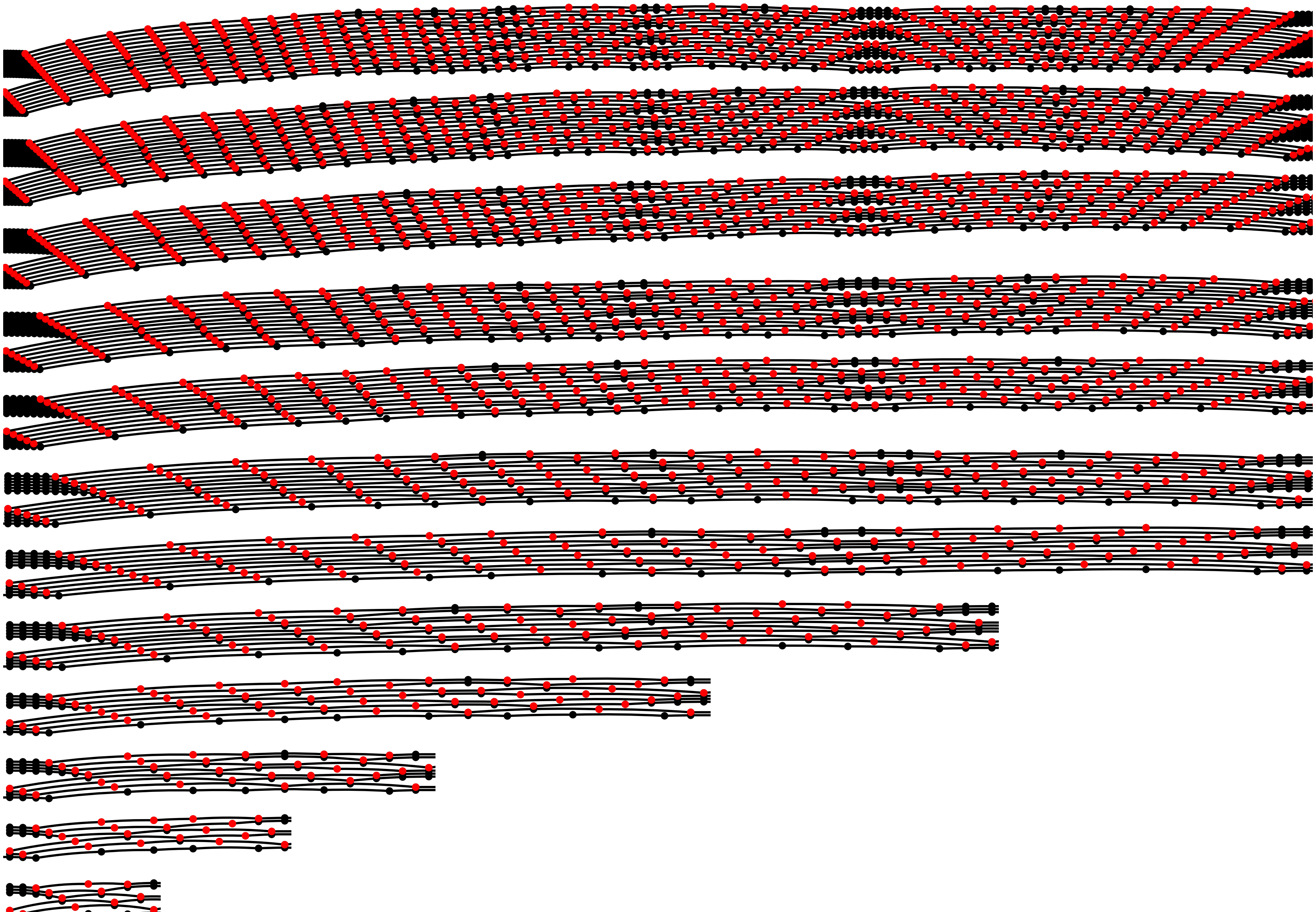


Conjecture: The polygon repeats after t steps, one level higher. (After $t+1$ steps if t is even.)



