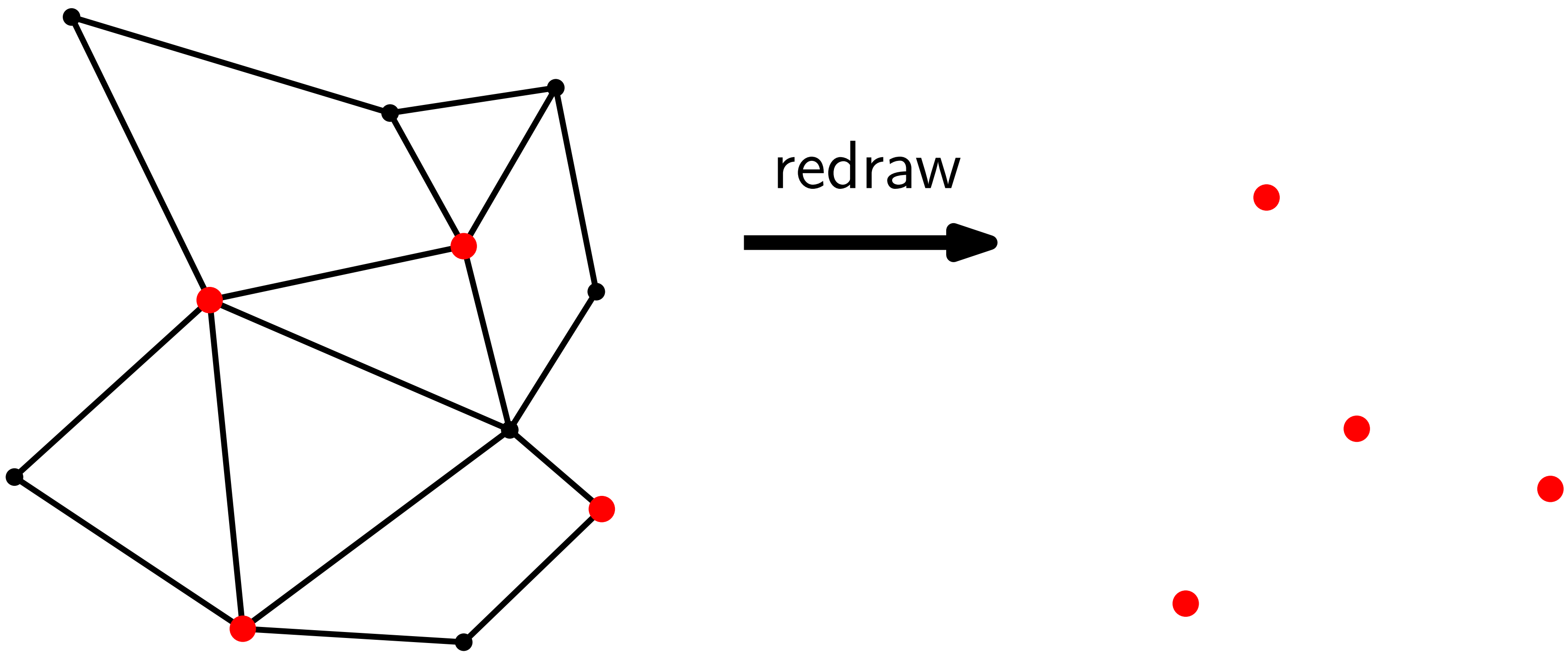
Every Collinear Vertex Set in a Planar Graph is Free

Vida Dujmović, Fabrizio Frati, Daniel Gonçalves, Pat Morin, Günter Rote



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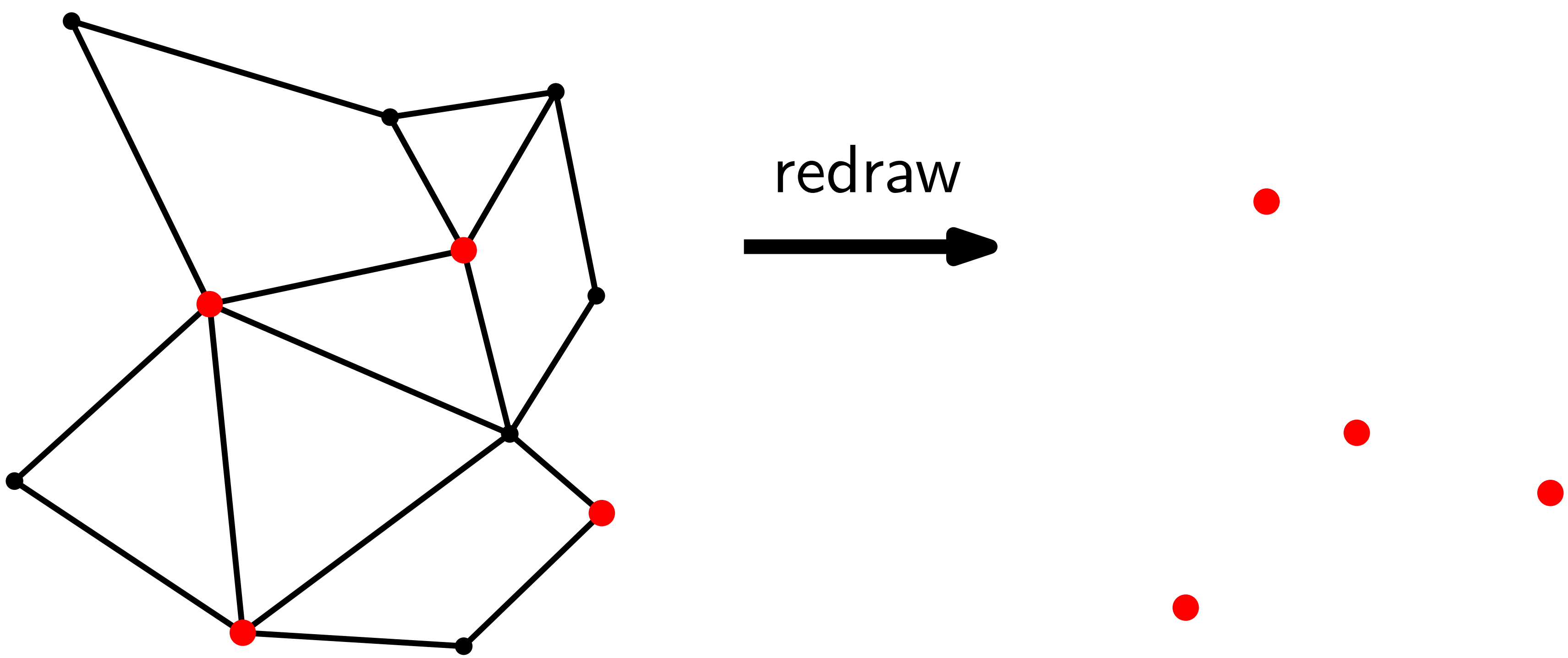

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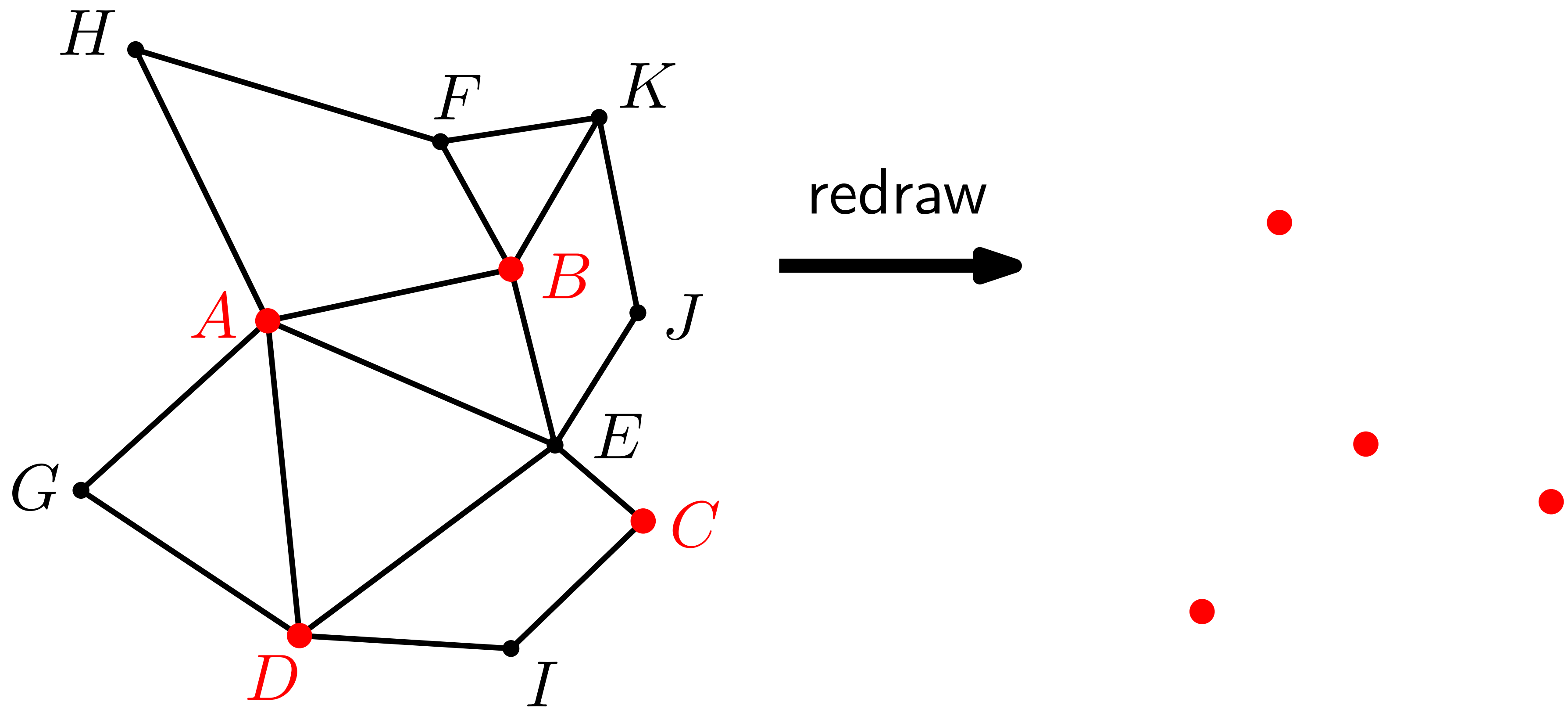

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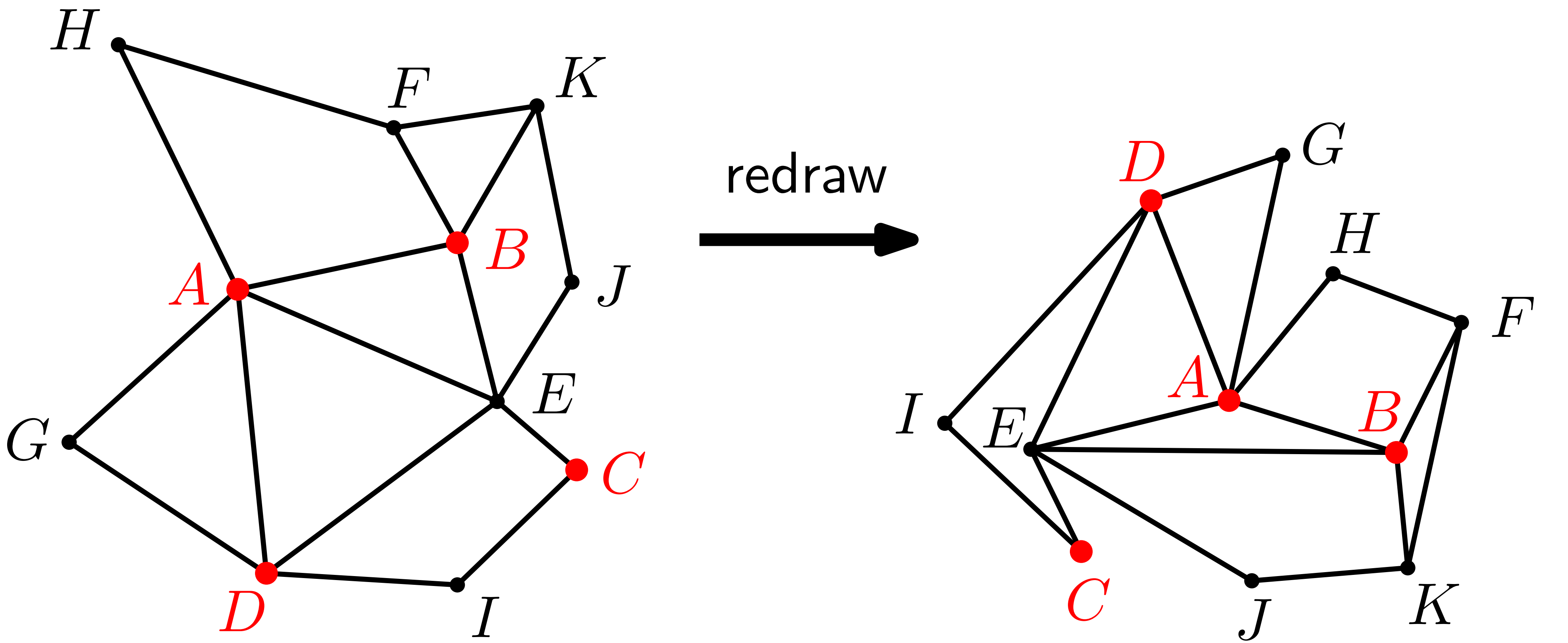

