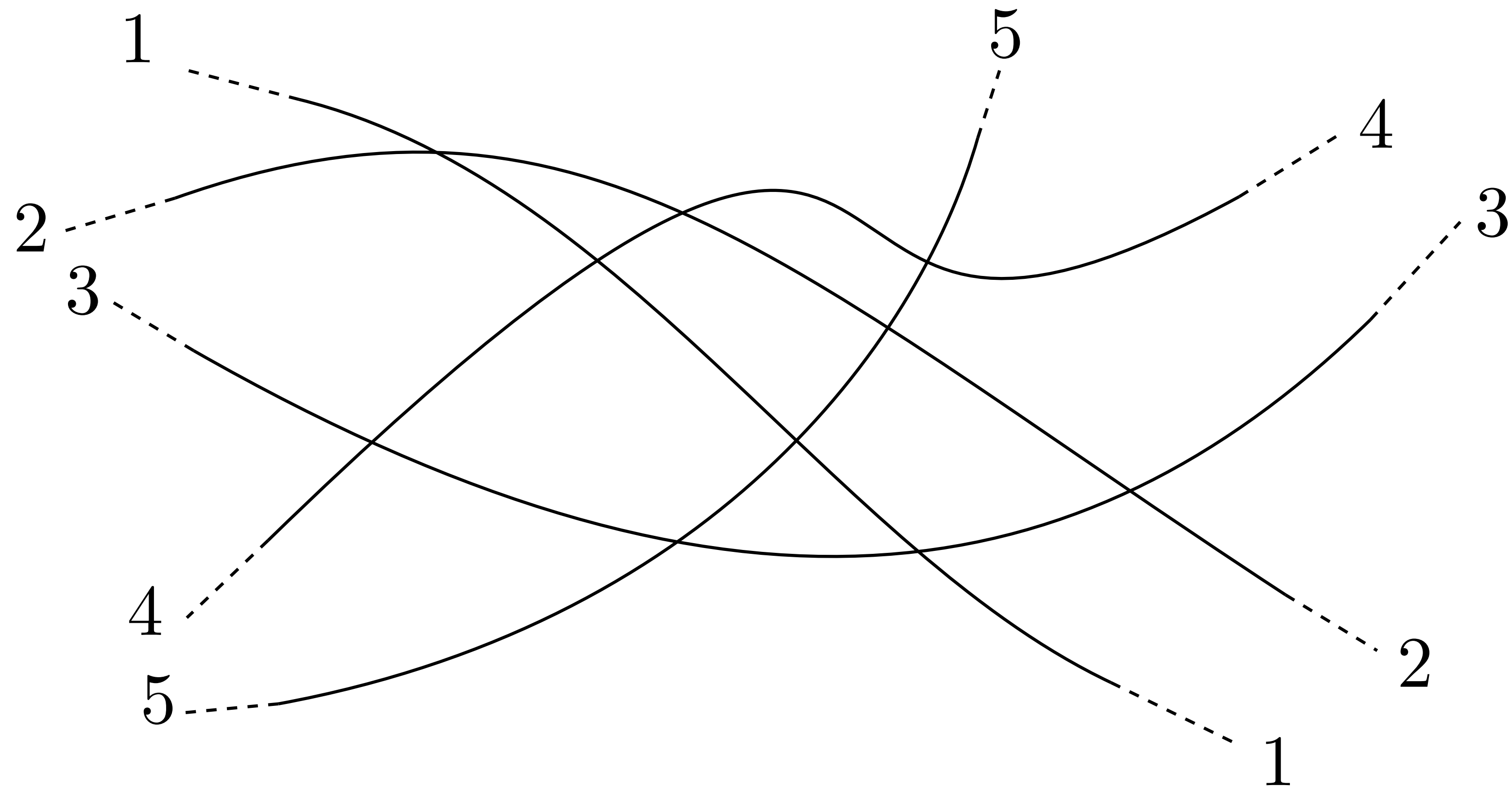
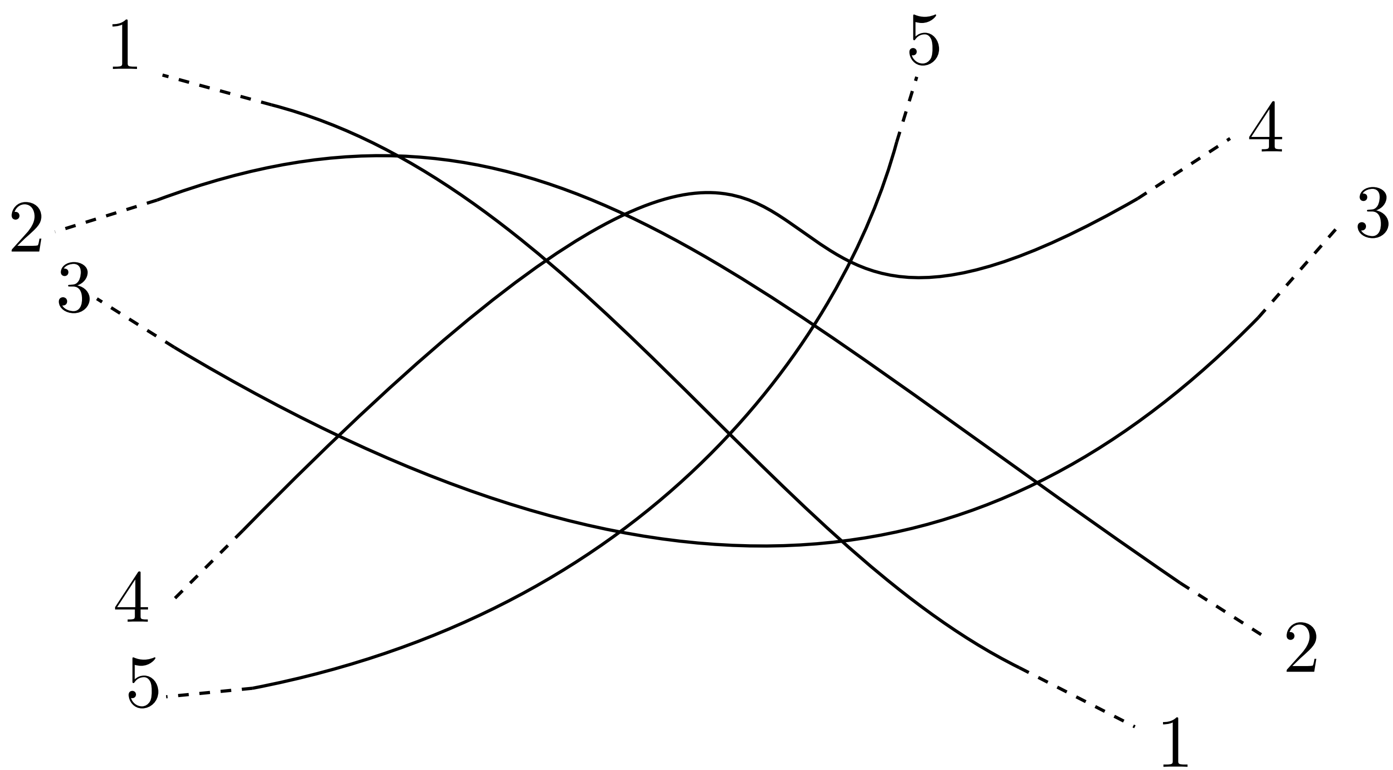
Enumeration and Counting of Pseudoline Arrangements

Günter Rote

Freie Universität Berlin



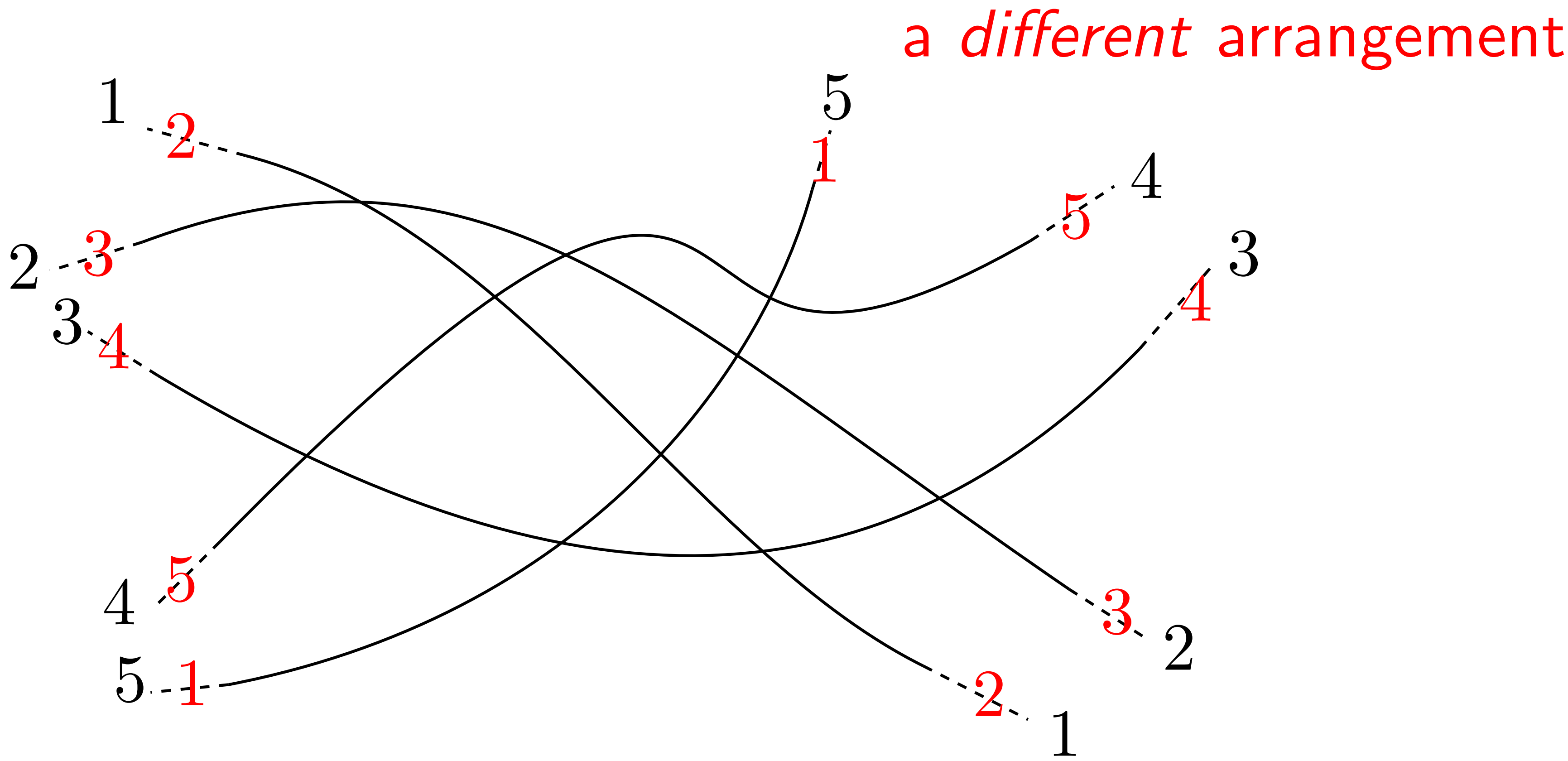
Pseudoline Arrangements



- n curves that go to infinity
- Two curves intersect exactly once, and they cross.