$t = 4, 5, 6, \ldots$



Asymptotic period

$$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$$

[OEIS A174405]

$$H_t := \sum_{\substack{0<y\le x\le t \\ \gcd(x,y)=1}} \left\lfloor \frac{t}{x} \right\rfloor x = \sum_{1\le i\le t} \sum_{d|i} d\varphi(d)$$

$$H_t = \frac{2\zeta(3)}{\pi^2} t^3 + O(t^2 \log t)$$

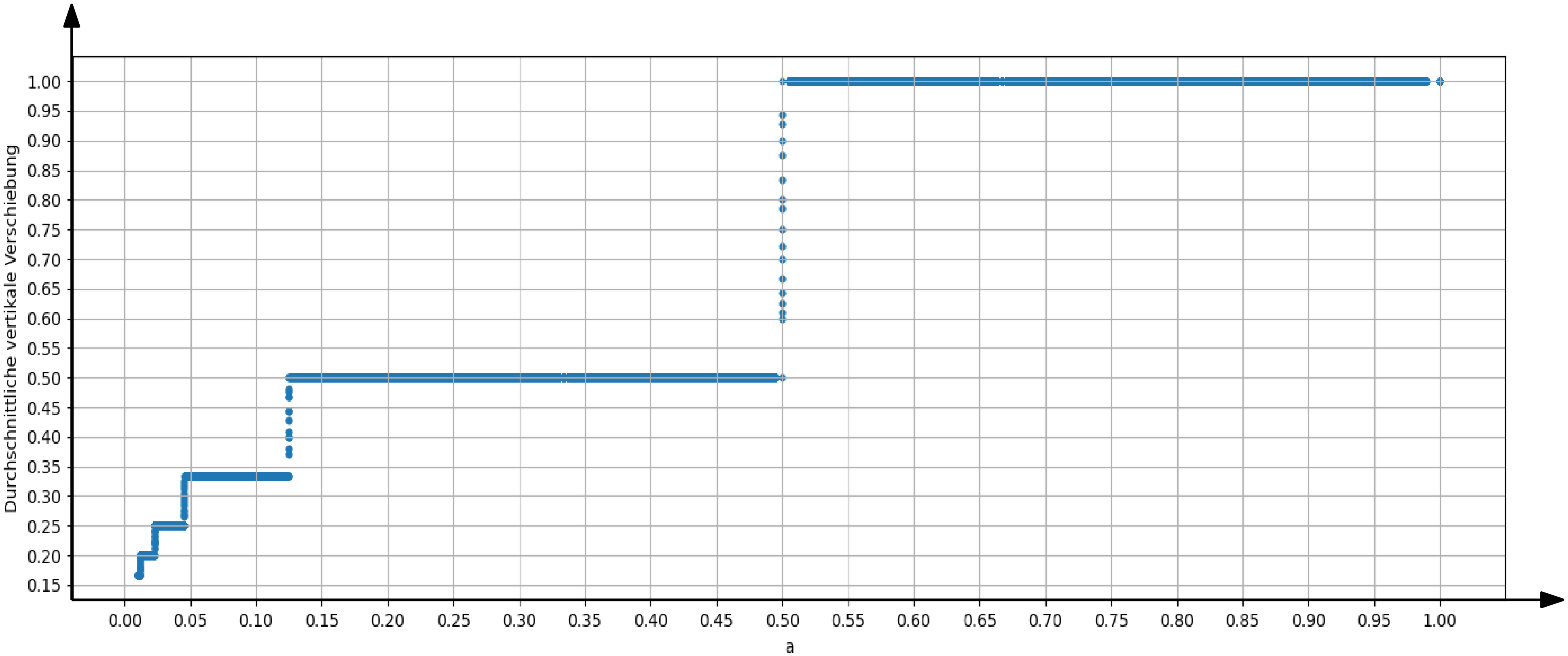
with $\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.2020569$

[Sándor and Kramer 1999]