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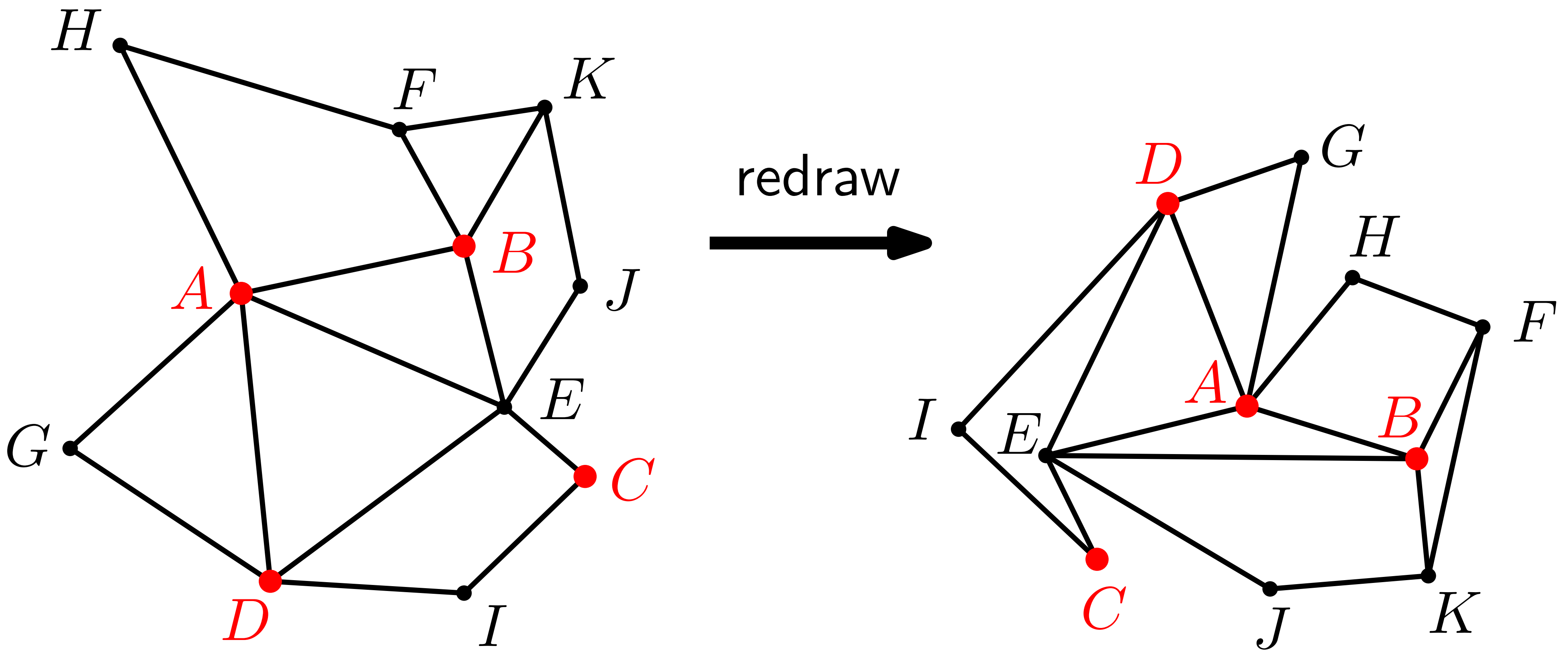

$\{A, B, C, D\}$ is *free* if it can be redrawn on *every* 4-point set.

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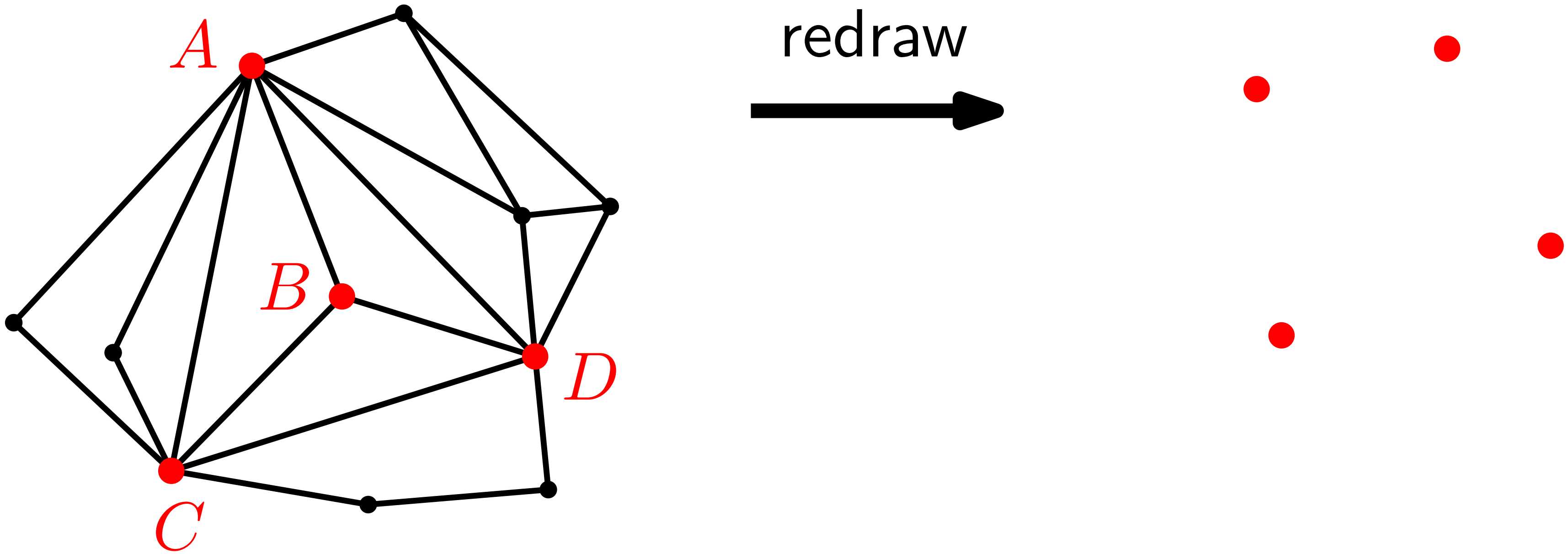
NOT a free set



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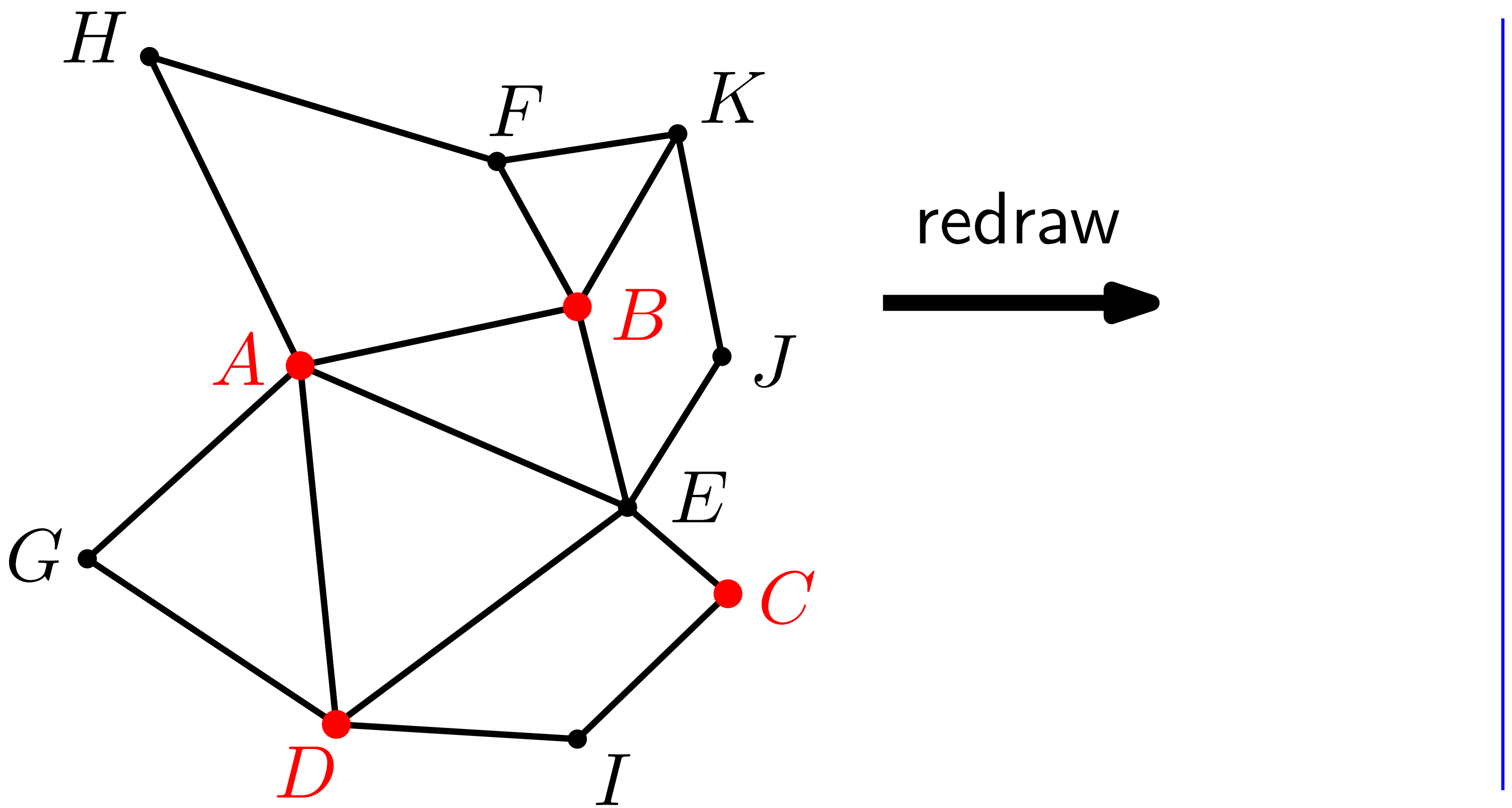

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$\{A, B, C, D\}$ is *collinear* if it can be redrawn on a line.

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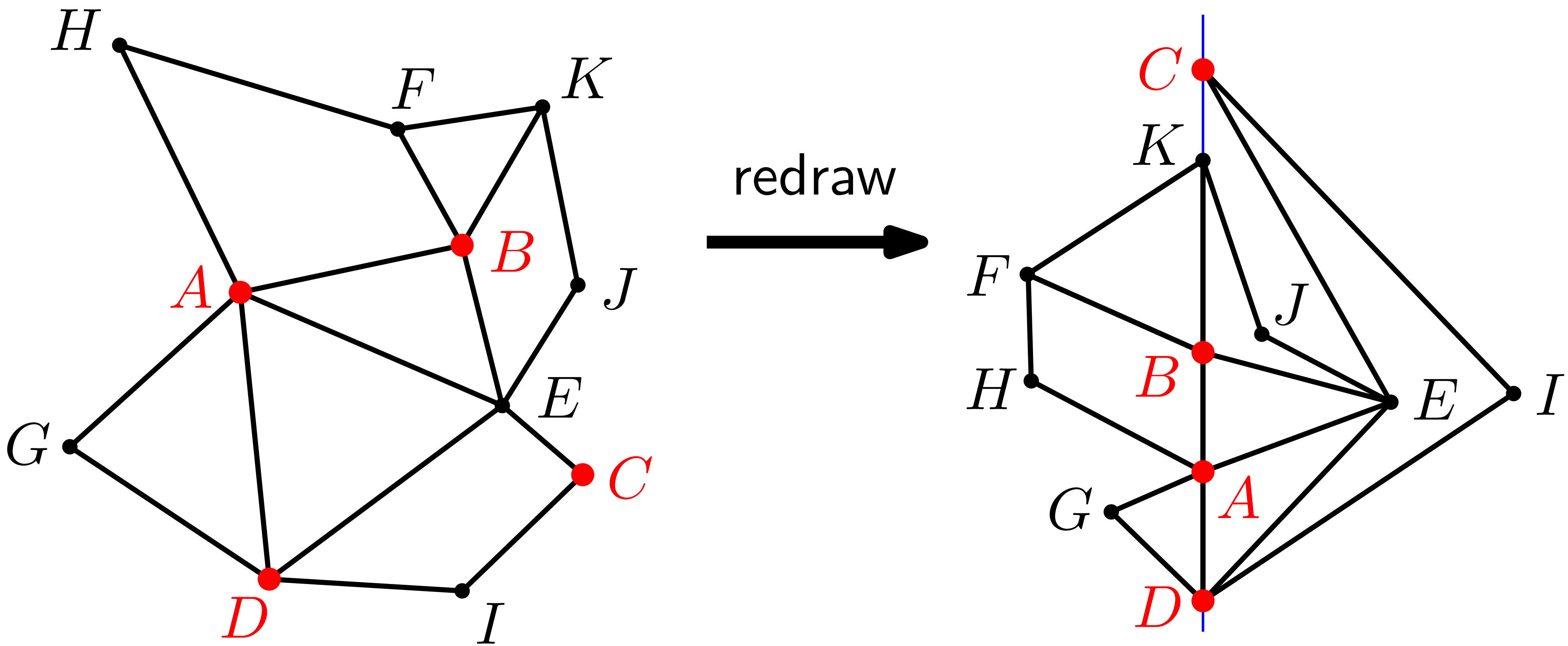