- *simple* pseudoline arrangements: no multiple crossings
- x-monotone curves

Pseudoline Arrangements



- n curves that go to infinity
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How many pseudoline arrangements?

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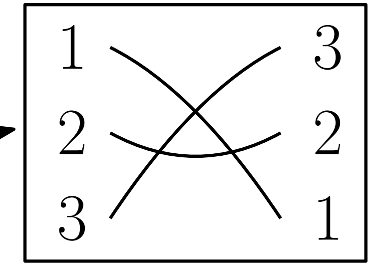
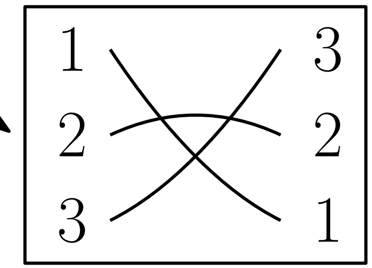
n	#PsA's with n pseudolines	
1	1	
2	1	
3	2	
4	8	
5	62	
6	908	
7	24698	
8	1232944	
9	112018190	
10	18410581880	
11	5449192389984	
12	2894710651370536	
13	2752596959306389652	
14	4675651520558571537540	[Yuma Tanaka, 2013]
15	14163808995580022218786390	
16	76413073725772593230461936736	[G. Rote, 2021]





OEIS A006245

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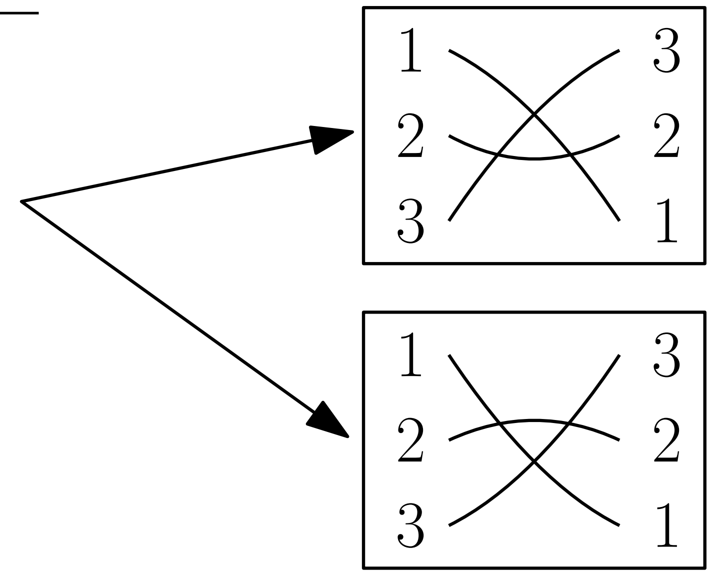
0.25 [Cortés Kühnast, Felsner, Scheucher 2023+]

$\geq 2^{\cancel{0.2083}n^2}$

[Dumitrescu, Mandal 2020]

$\leq 2^{0.6571n^2}$

[Felsner, Valtr 2012]



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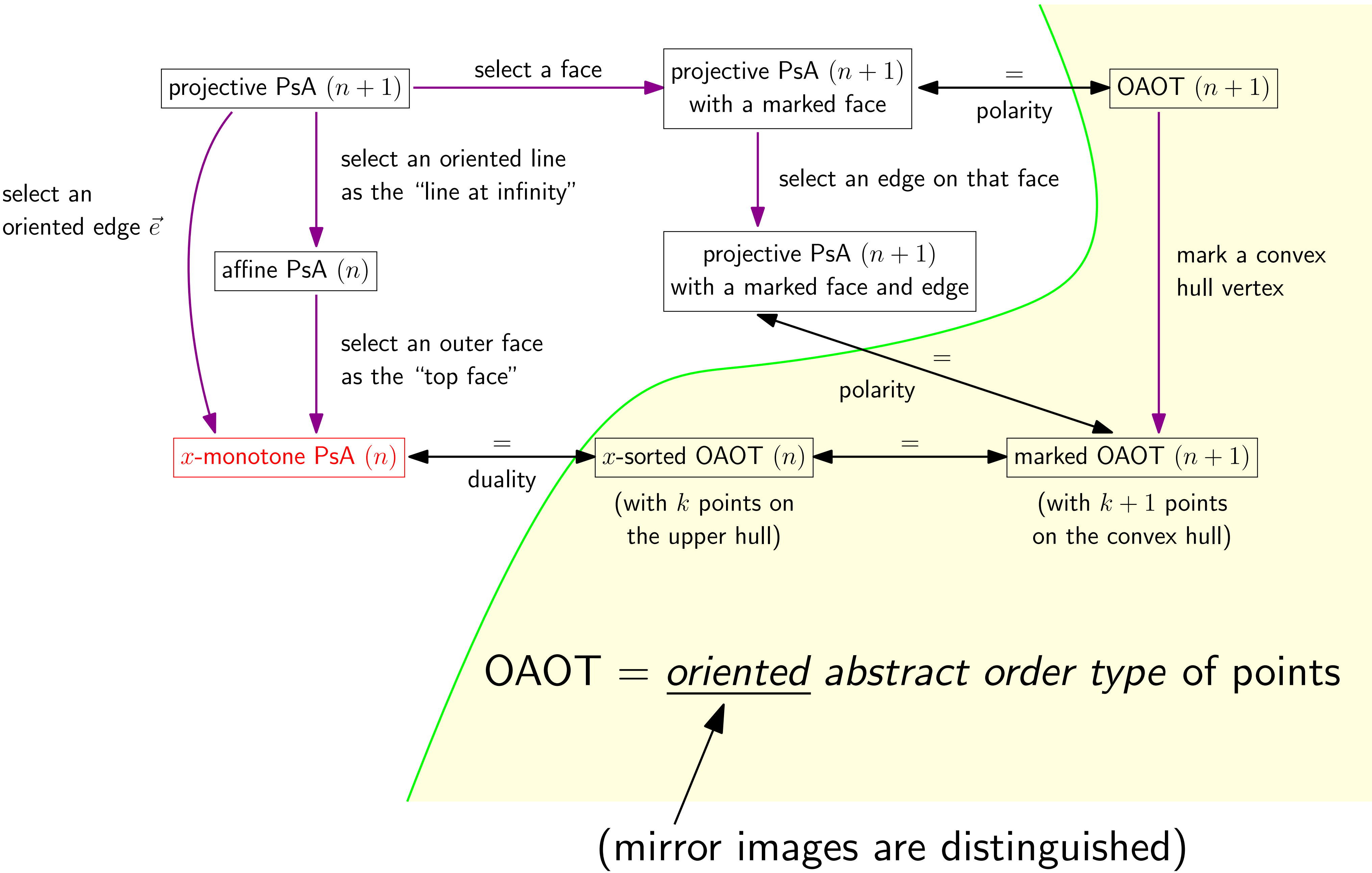
$n = 14, 15$: [Yuma Tanaka, 2013]

$n = 16$: [G. Rote, 2021]