Time period for various parabolas

$$y = ax^2 + bx$$

speed depending on a (various values of b)



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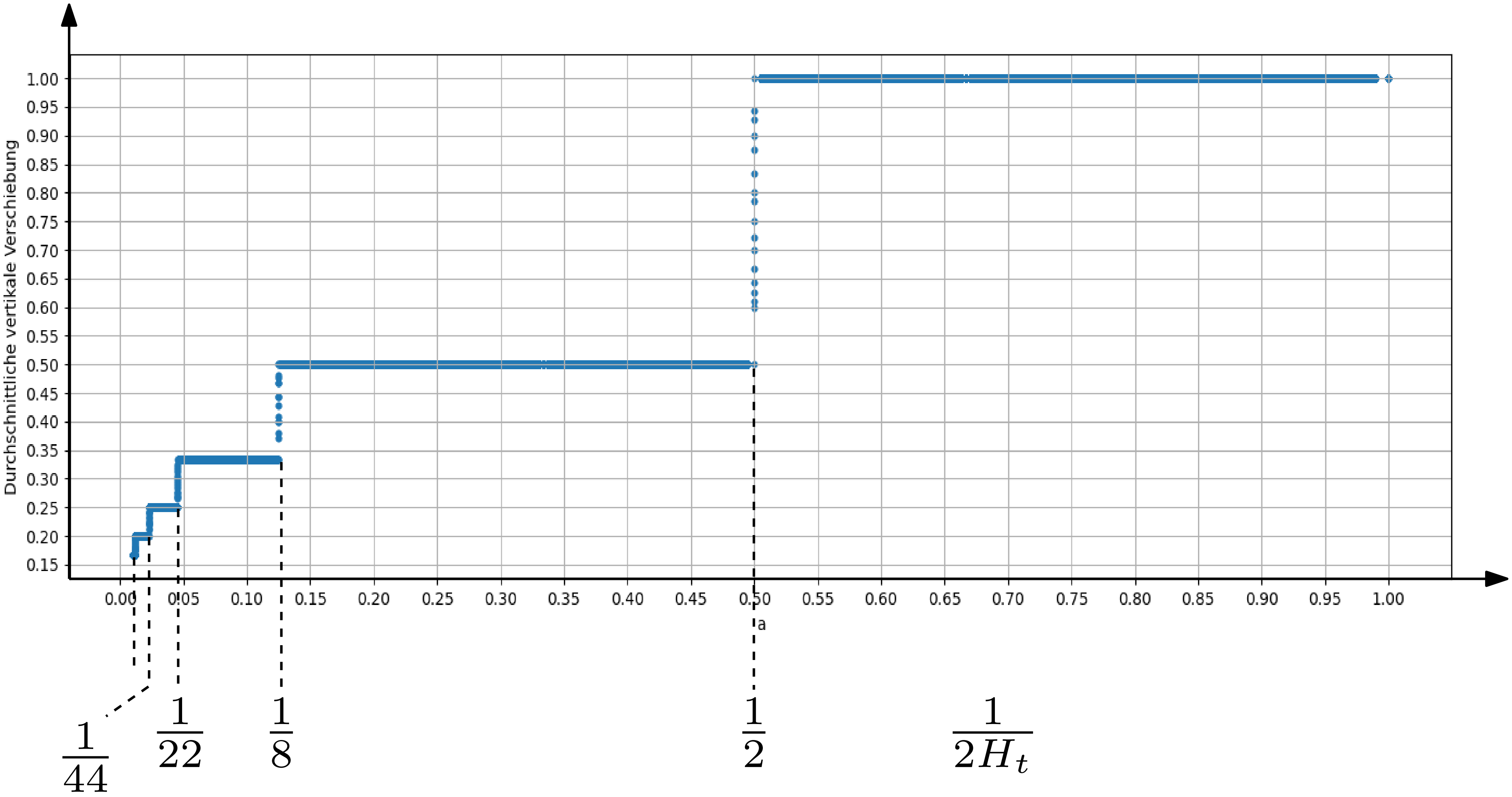