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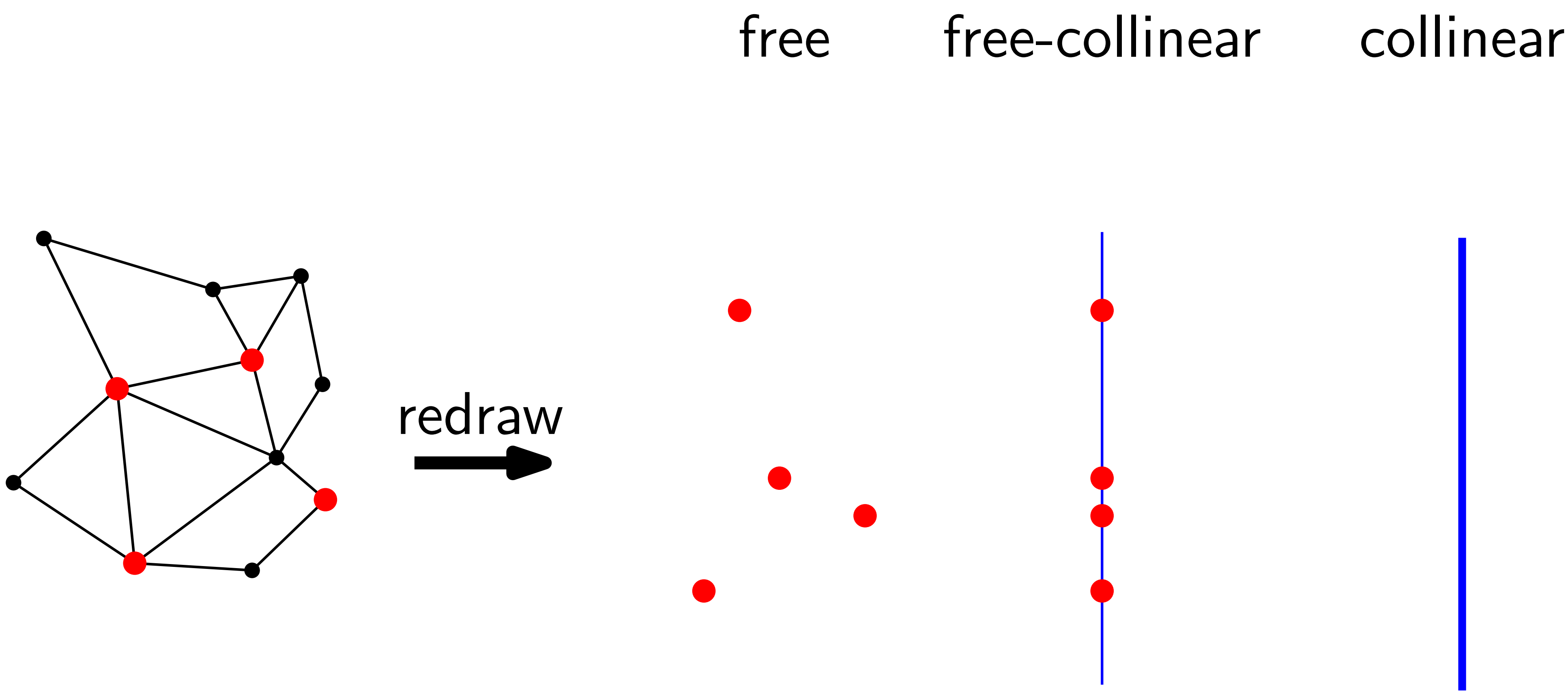
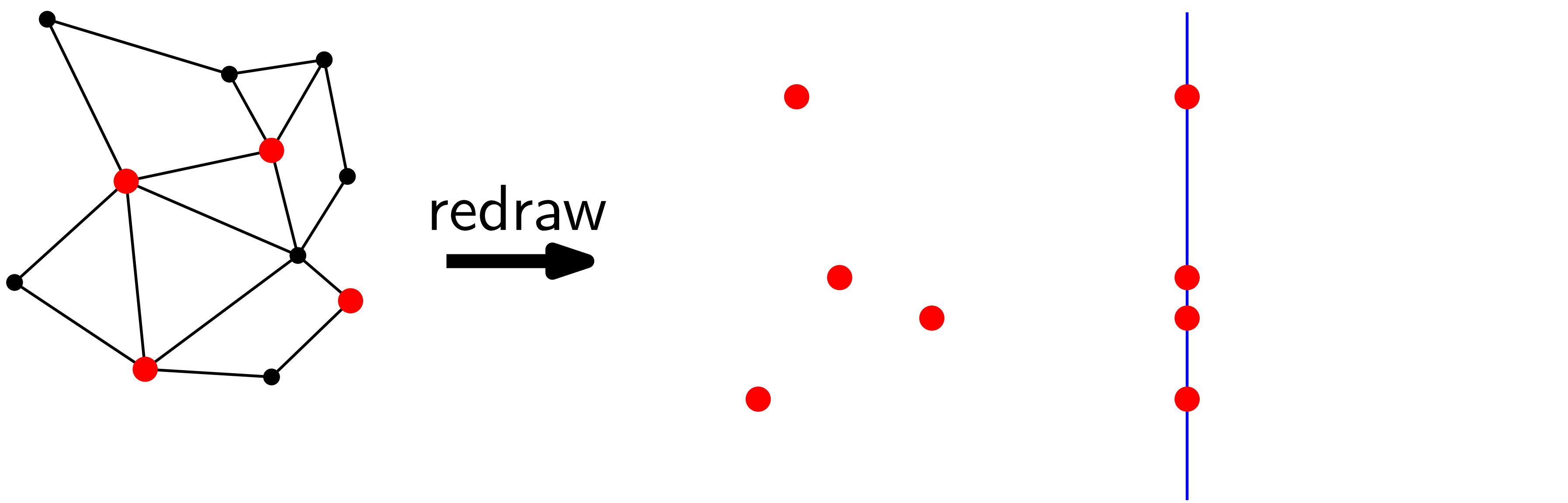
Free, Free-Collinear, Collinear



Free, Free-Collinear, Collinear

(trivial) (trivial)

free $\Rightarrow$ free-collinear $\Rightarrow$ collinear



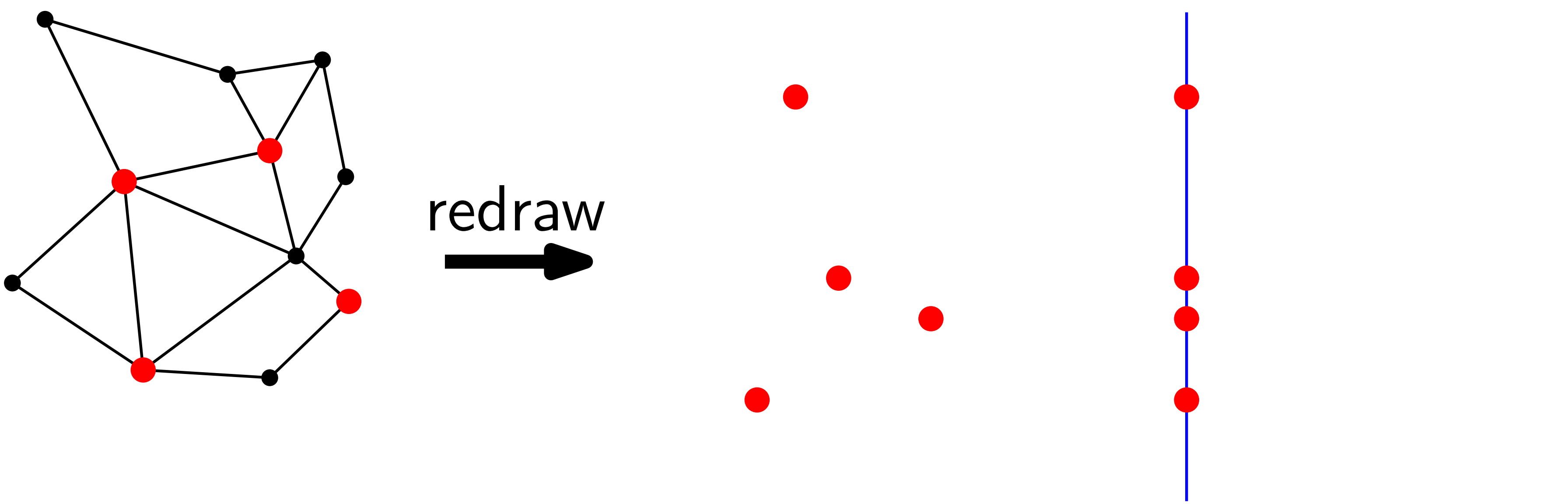
Free, Free-Collinear, Collinear

(trivial) (trivial)

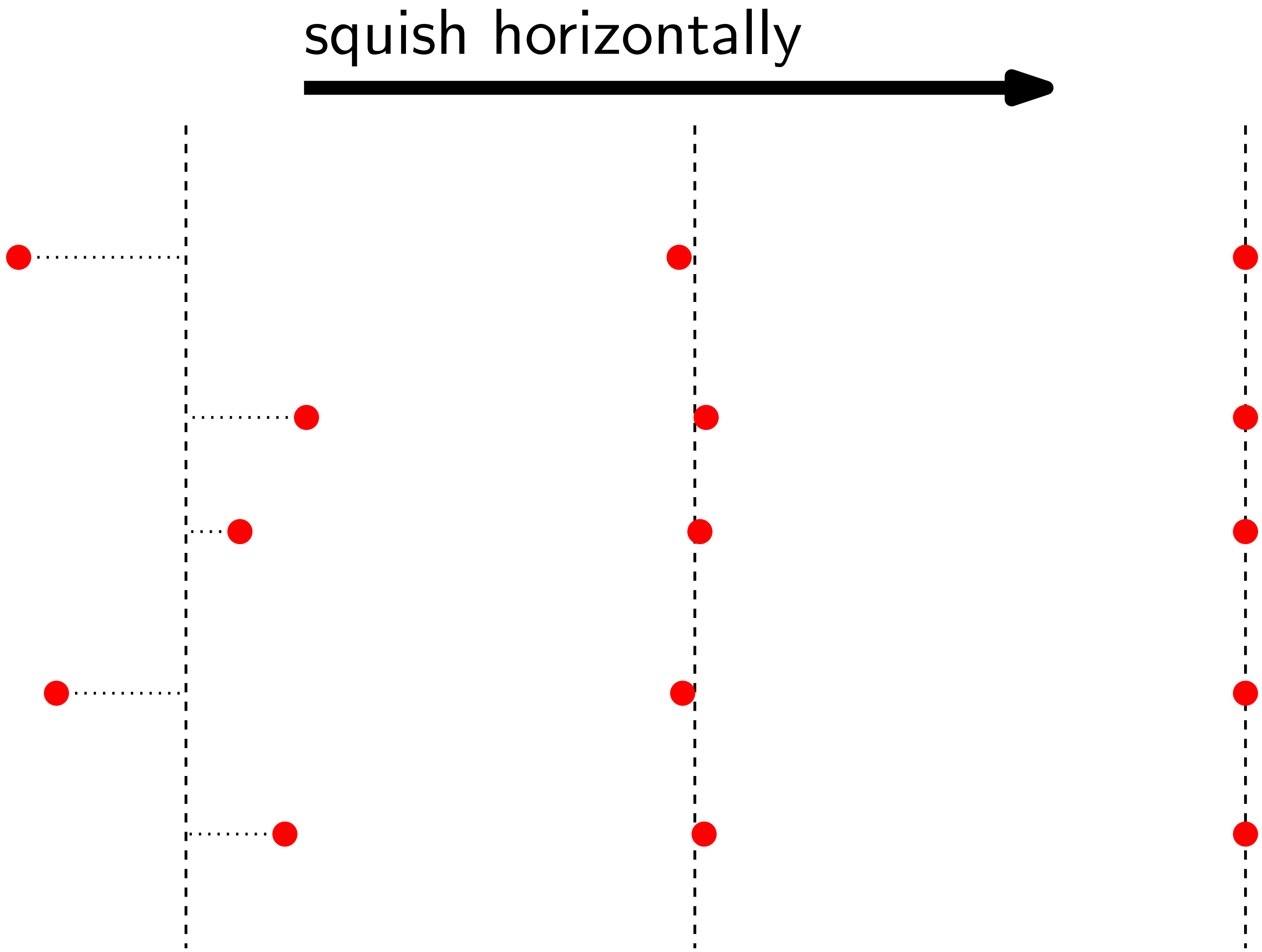
free $\Rightarrow$ free-collinear $\Rightarrow$ collinear