Outline

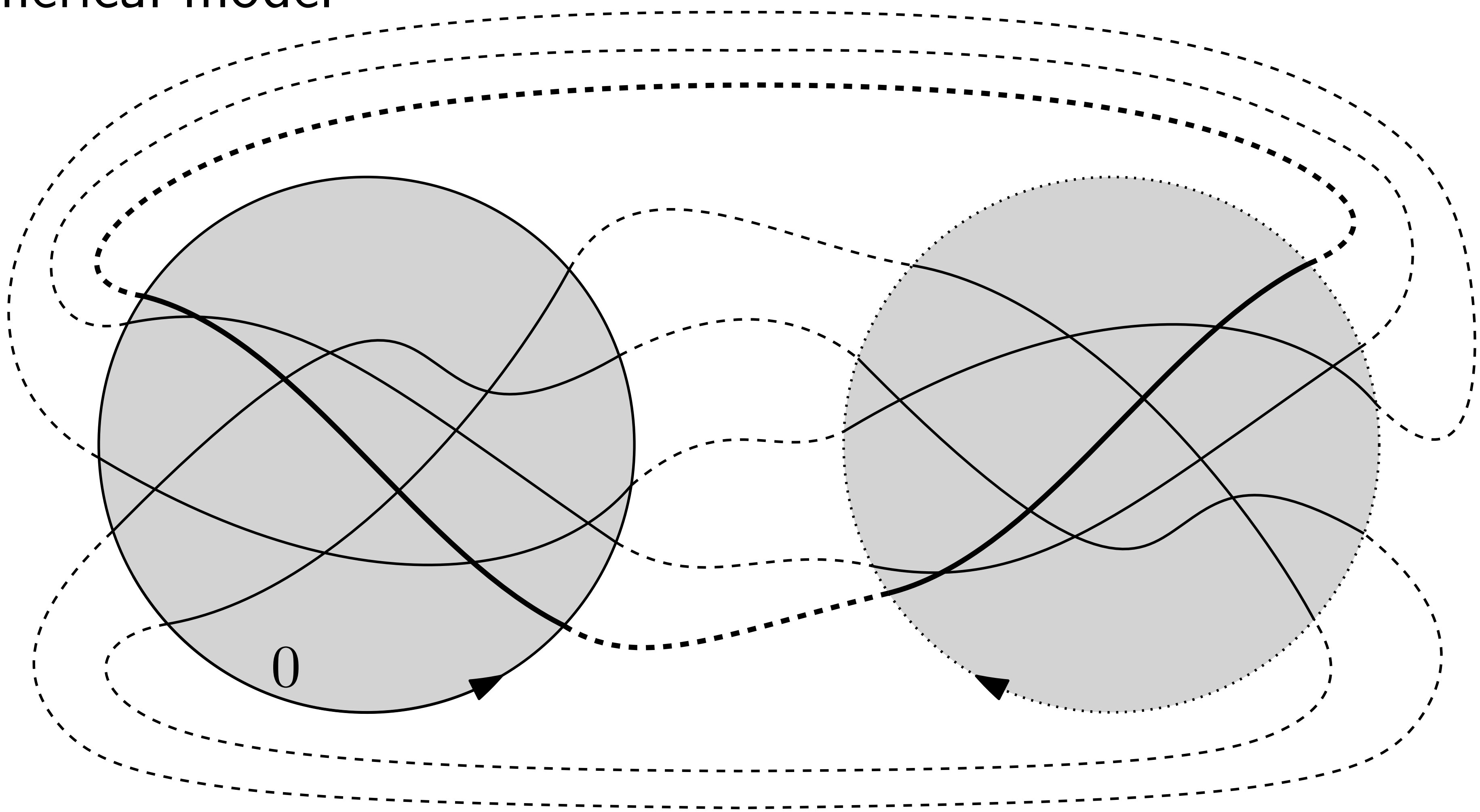
- Counting versus Enumeration
- What are we counting?
- 2-level-approach:
  Threading ℓ extra strands through
  a fixed arrangements of k pseudolines
- Partial pseudoline arrangements
- Sweep an arrangement (a bipolar orientation) with a rope

Related concepts



Projective pseudoline arrangements

spherical model



Projective pseudoline arrangements

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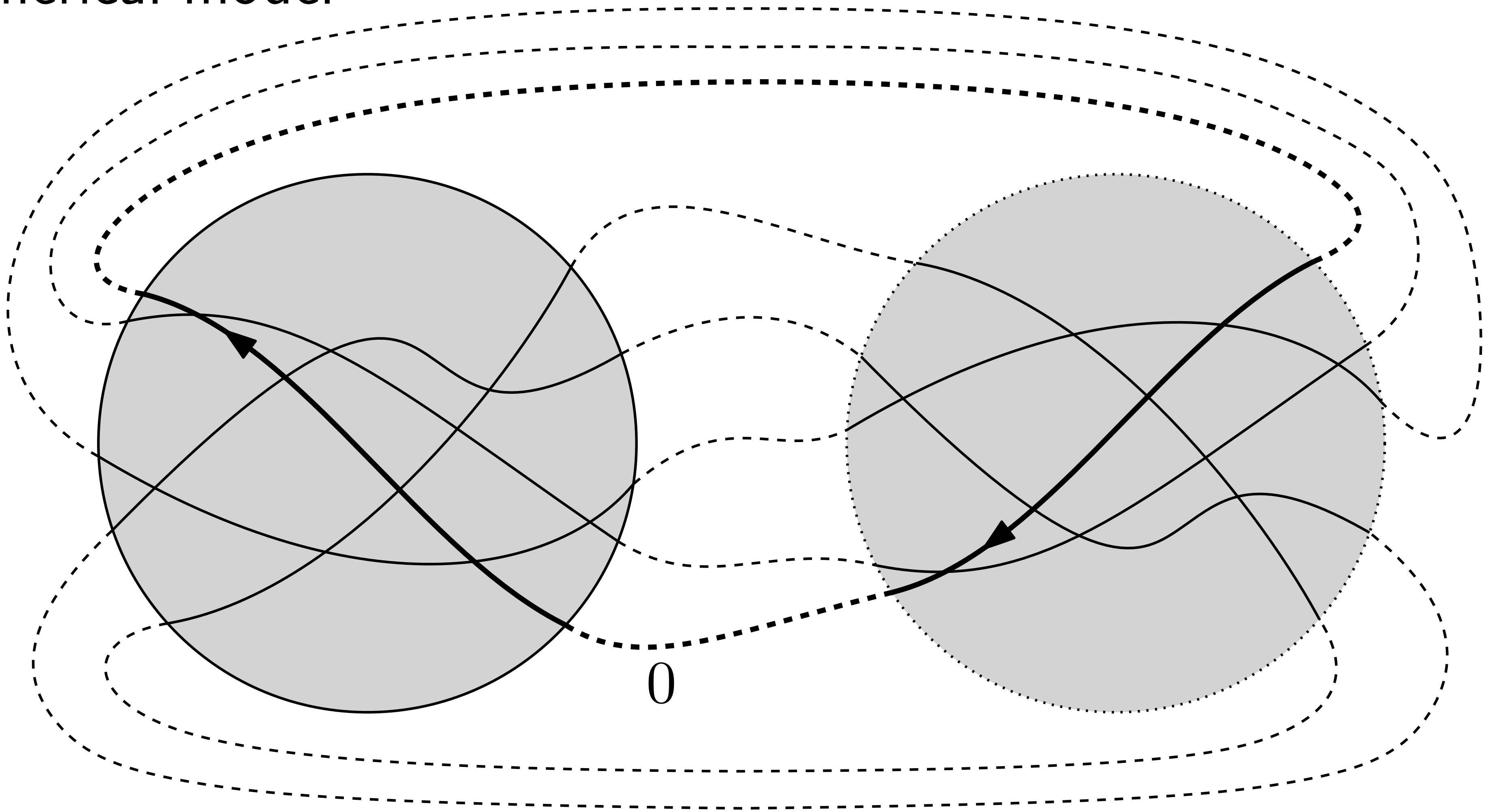
spherical model



- *affine* pseudoline arrangement

Projective pseudoline arrangements

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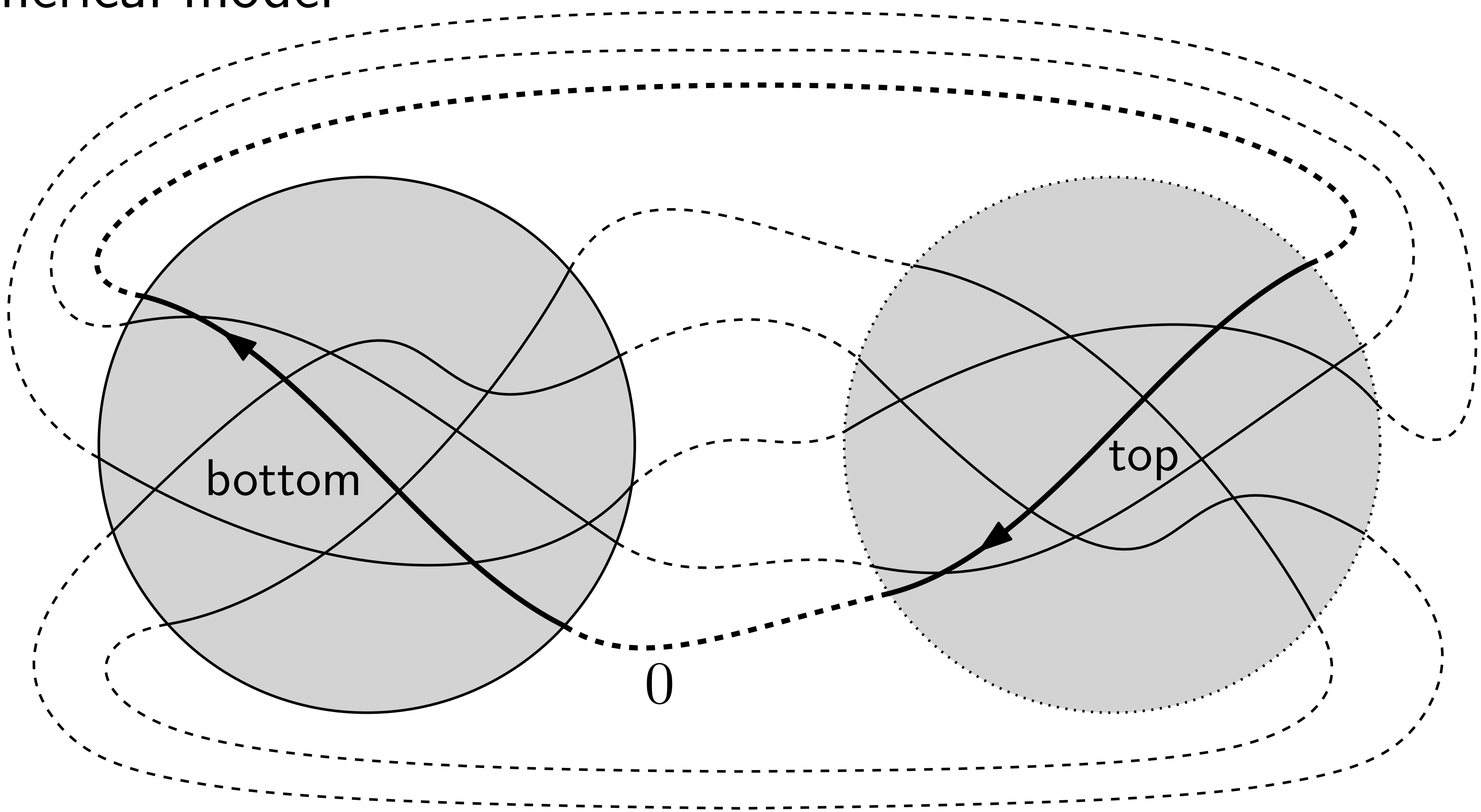
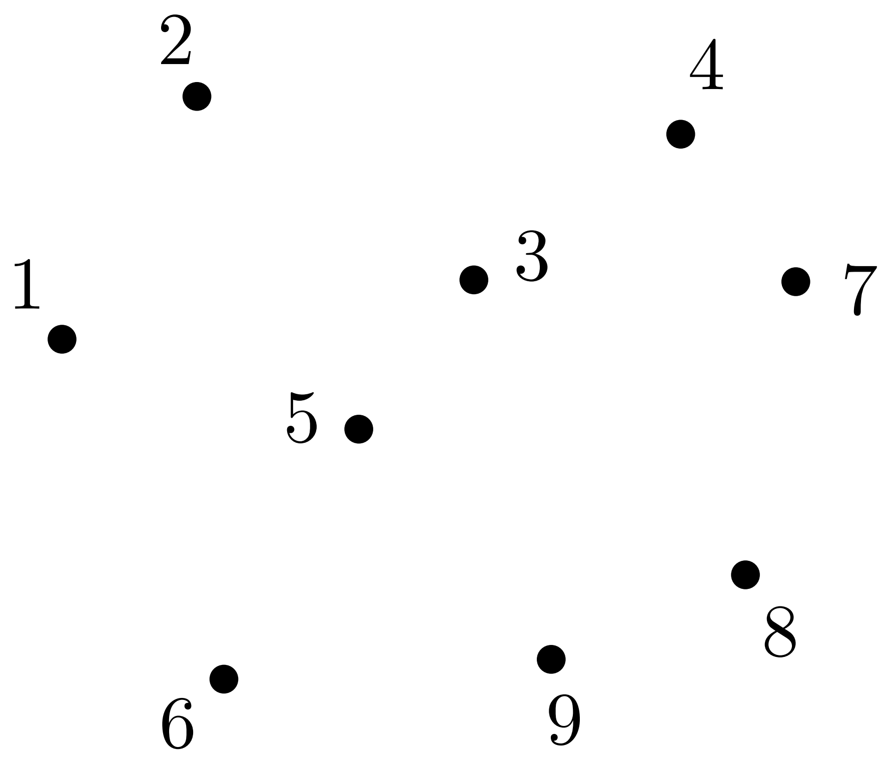