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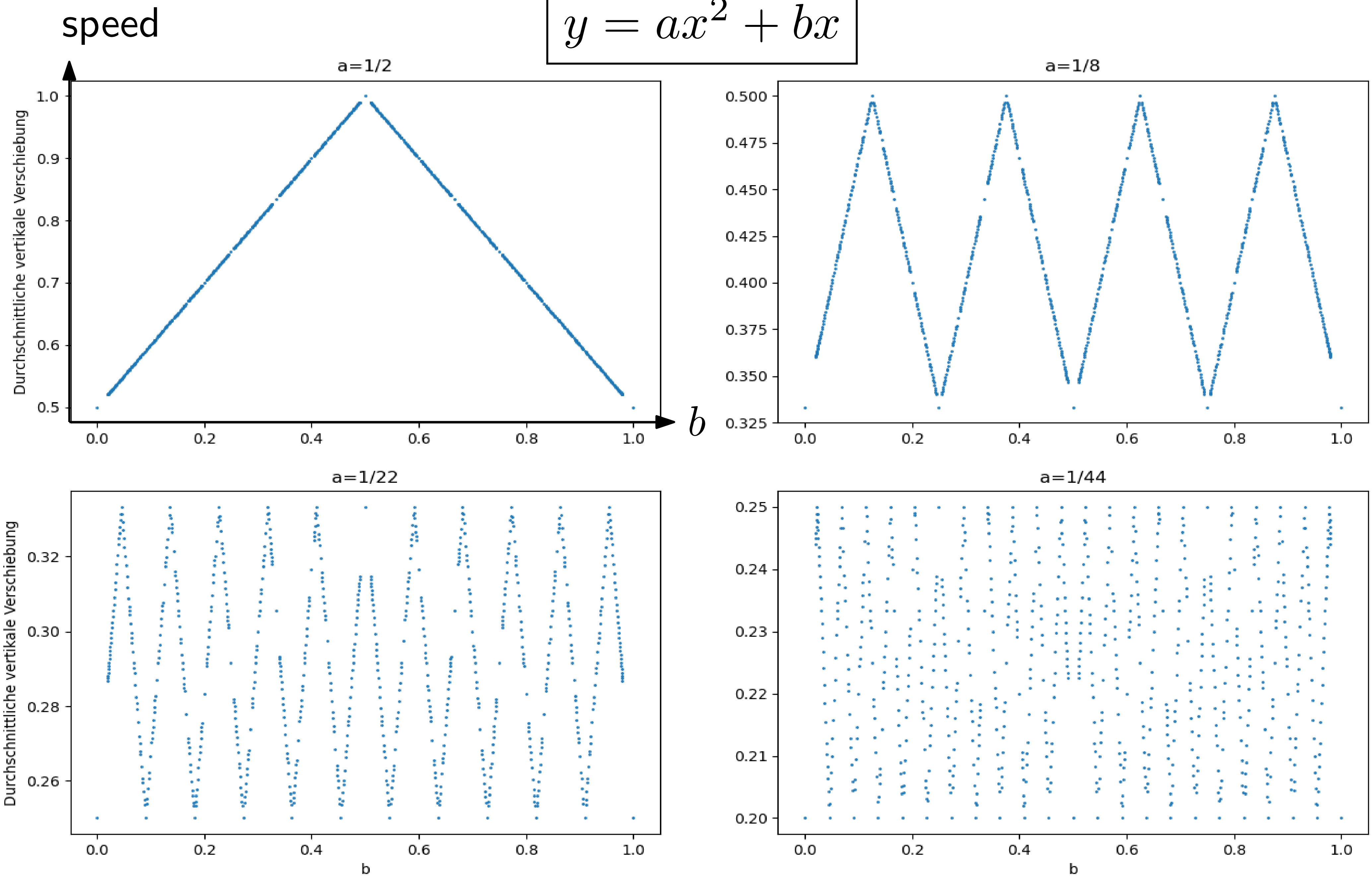