(easy) this talk

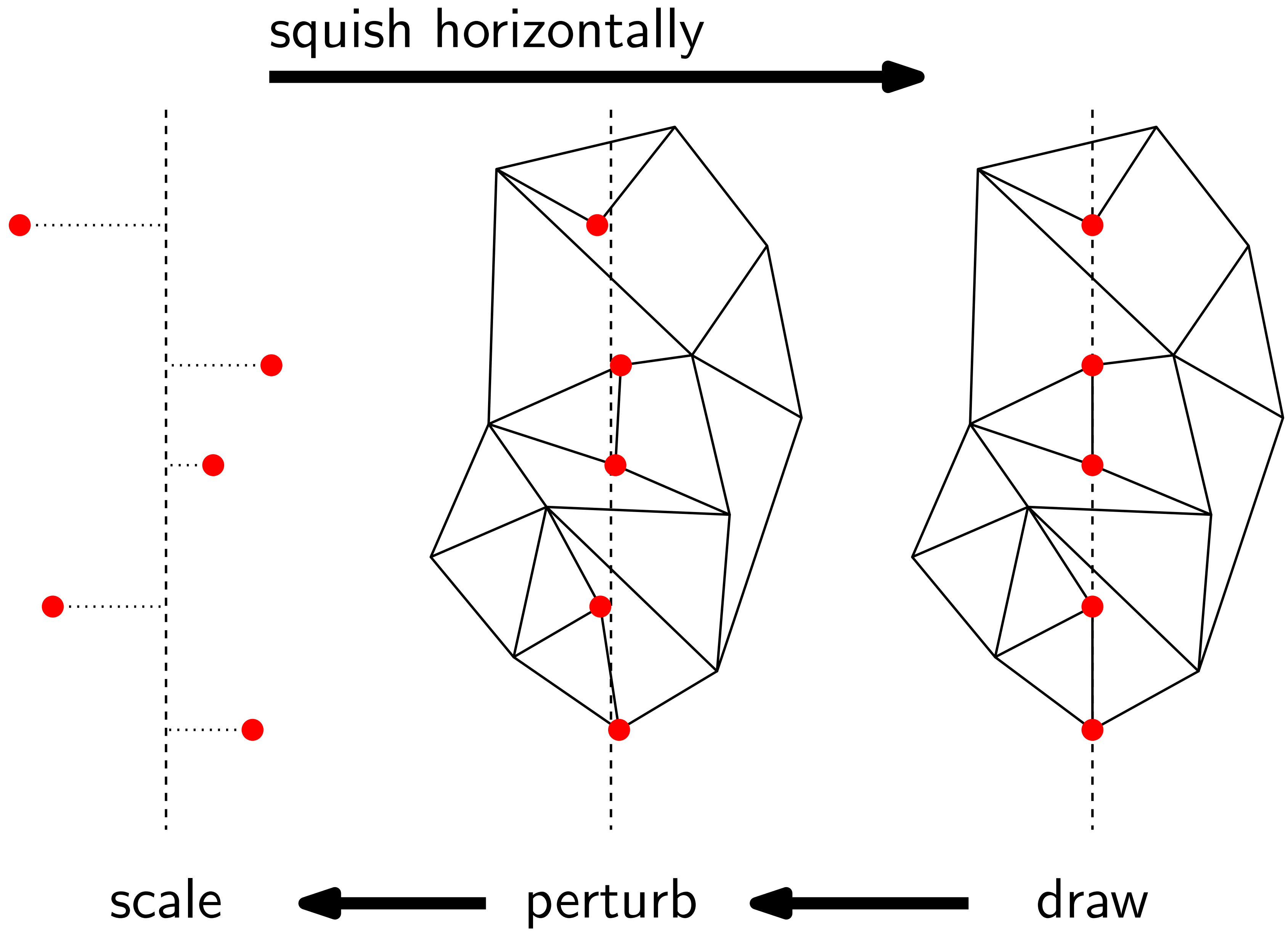
$\Leftarrow$ $\Leftarrow$



Free-Collinear $\Longrightarrow$ Free

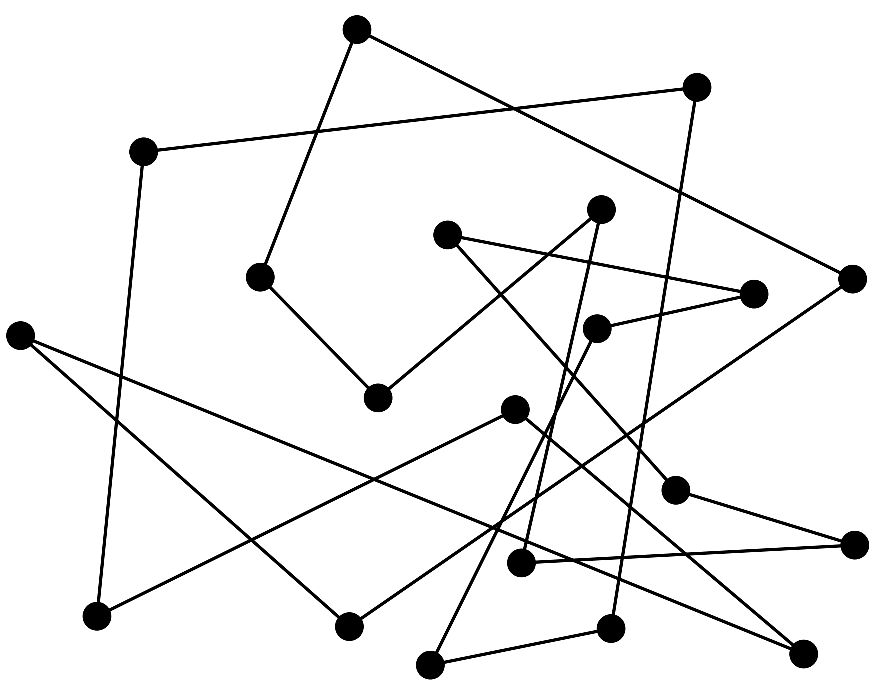


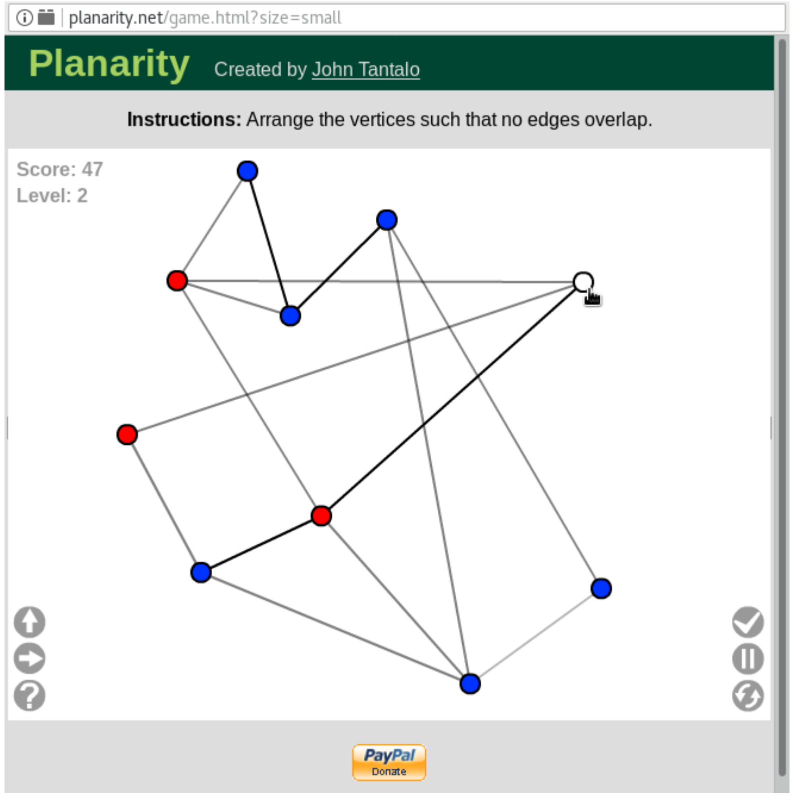
Free-Collinear $\Longrightarrow$ Free



Applications

- Partial simultaneous geometric embeddings
- Universal point subsets
- Column planarity
- Untangling:
  Given a drawing of a planar graph with crossings:
  Find a *plane* redrawing while keeping *many vertices fixed*.





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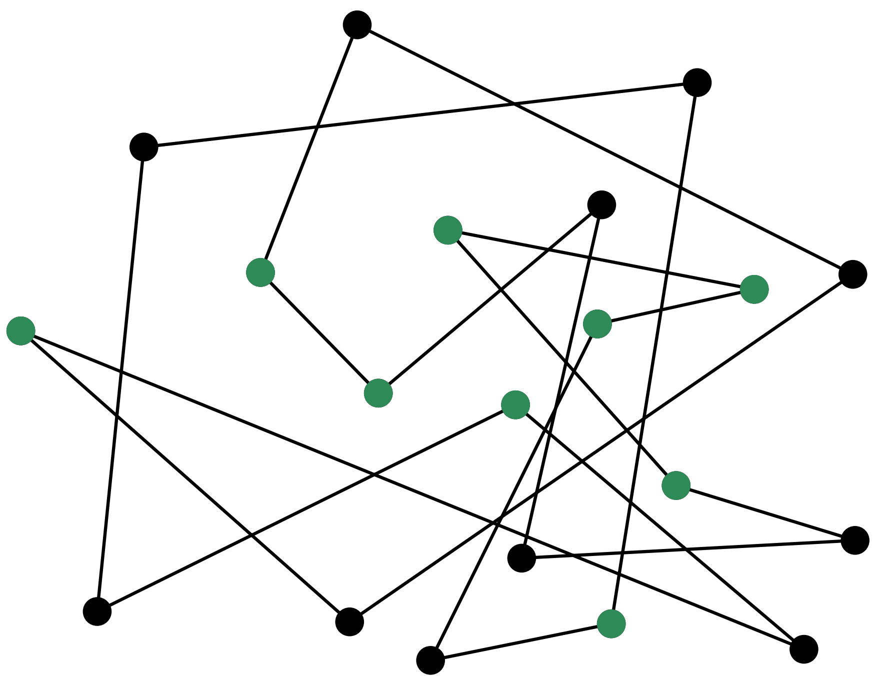

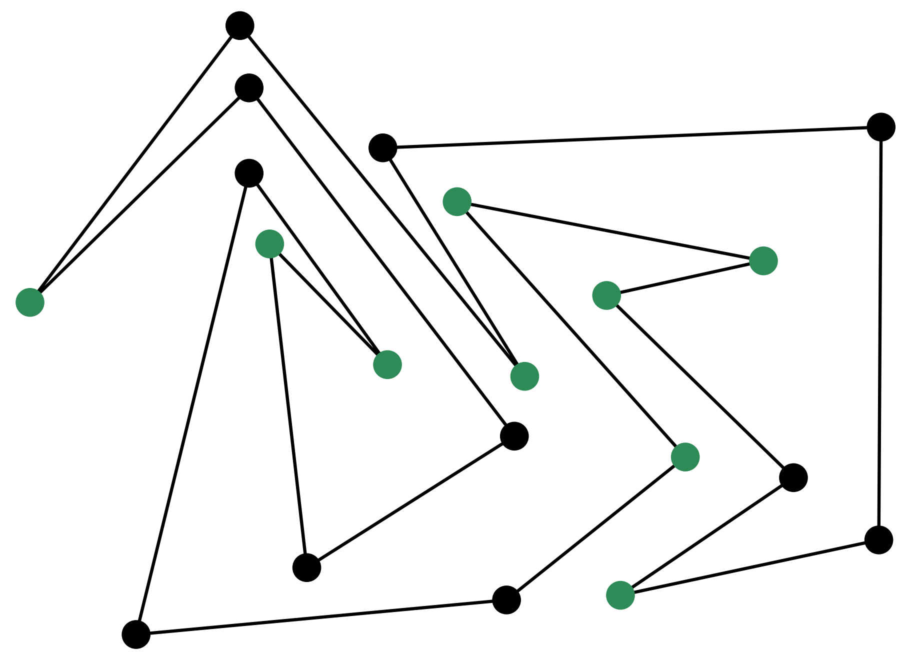

Untangling

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Given a drawing of a planar graph with crossings:
Find a *plane* redrawing while keeping *many vertices fixed*.