- *affine* pseudoline arrangement
- x-*monotone* pseudoline arrangement

(Abstract) order types of points



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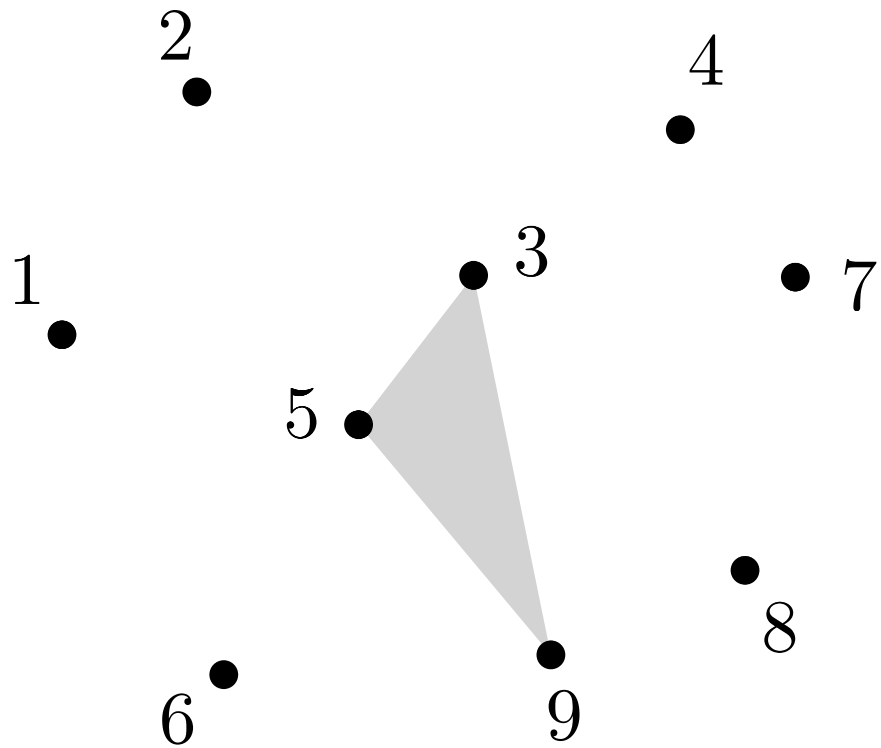


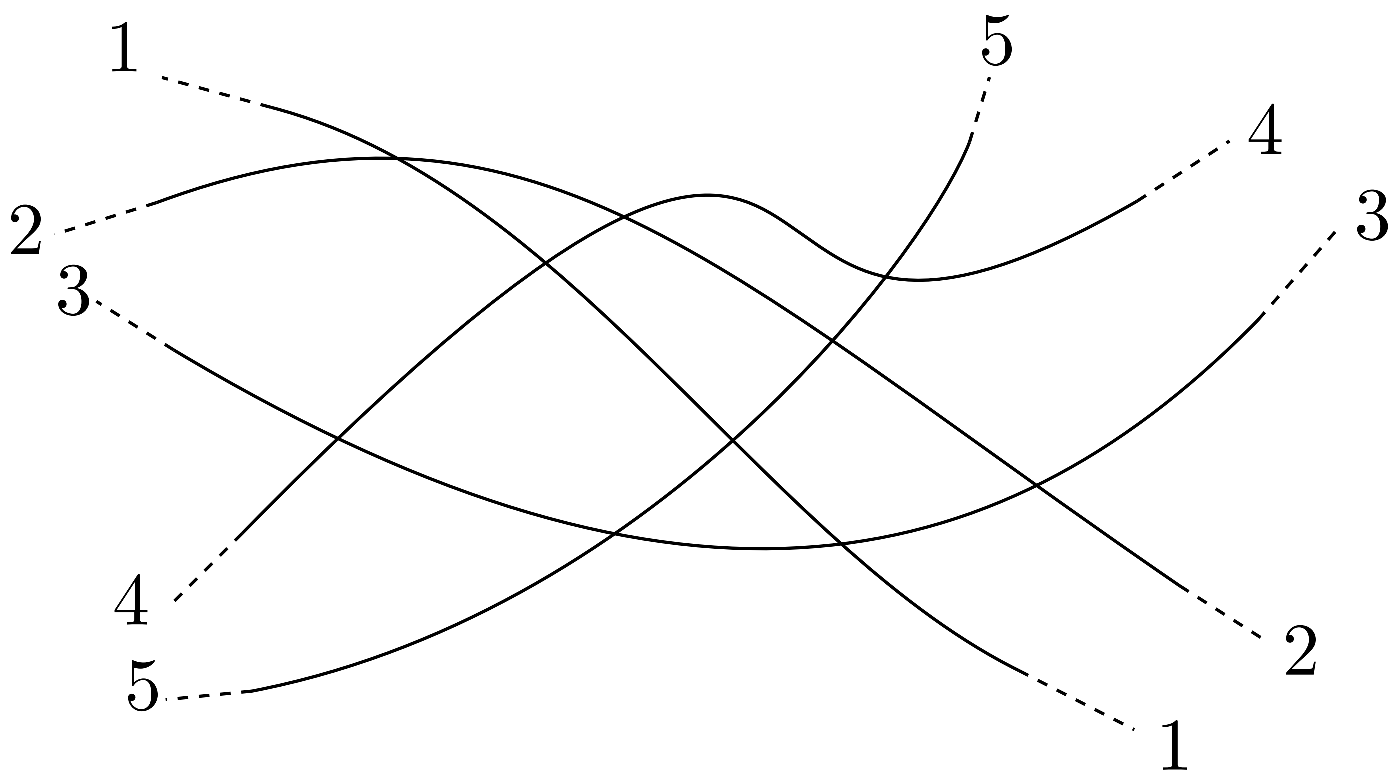
orientation

$593 = +$

$359 = +$

$539 = -$

Inductive Enumeration of PsA's



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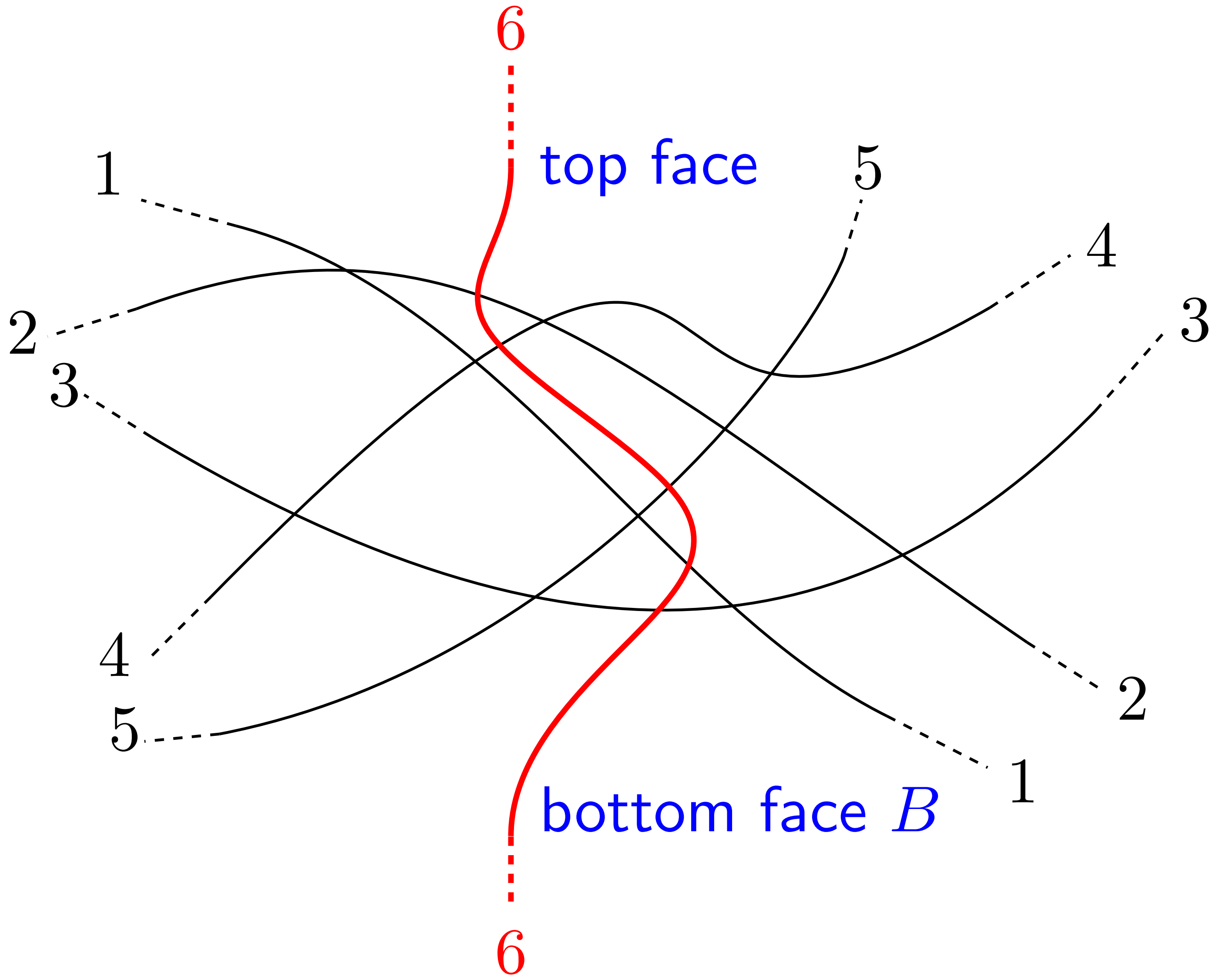