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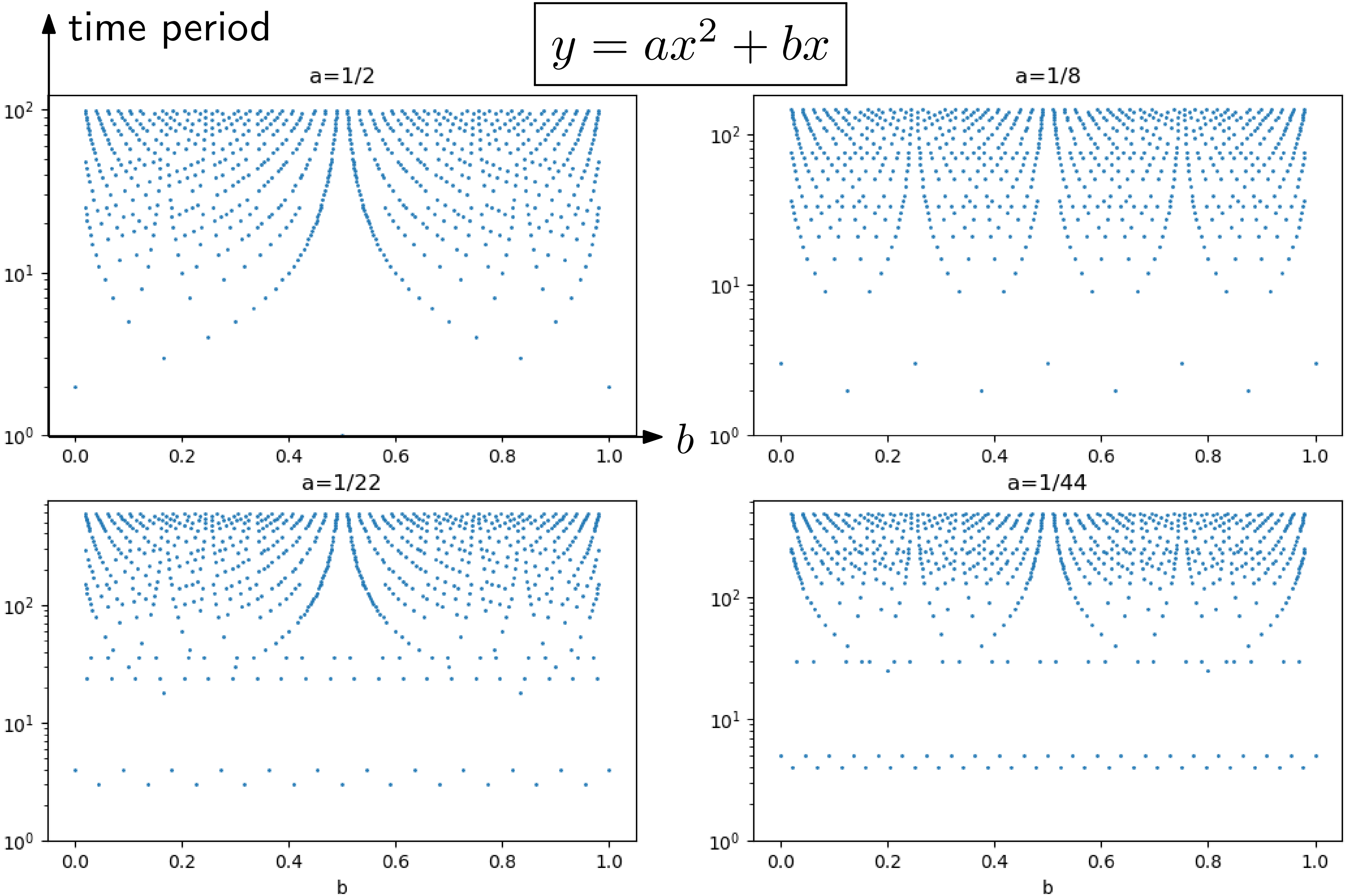