THEOREM. If G has a free (*now:* collinear) set S, then we can leave $\Omega(\sqrt{|S|})$ vertices fixed. [Ravsky and Verbitsky 2011]
[Bose, Dujmović, Hurtado, Langerman, Morin, Wood 2009]

COROLLARY: If G is a 3-connected triangulation, then $\Omega(\sqrt{n})$ vertices (*previously:* $\Omega(n^{1/4})$ vertices) can remain fixed.

(Upper bound: $O(\sqrt{n \log^3 n})$ [Cibulka 2010])

Largest Free Sets

Every n-vertex planar graph has a collinear/free set of size $\Omega(\sqrt{n}\,)$. [Bose, Dujmović, Hurtado, Langerman, Morin, Wood 2009]

Proof by canonical order or by Schnyder decomposition.

Upper bound: $O(n^{0.986})$ [Ravsky and Verbitsky 2011]

follows from 3-regular planar graphs without long cycles [Grünbaum and Walther 1973]

Every planar graph of *bounded degree* has a collinear/free set of size $\Omega(n^{0.8})$. [Dujmović and Morin 2019]

Testing whether a set is collinear (*now also:* free) is NP-hard. [Mchedlidze, Radermacher, Rutter 2017]

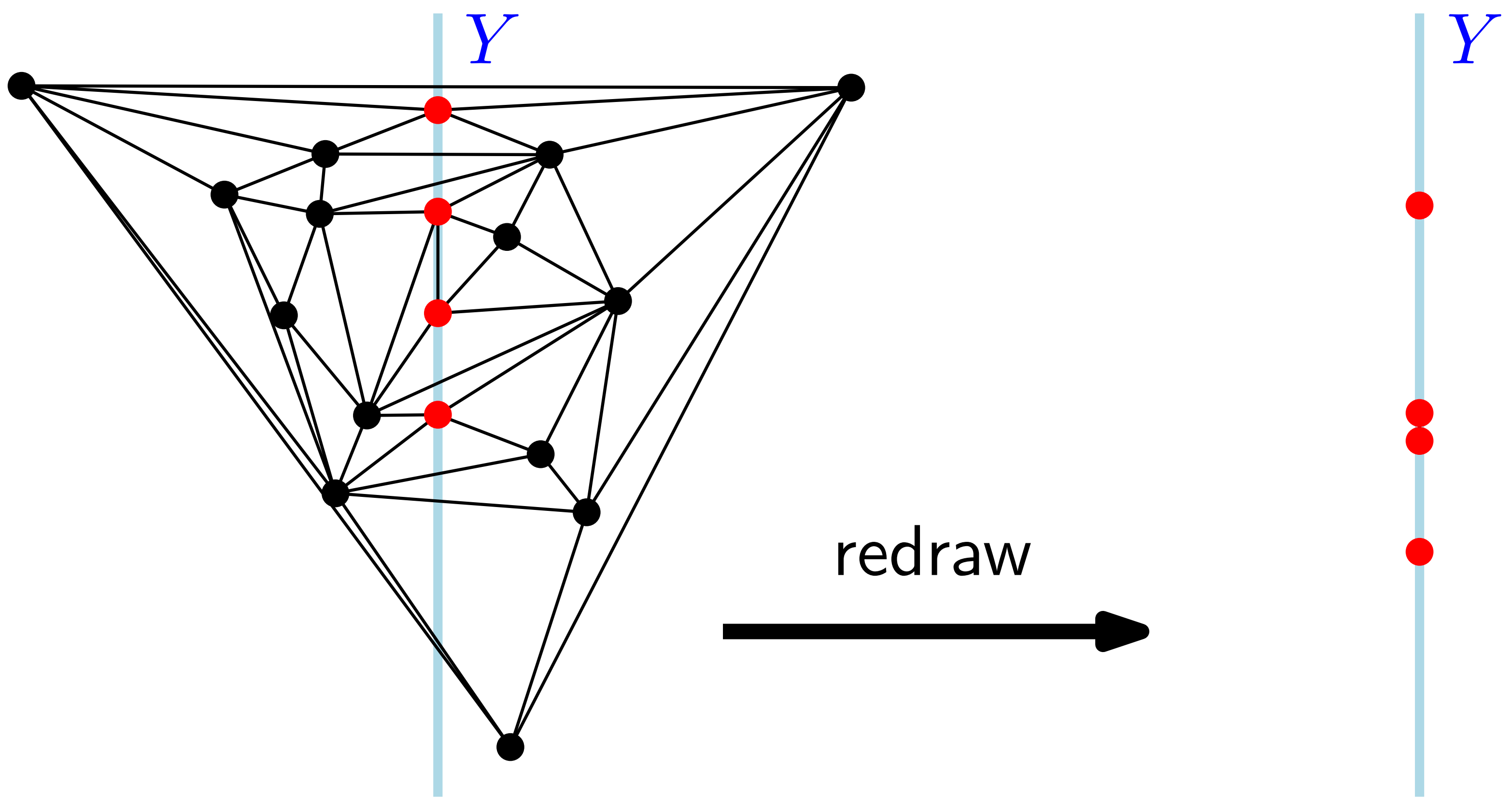
Proof Outline

Given:

- a drawing with some vertices $v_1, \ldots, v_k$ on a vertical line Y
- new y-coordinates $b_1 < \cdots < b_k$ for these vertices

Wanted:

- a redrawing with these y-coordinates



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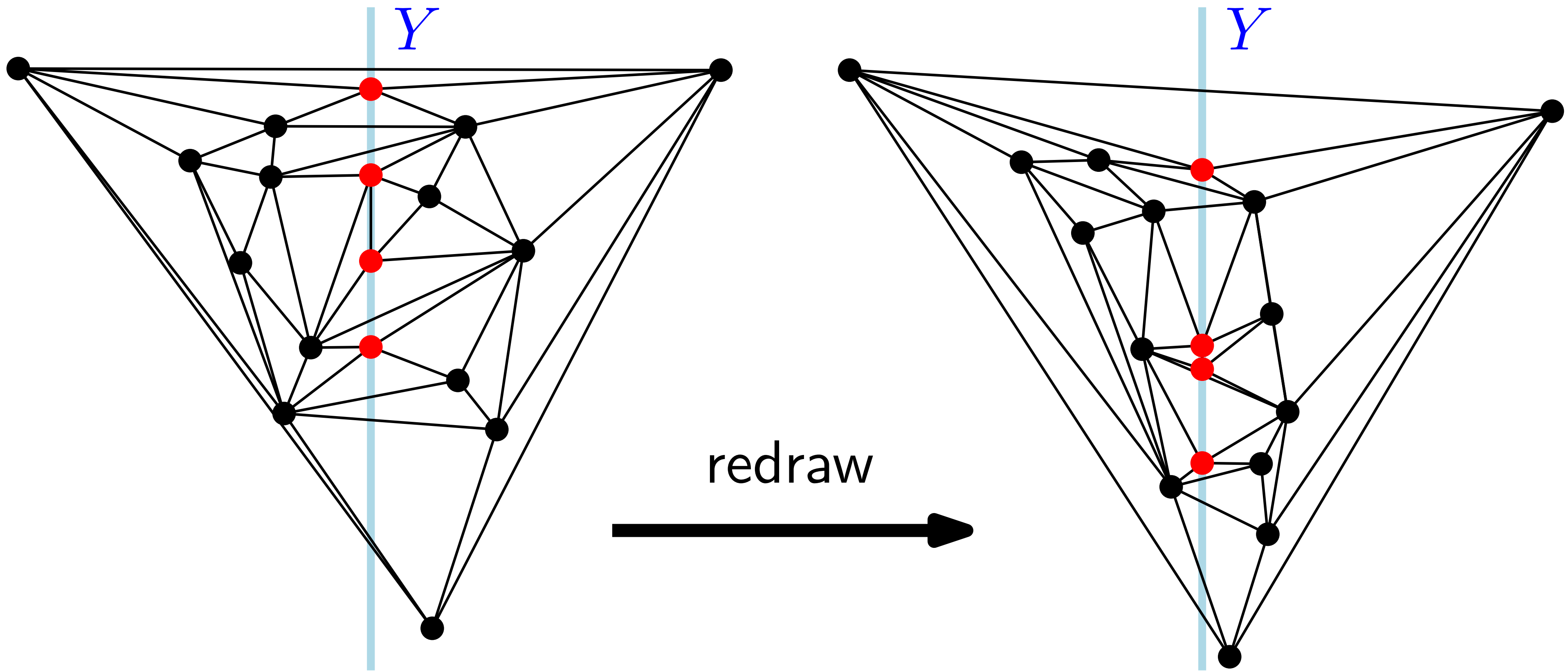

Proof Outline

Given:

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- new y-coordinates $b_1 < \cdots < b_k$ for these vertices

Wanted:

- a redrawing with these y-coordinates

(i) Special case:
Maximal graph such that all edges intersect Y.

(ii) Treat edges e that do not intersect Y:
Use several types of reductions

A. a) *contract* e into a single vertex v
   b) draw the resulting graph
   c) perturb v into two vertices with an edge between

B. *flip* the edge

C. Induction on separating triangles

Quadrilateralizations with no point on Y

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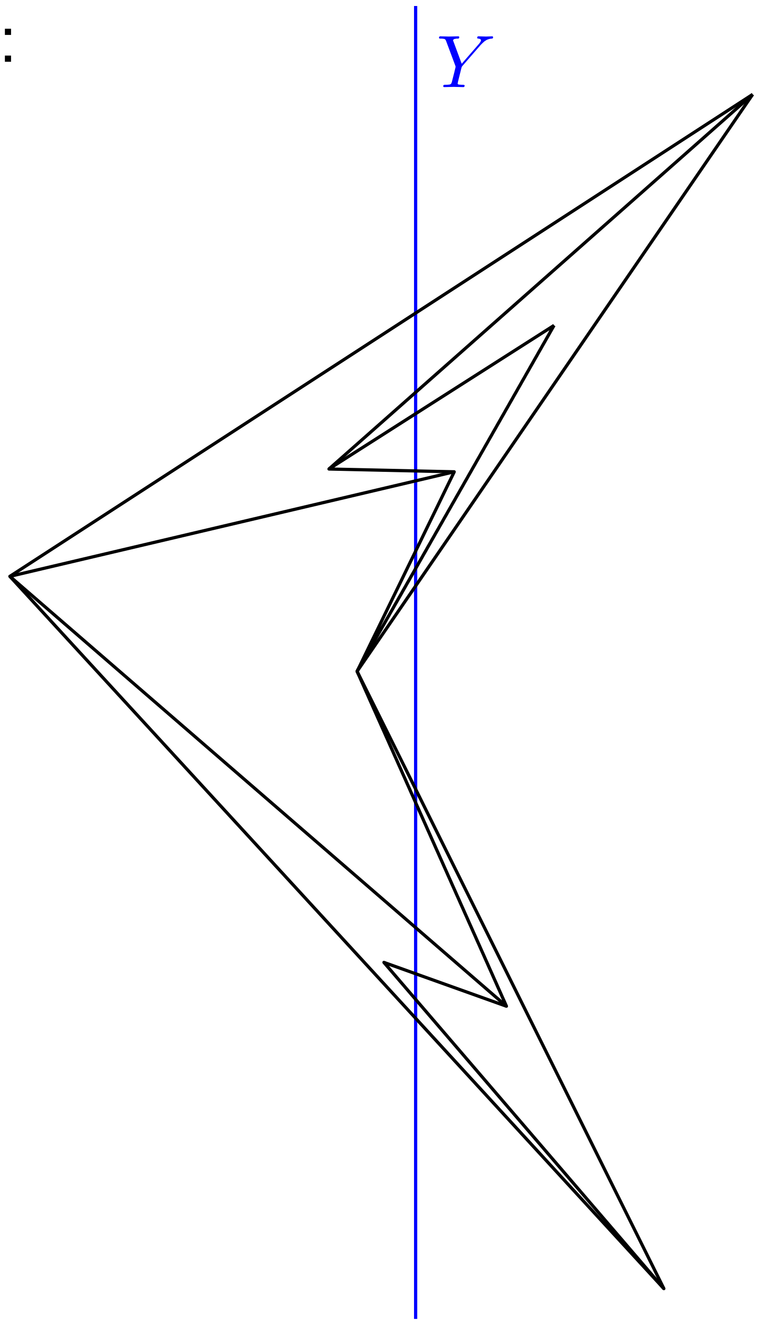
(i.0) An even more special special case:

- 1. Every edge crosses Y.
- 2. No vertex lies on Y.
- *Edge* e_i should intersect Y at b_i.