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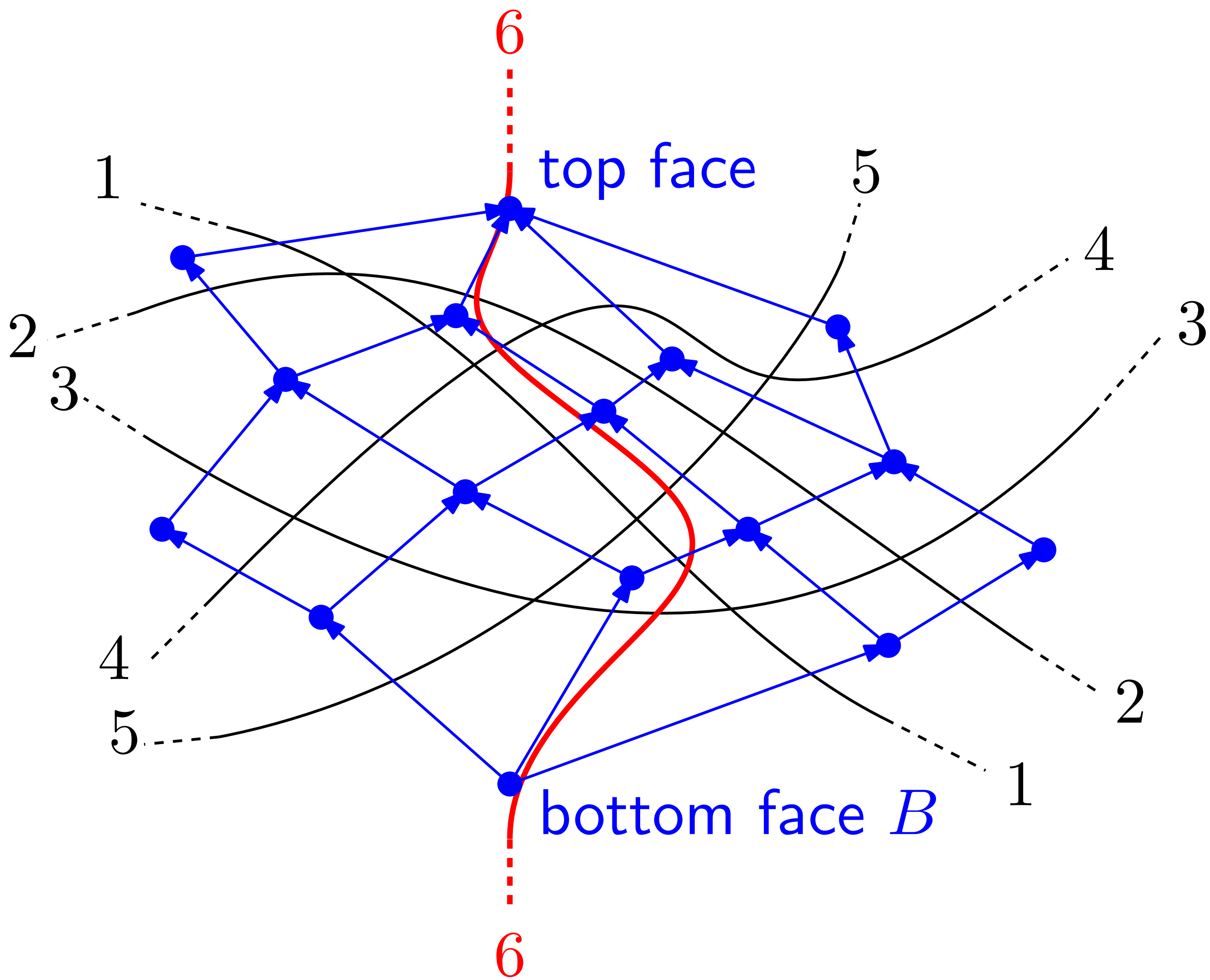
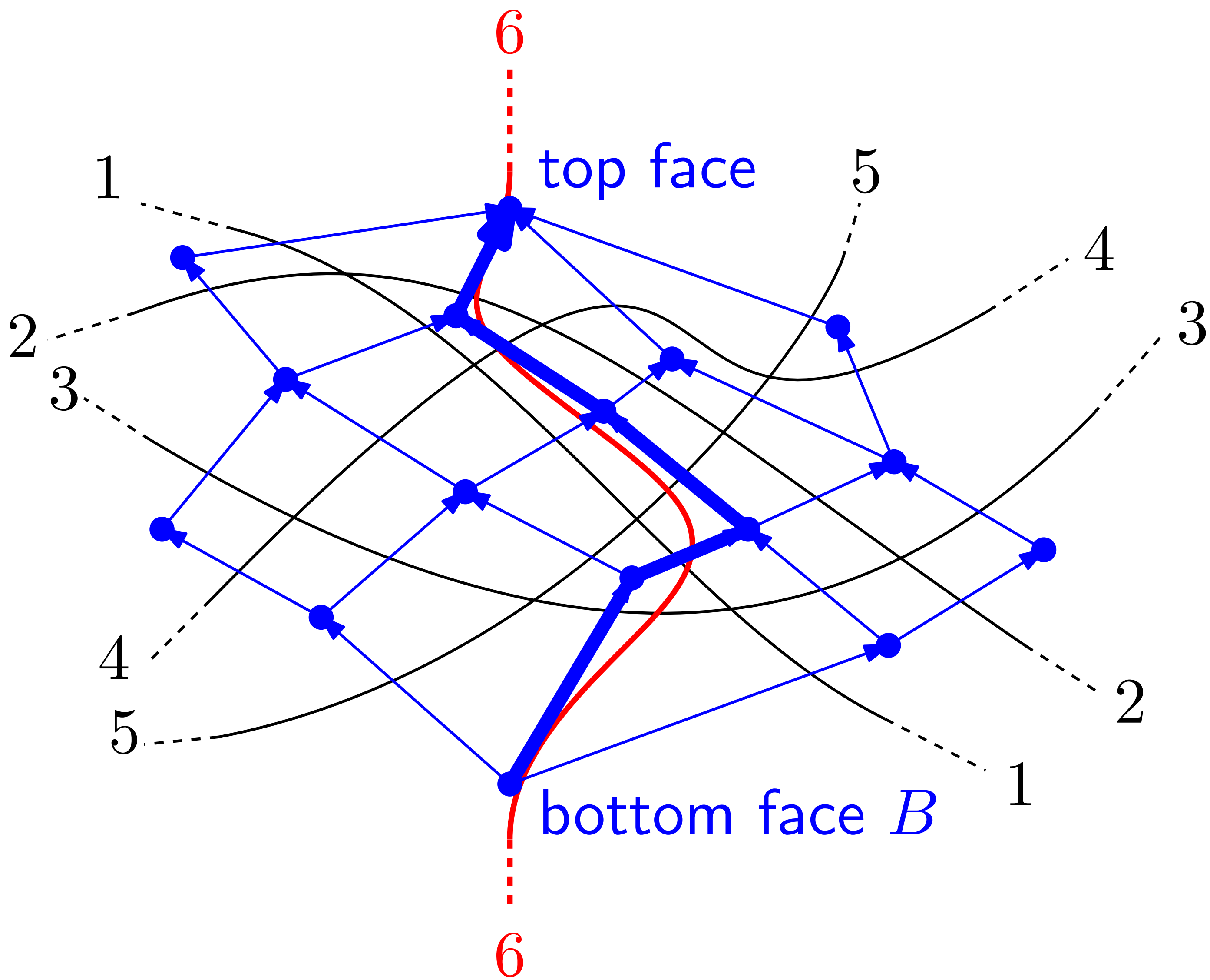


pseudoline $n+1$ = path in the dual DAG

Inductive Enumeration of PsA's



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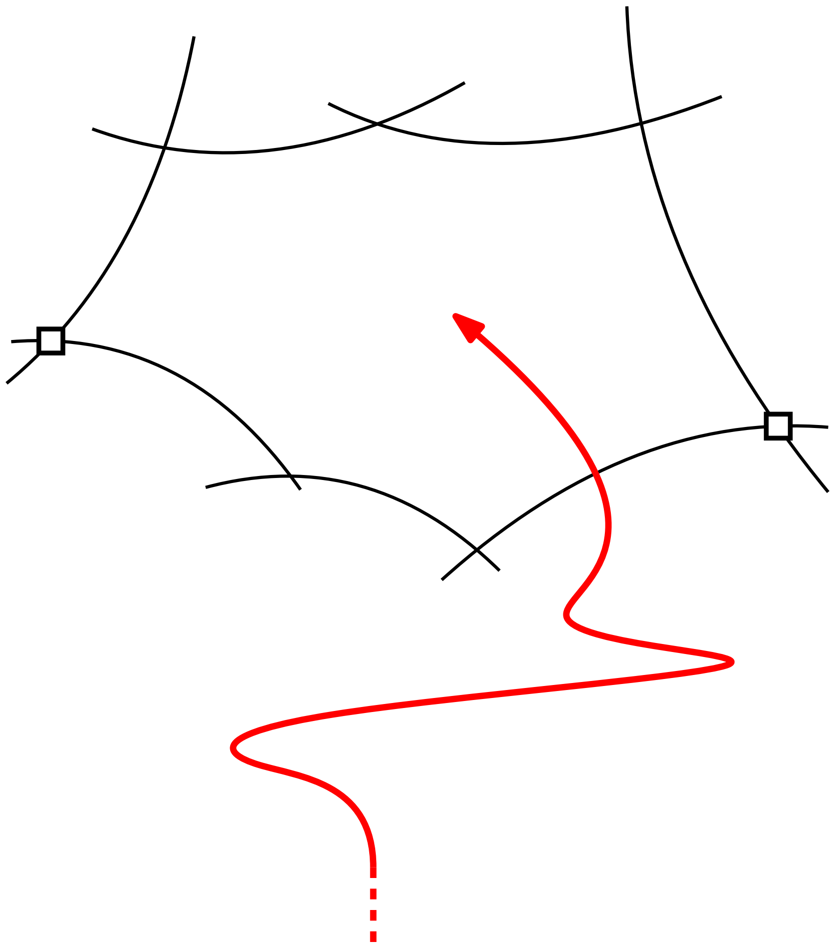
Inductive Enumeration of PsA's