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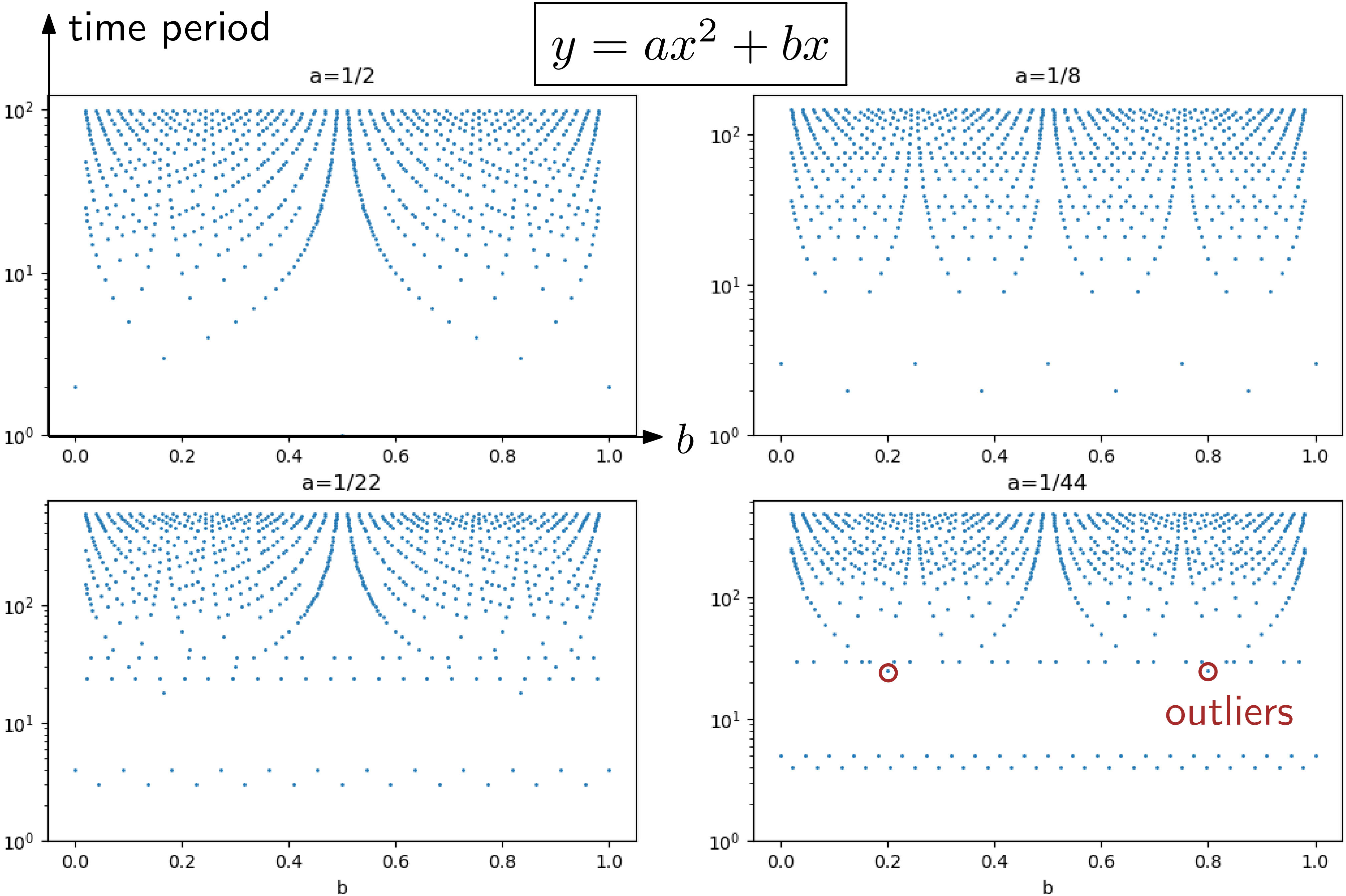
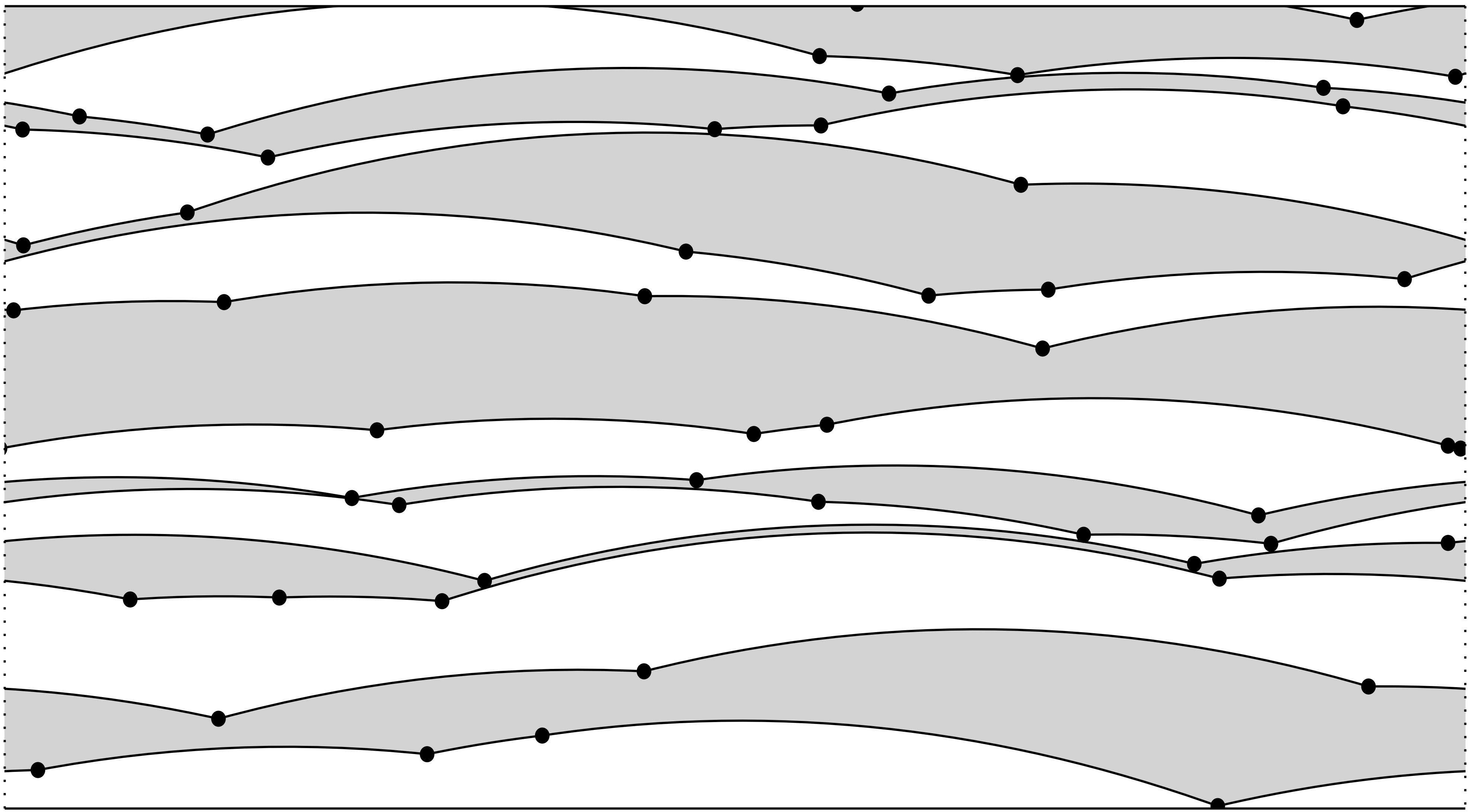


Random-set peeling

Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020



semiconvex peeling, on a cylinder