Quadrilateralizations with no point on Y

(i.0) An even more special special case:

- 1. Every edge crosses Y.
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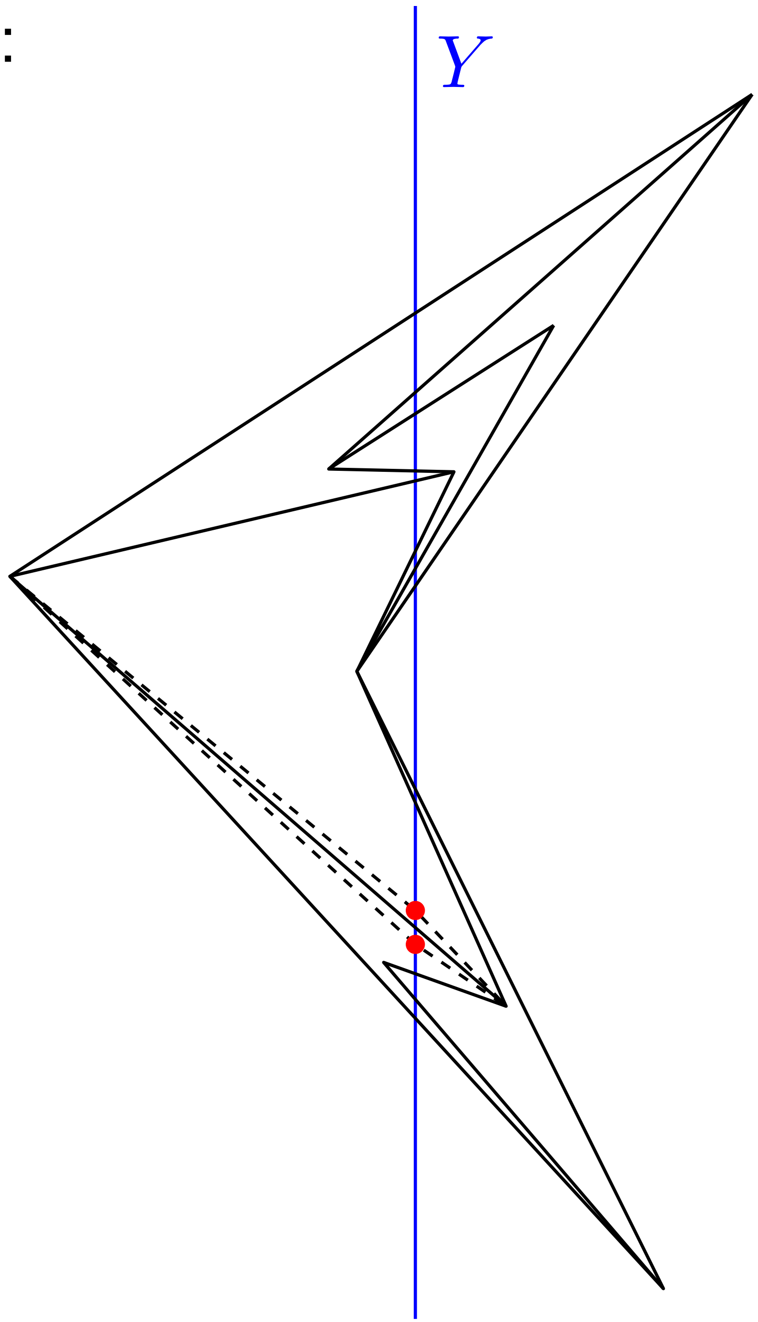
- *Edge* e_i should intersect Y at b_i.

  Think of two close points on Y straddling e_i.



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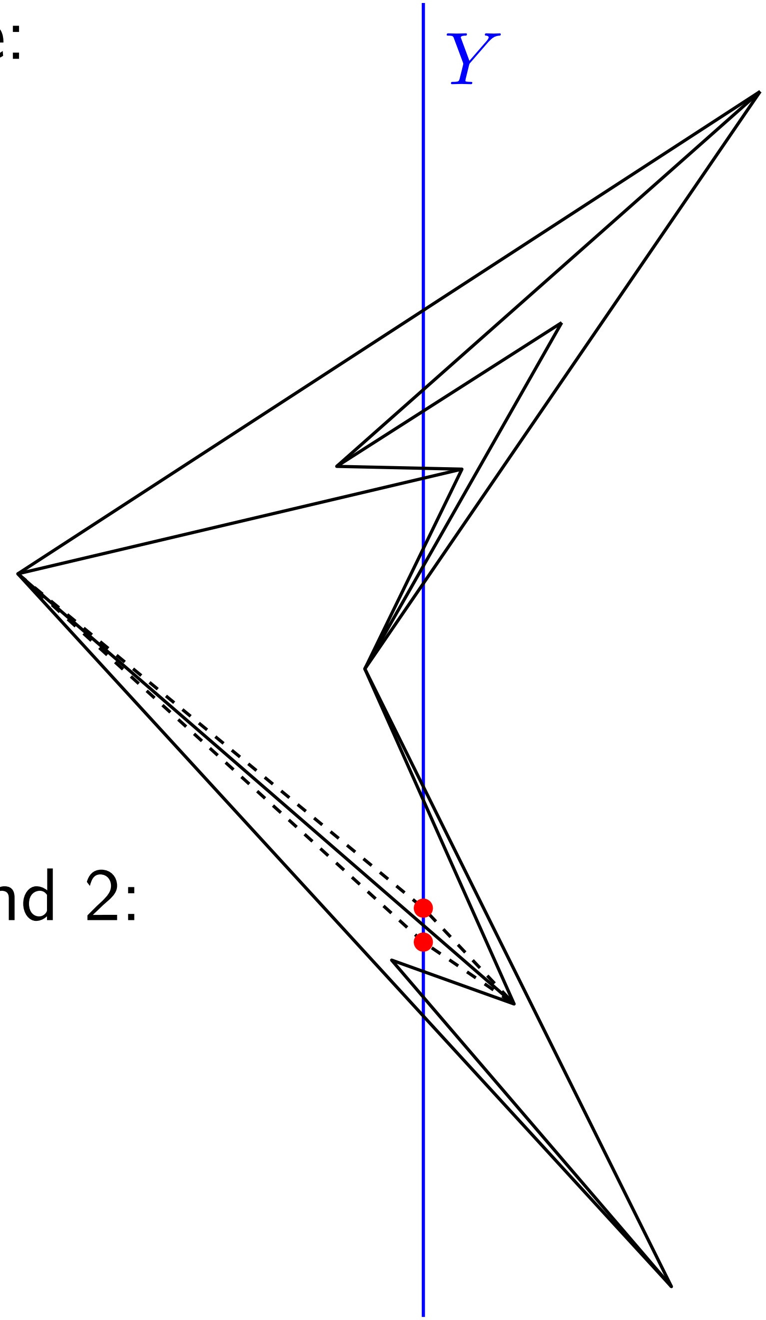
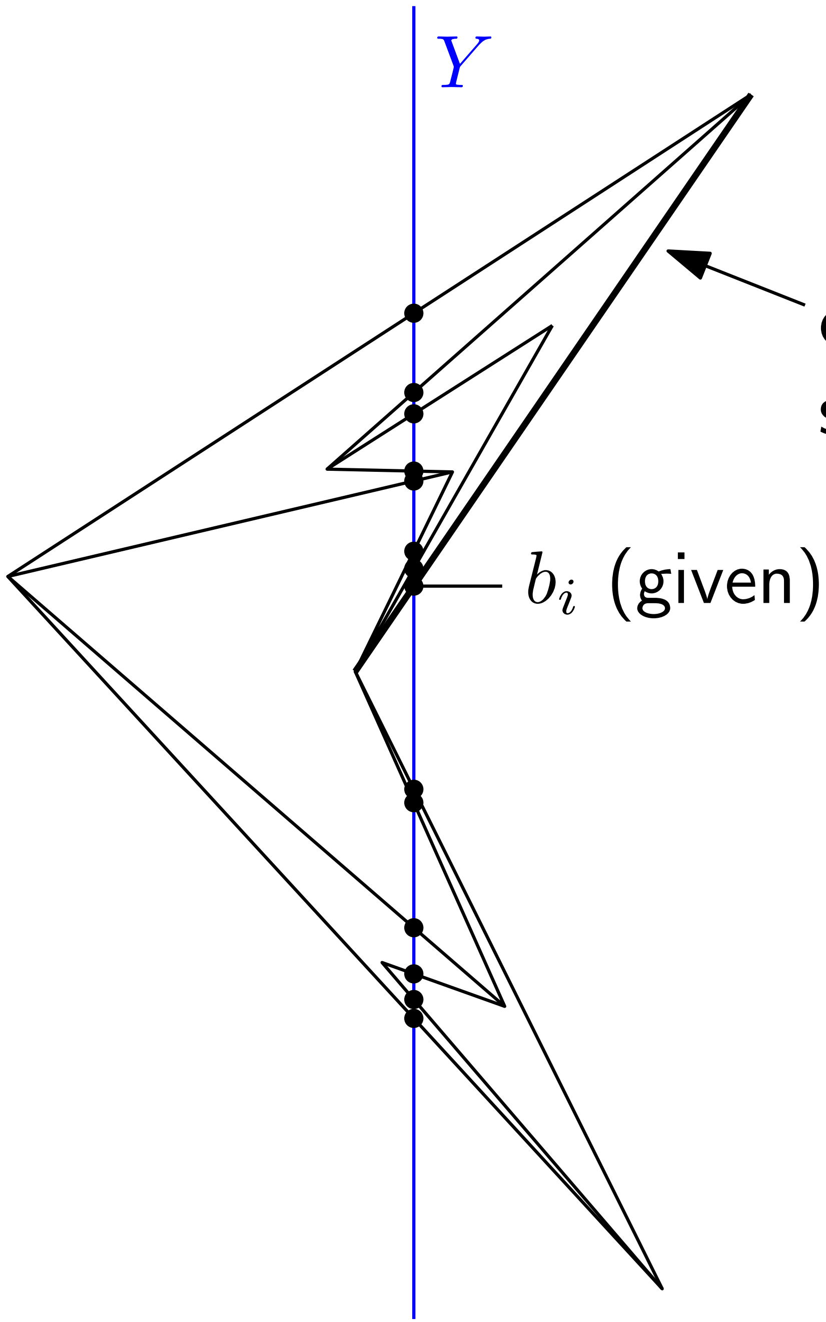
  Think of two close points on Y straddling e_i.

- maximal graph with properties 1 and 2:
  $\rightarrow$ quadrilateralization.