Generation (enumeration) is straightforward. (No dead ends!)



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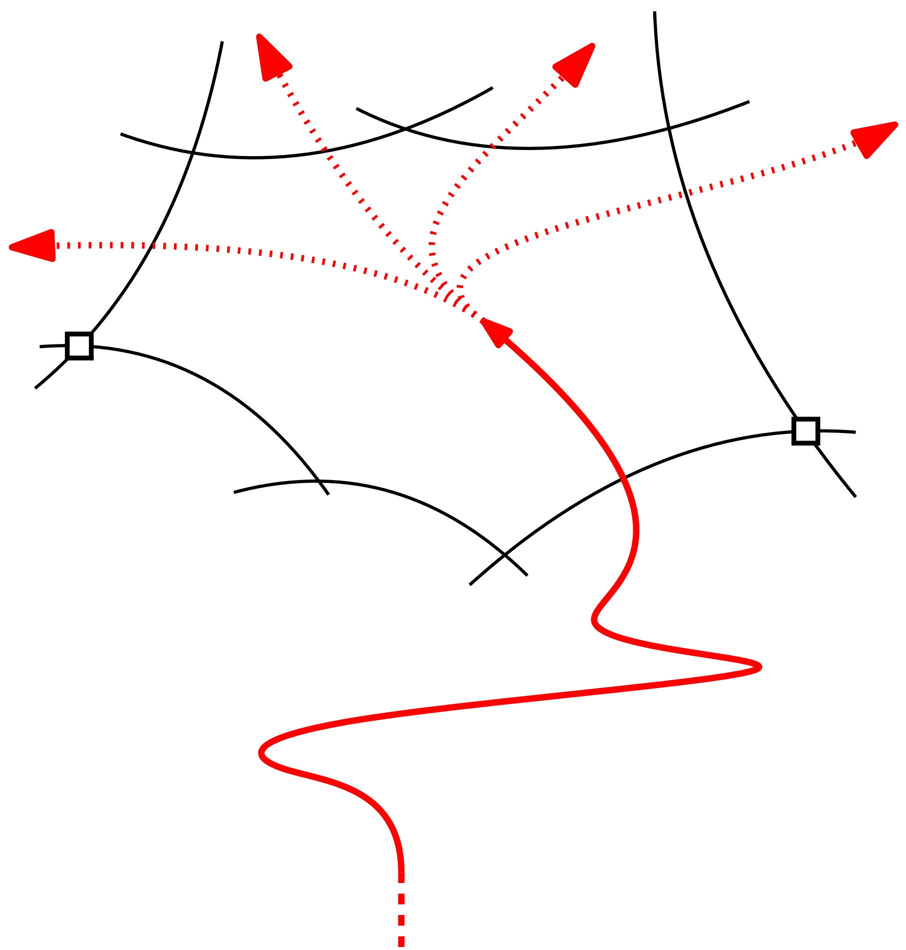
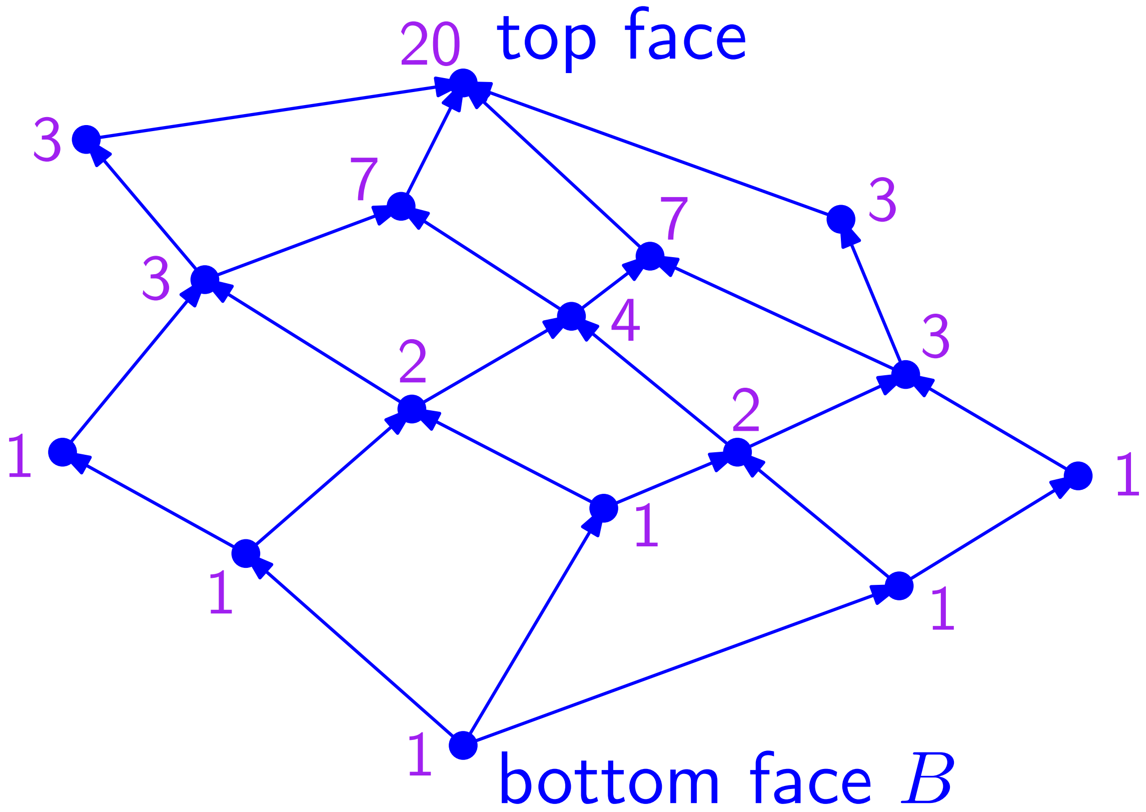


Inductive Enumeration of PsA's

Counting is straightforward. (#paths from B in a DAG)



#paths $\leq 2.49^n$
[Felsner, Valtr 2012]

#paths can be as large as 2.076^n.
[O. Bílka 2010]

pseudoline $n+1$ = path in the dual DAG

Threading several pseudolines at once

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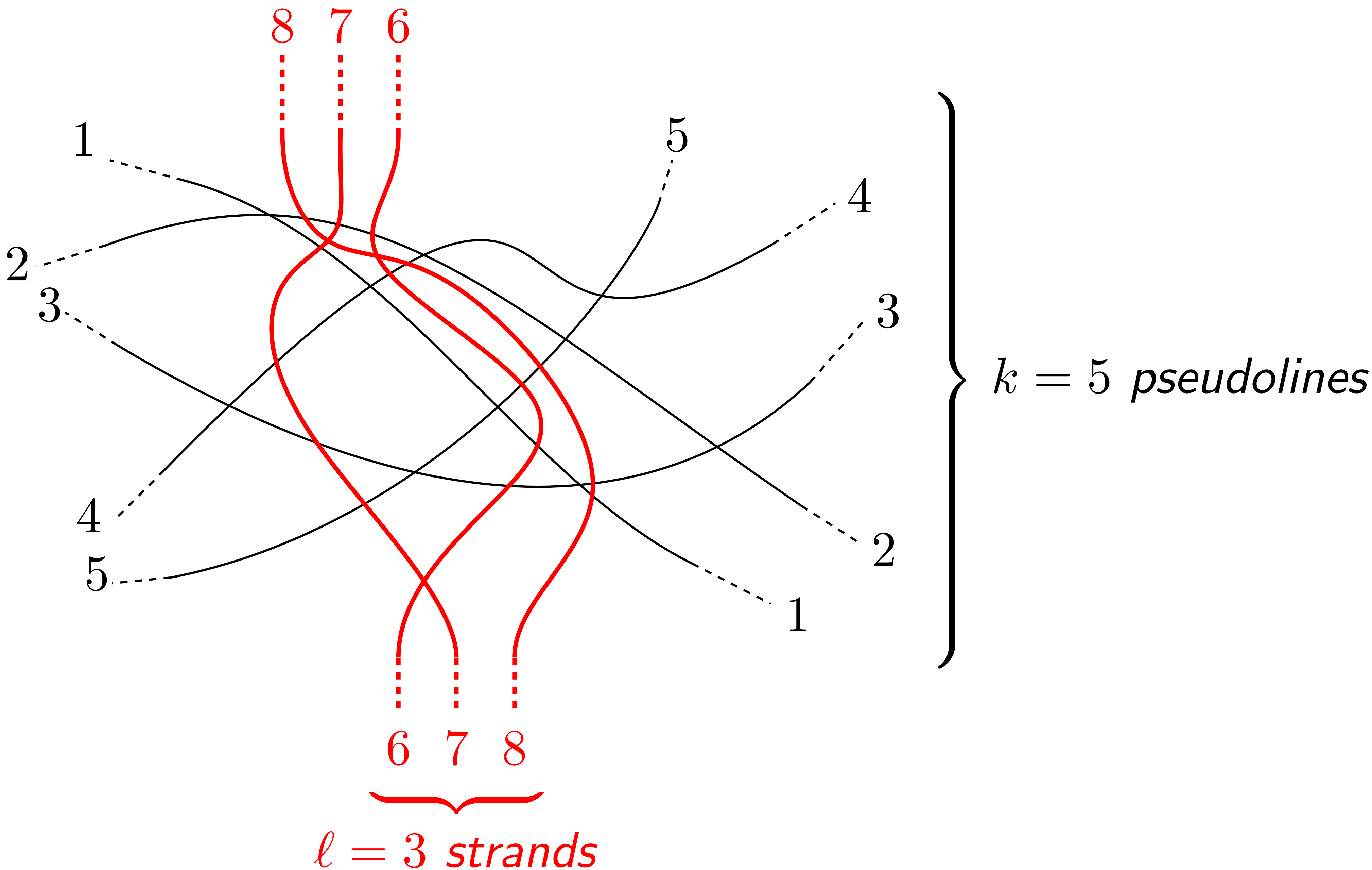

2-Level approach

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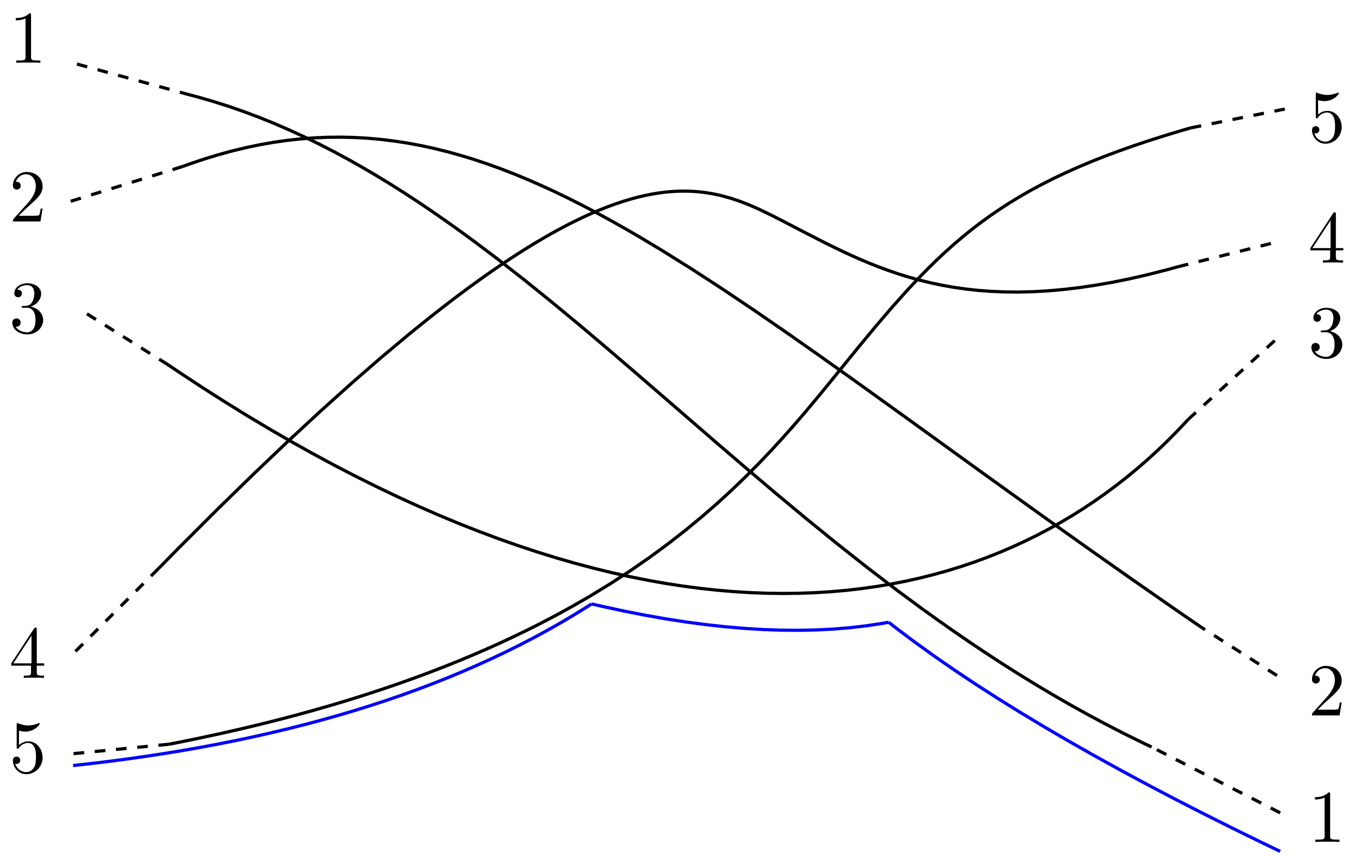
- *Enumerate* all arrangements of k pseudolines
- For each arrangement of k pseudolines:
  - *Count* the possibilites to thread ℓ extra strands

Preprocessing:
to deal with (partial) arrangements with ℓ strands fast

A sequence of ropes



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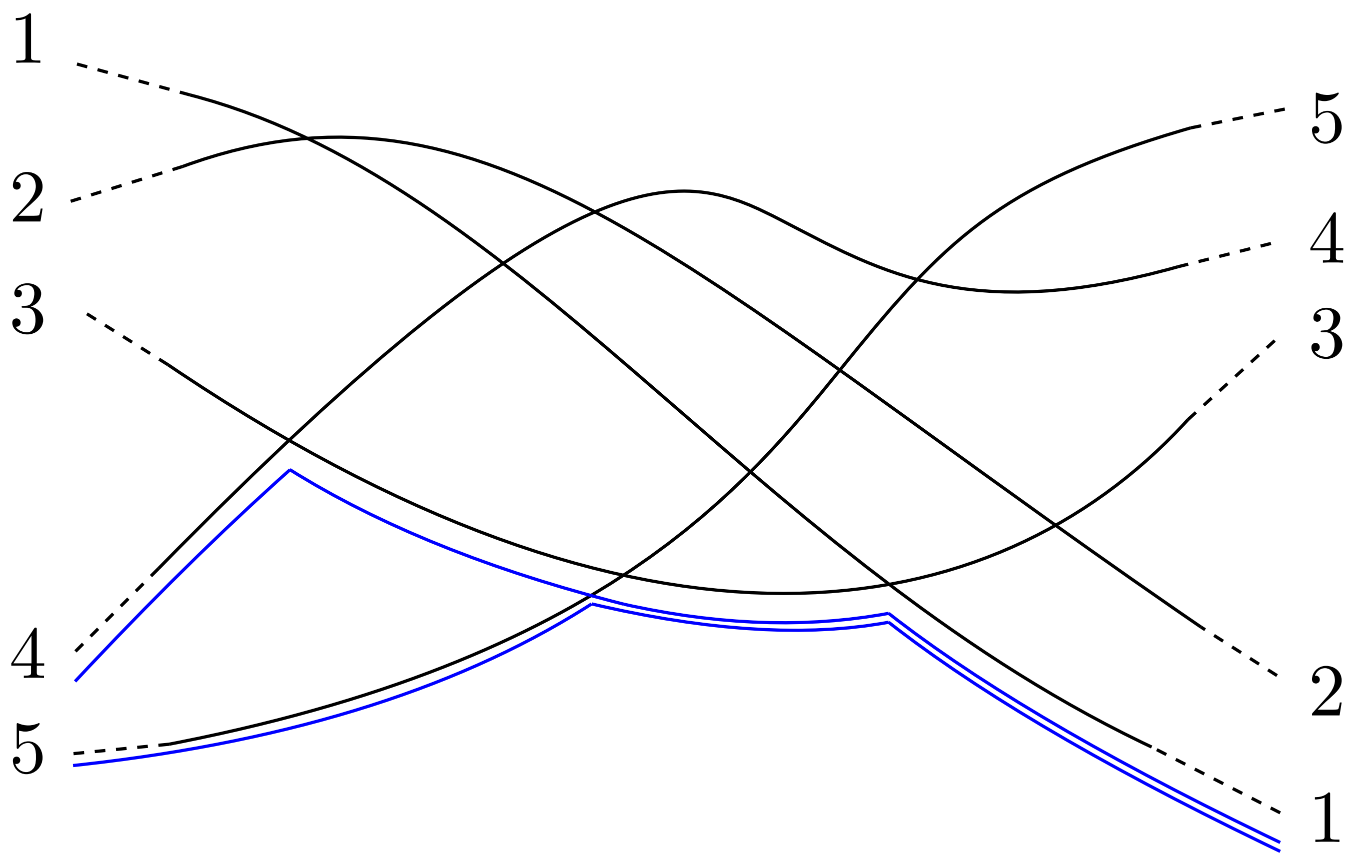