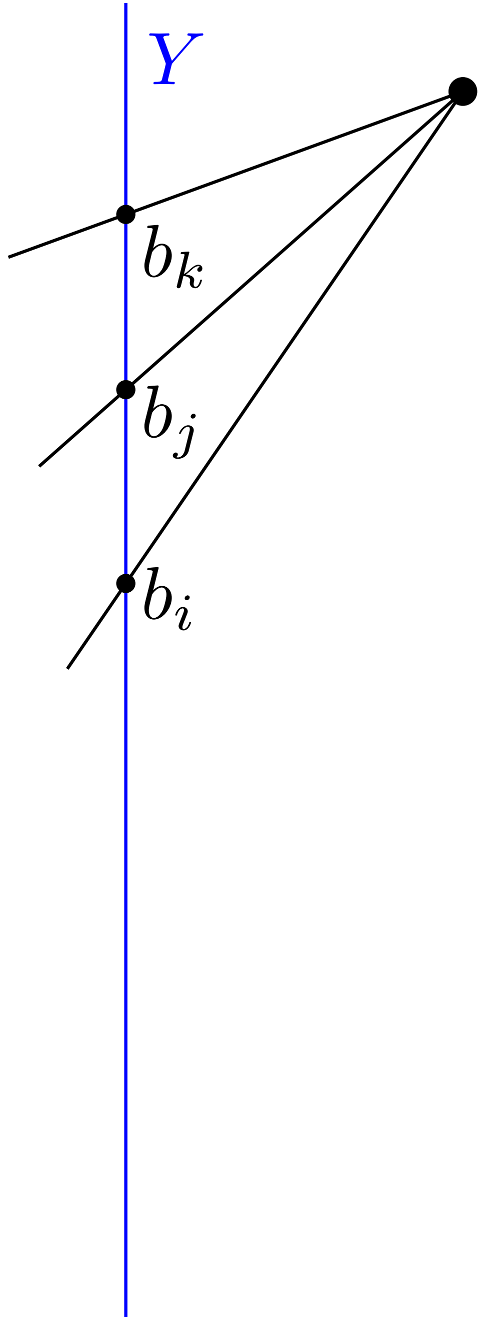
Parameterization by slope



edge e_i: $y = b_i + s_i x$
slope $= s_i$ (unknown)

m edges $e_1, \ldots, e_m$
Given: $b_1 < b_2 < \cdots < b_m$
m variables: $s_1, \ldots, s_m$

Concurrency constraints



$$\begin{vmatrix} 1 & 1 & 1 \\ b_i & b_j & b_k \\ s_i & s_j & s_k \end{vmatrix} = 0$$

$$s_i(b_k - b_j) + s_j(b_i - b_k) + s_k(b_j - b_i) = 0$$

a *linear* equation in s_i, s_j, s_k

Concurrency constraints

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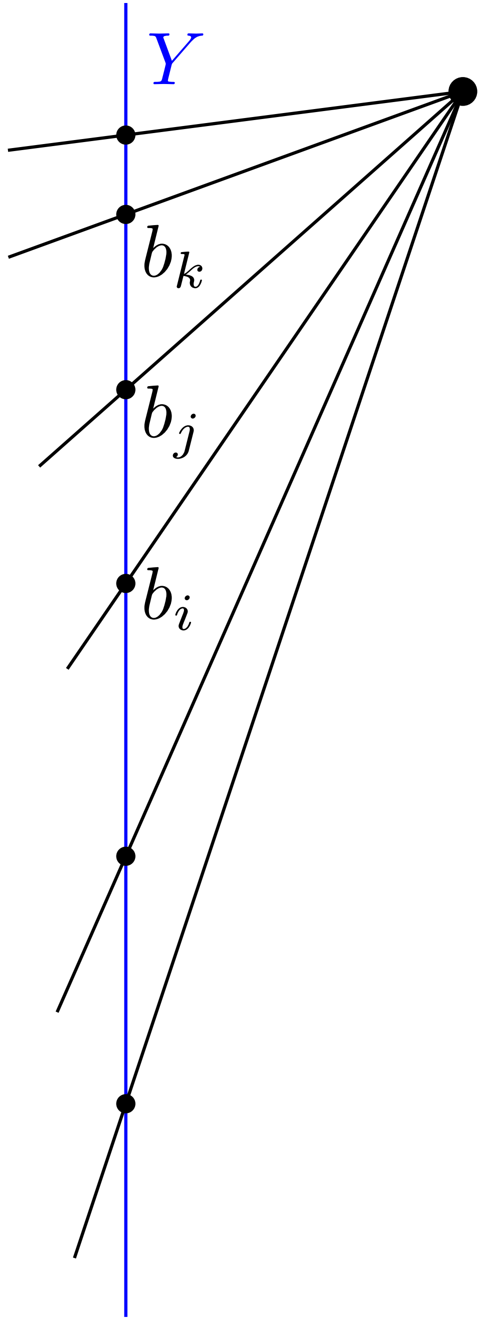

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$$s_i(b_k - b_j) + s_j(b_i - b_k) + s_k(b_j - b_i) = 0$$

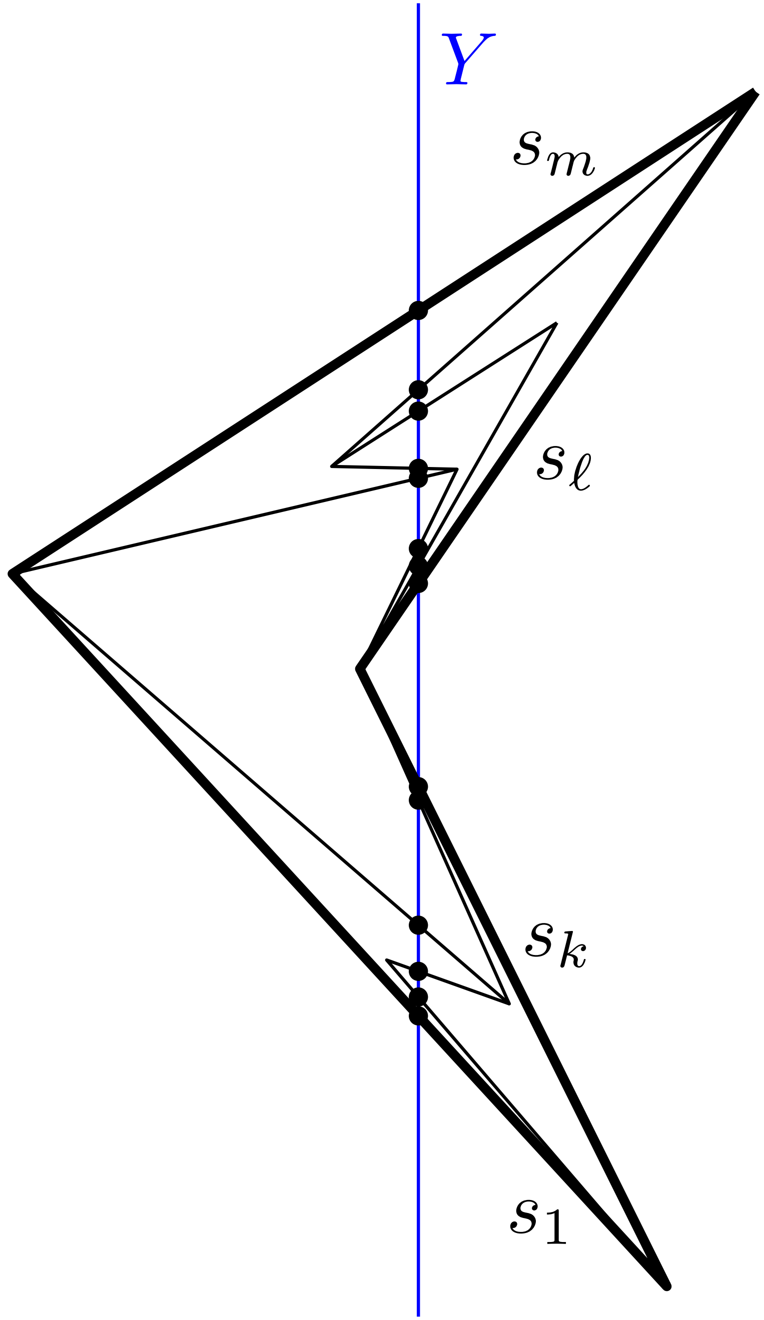
a *linear* equation in s_i, s_j, s_k

A vertex of degree d gives rise to $d - 2$ concurrency constraints.

Euler's formula
$\rightarrow m - 4$ equations in m variables

4 boundary equations to the rescue!

Boundary constraints

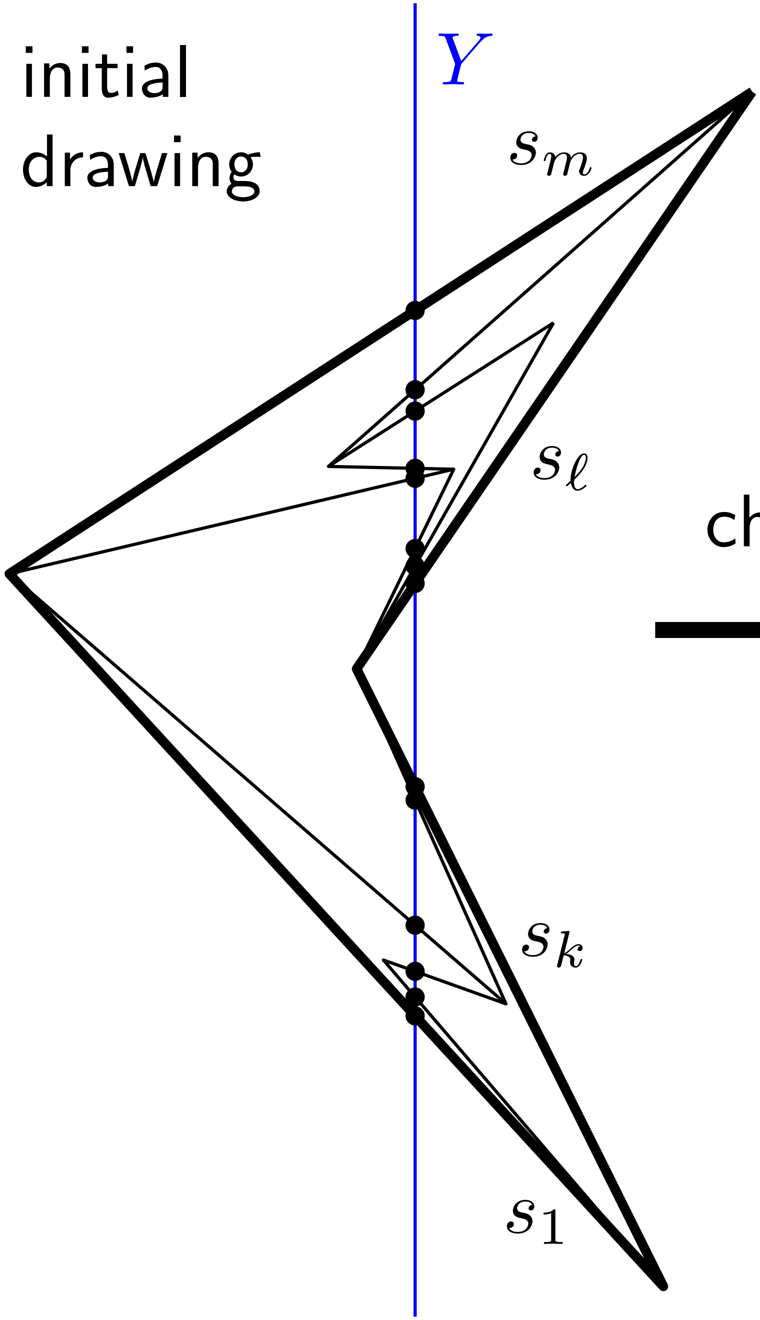


Set the 4 slopes s_1, s_k, s_ℓ, s_m of the outer boundary to fixed values.

$\rightarrow$ m equations in m variables

Morphing to the target solution

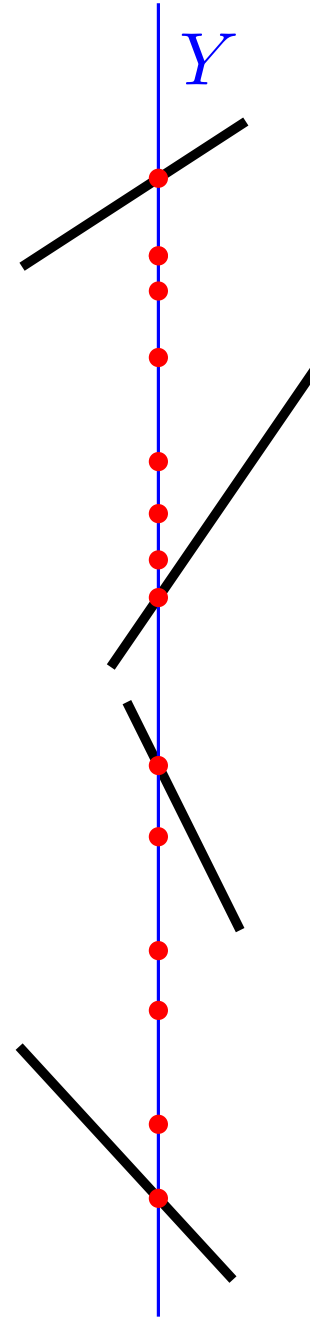
initial
drawing



change b_i continuously

follow the solution
$(s_1, \ldots, s_m)$ of the
linear system

target
drawing

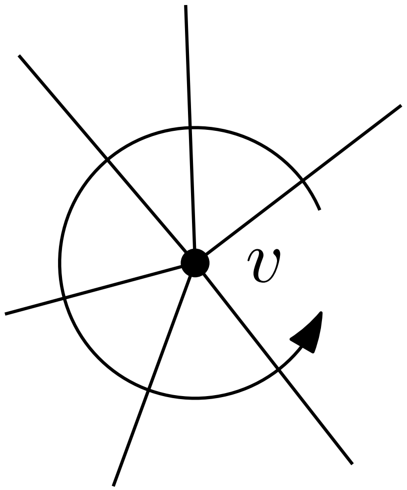


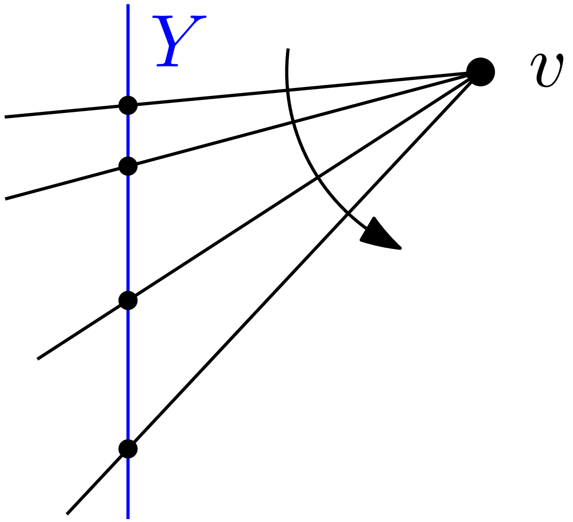
Solution of the linear system

- The solution $(s_1, \ldots, s_m)$ might not exist.
- The solution $(s_1, \ldots, s_m)$ might not be unique.
- The drawing might have crossings. ←

LEMMA: In a *straight-line drawing*, if
- the cyclic order of edges around each vertex agrees with a given planar drawing,
- and every face is a non-crossing polygon,

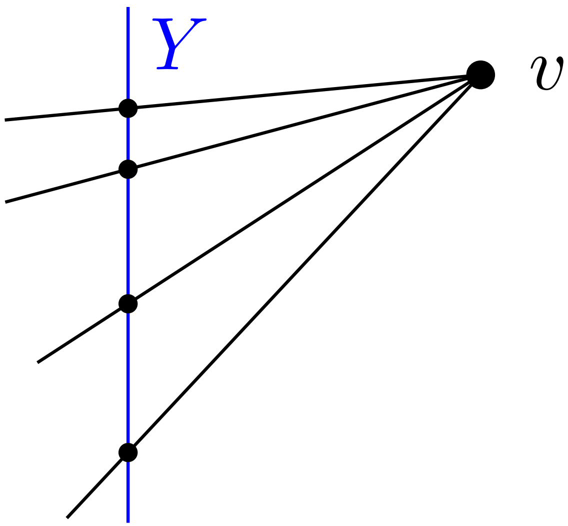
then the edges do not cross.





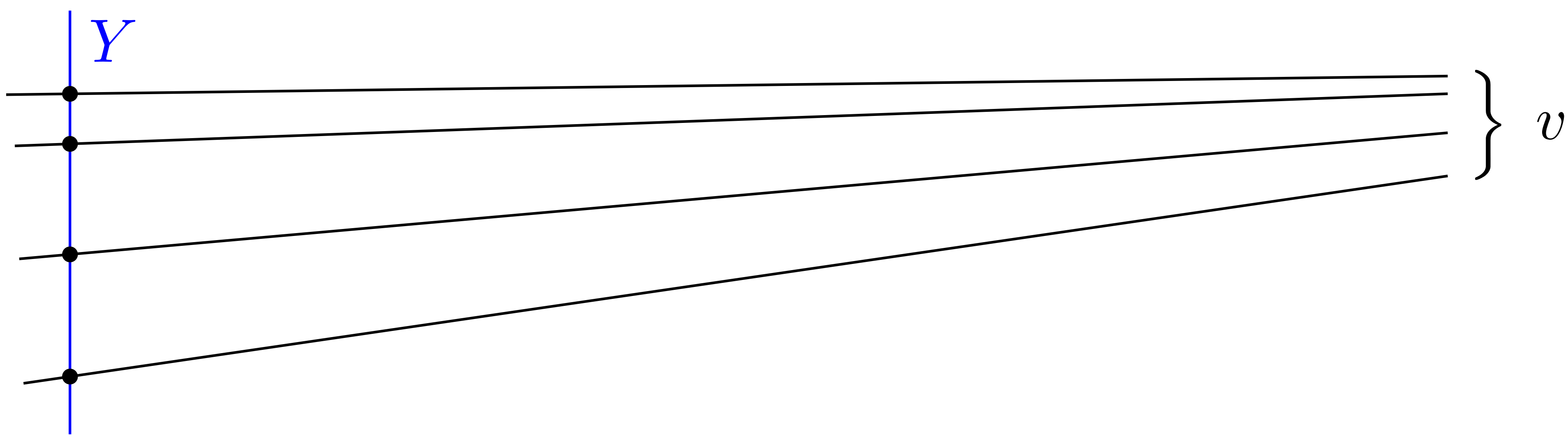
The cyclic order is correct because the order of b_i is fixed,
EXCEPT if the edges intersect on the *other side* of Y.

Flipping to the other side?



Continuity argument:

A vertex v that flips to the other side must first run off to infinity.



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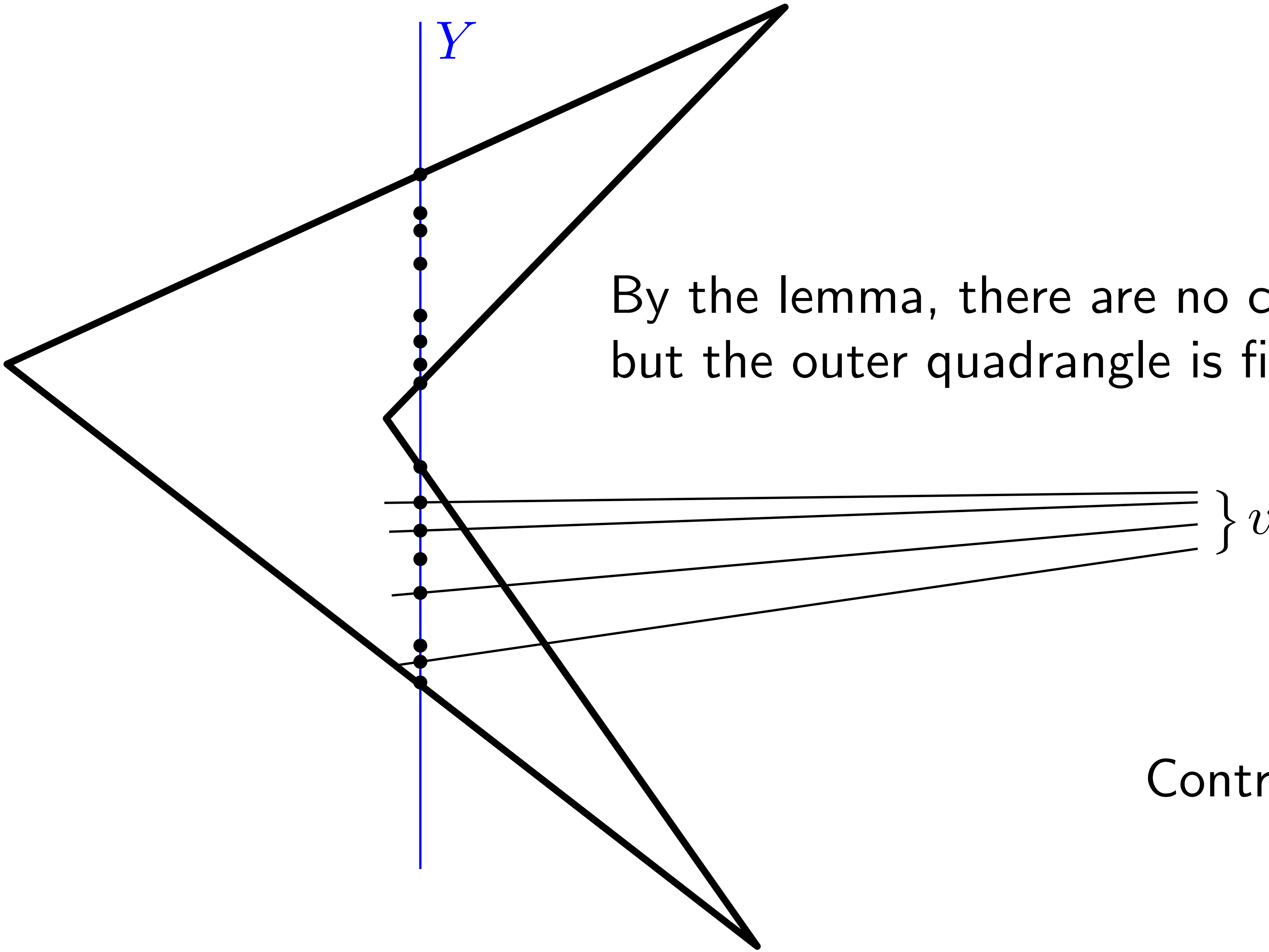


By the lemma, there are no crossings, but the outer quadrangle is fixed.

Contradiction.

Solution of the linear system

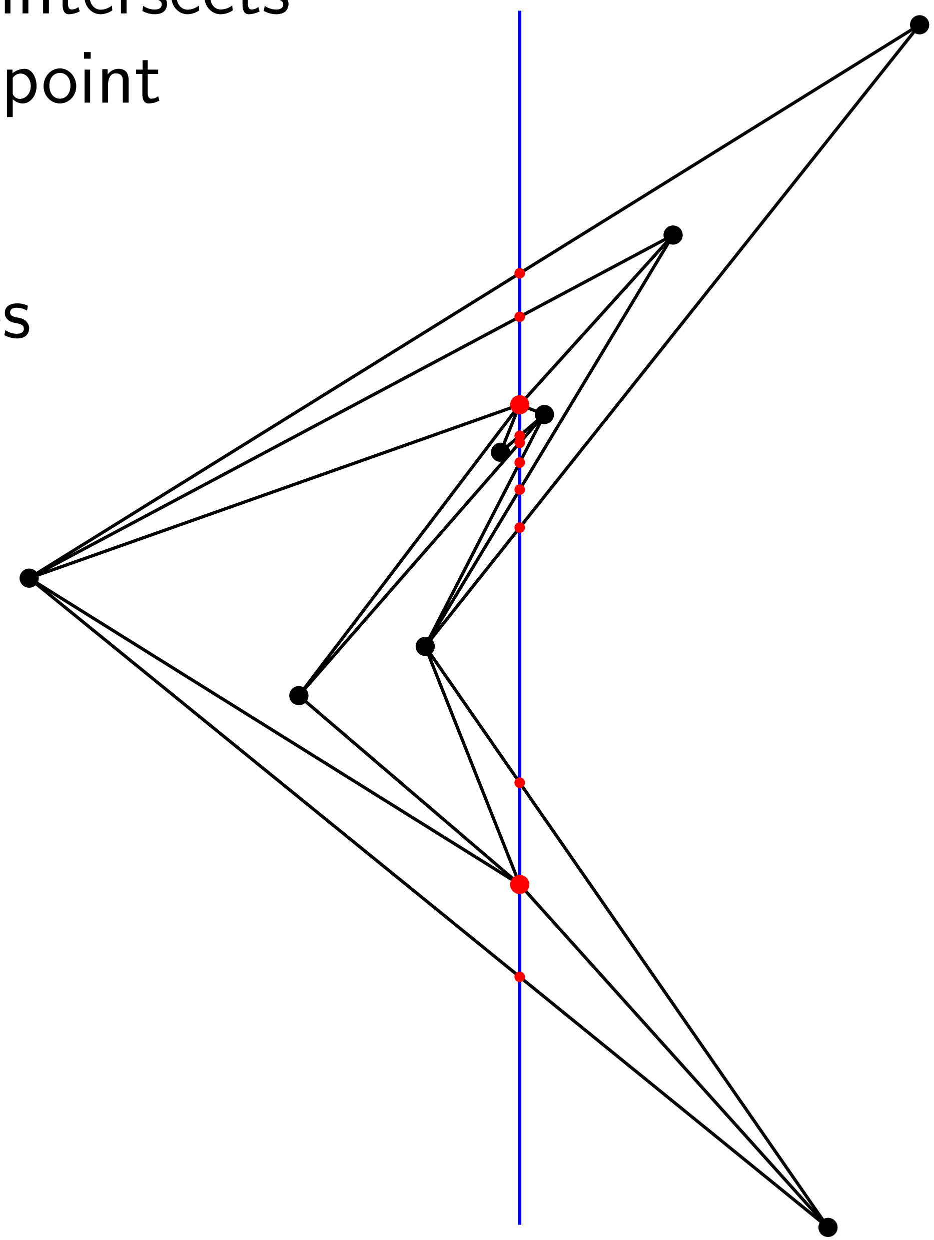
Existence and uniqueness are established with similar arguments.

- The solution $(s_1, \ldots, s_m)$ might not exist.
- The solution $(s_1, \ldots, s_m)$ might not be unique.
- The drawing might have crossings. ✓

(i) Extension to points on Y

- maximal graphs such that Y intersects every edge, at least in an endpoint (so-called "A-graphs")

$\rightarrow$ triangle and quadrilateral faces



Add some artificial *proportionality constraints*, in order to get m equations in m variables.

(Ensure at the same time the correct cyclic order)

(ii) Edges that don't intersect Y

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Use several types of reductions

A. a) *contract* xy into a single vertex v

b) draw the resulting graph

c) perturb v into two vertices with an edge between

B. *flip* the edge xy

C. Induction on separating triangles