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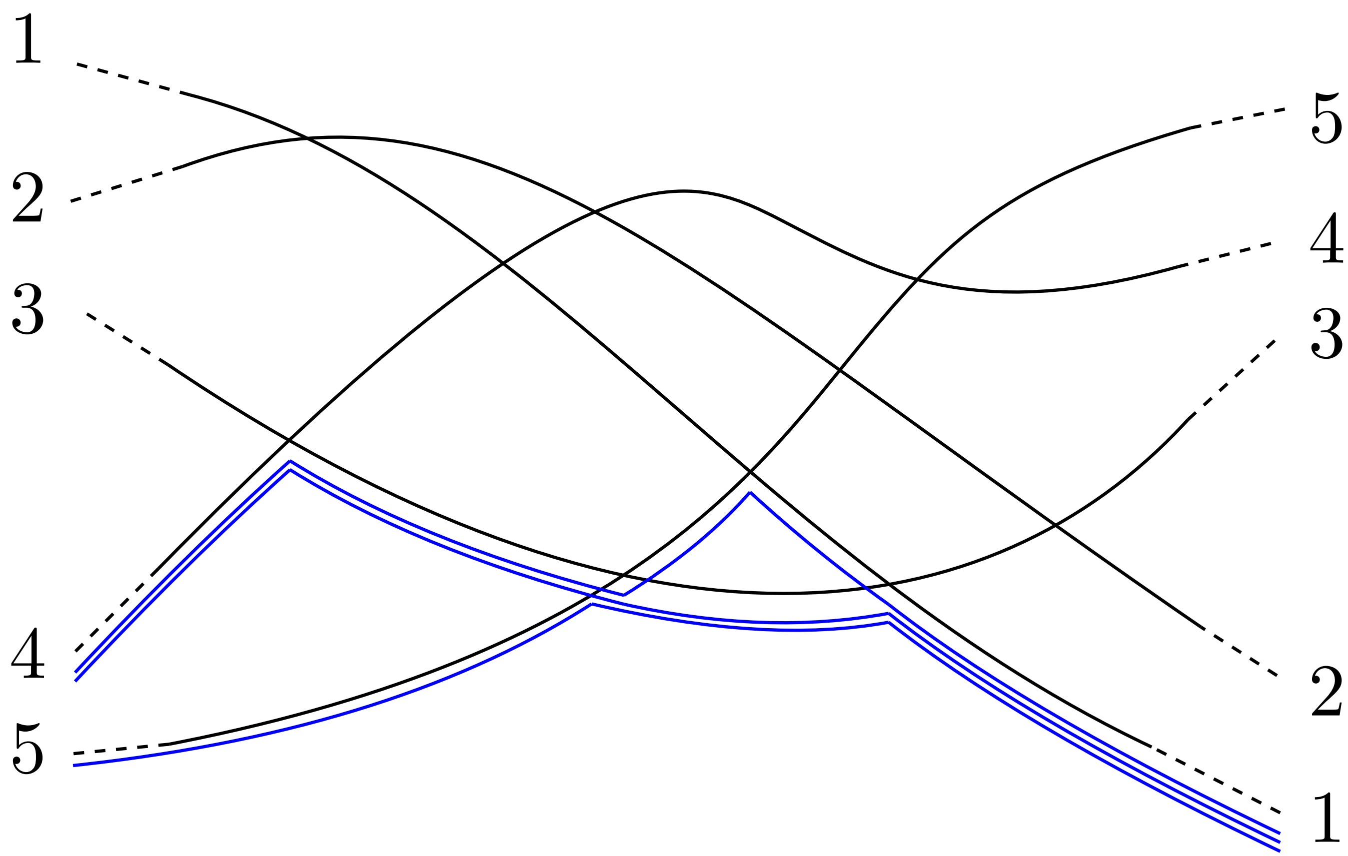

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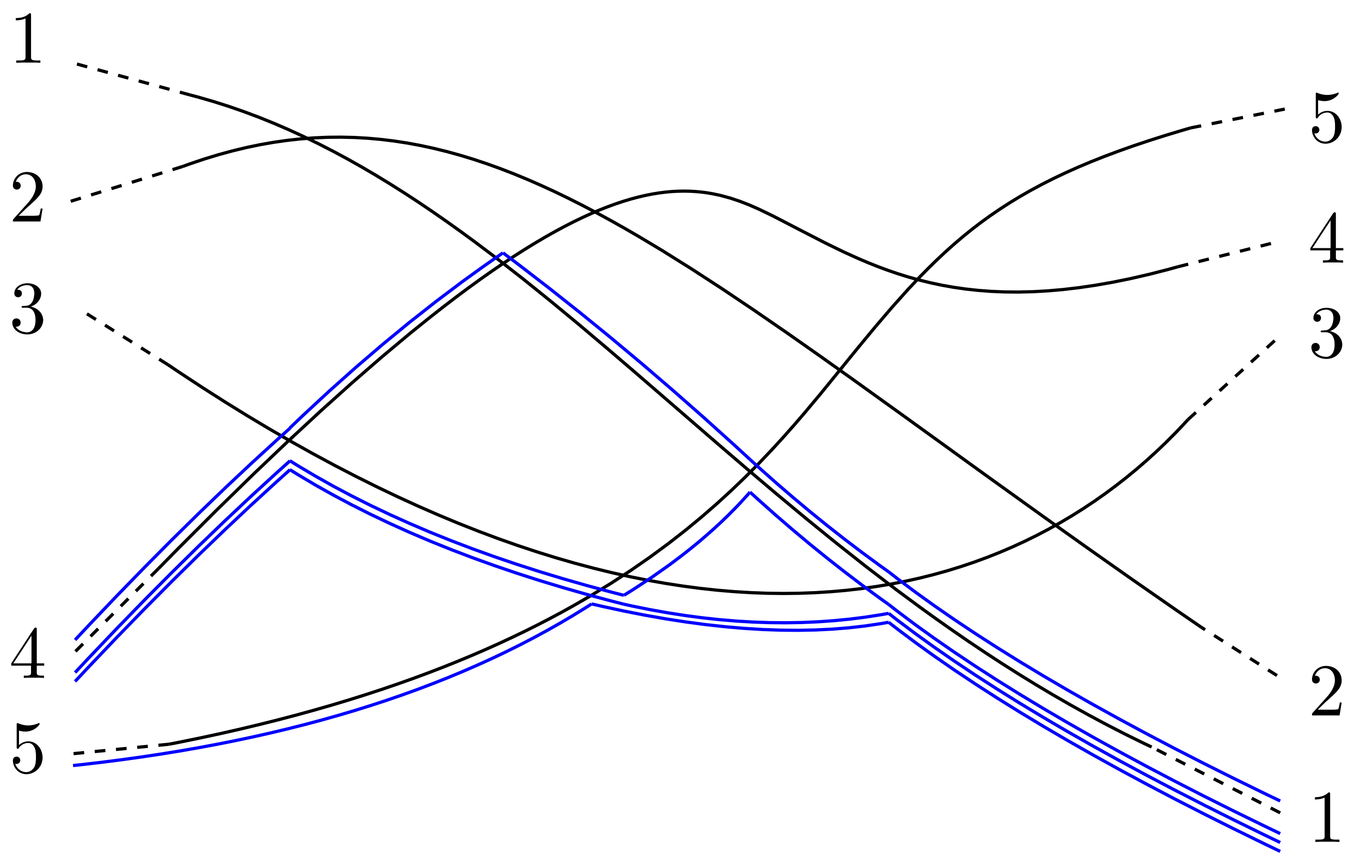

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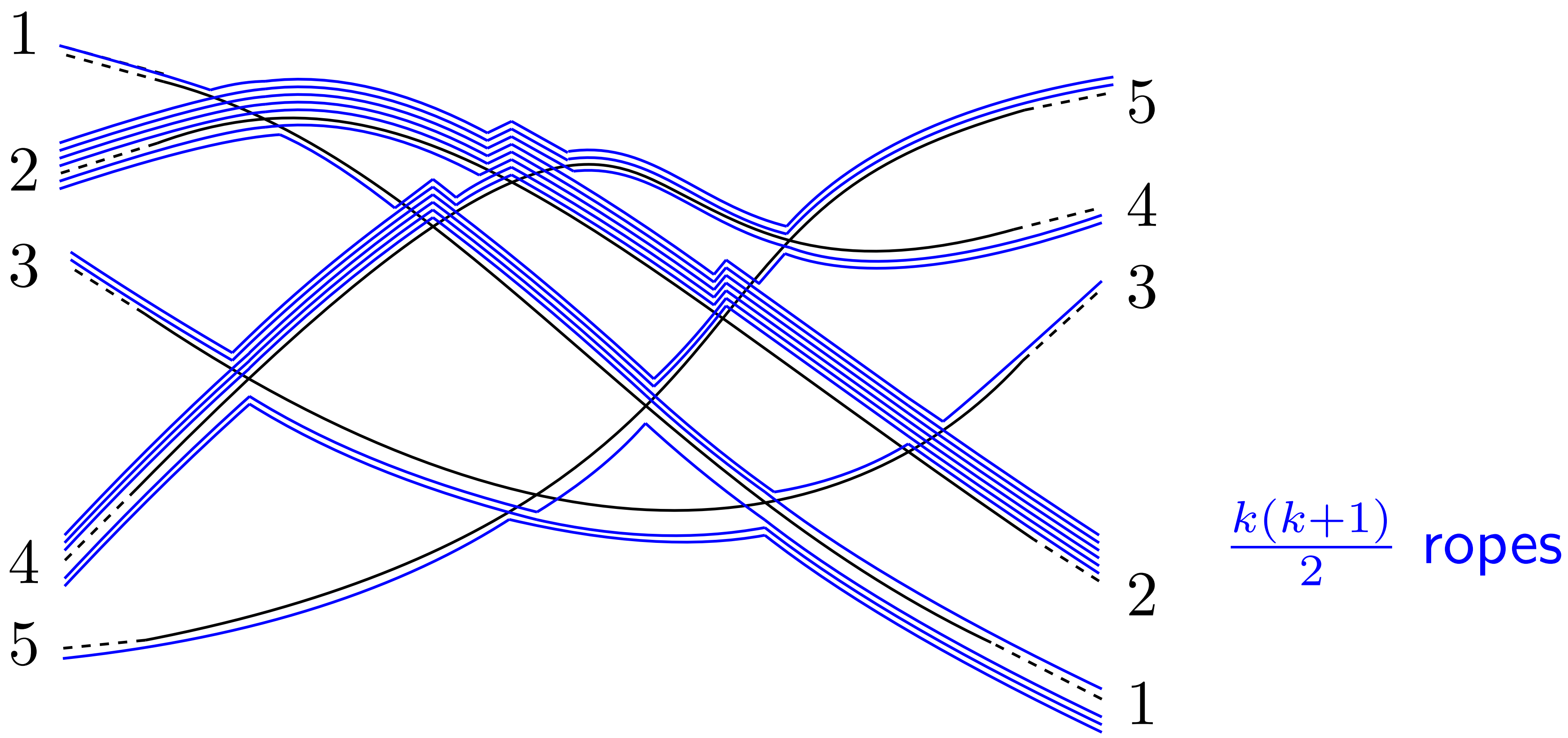
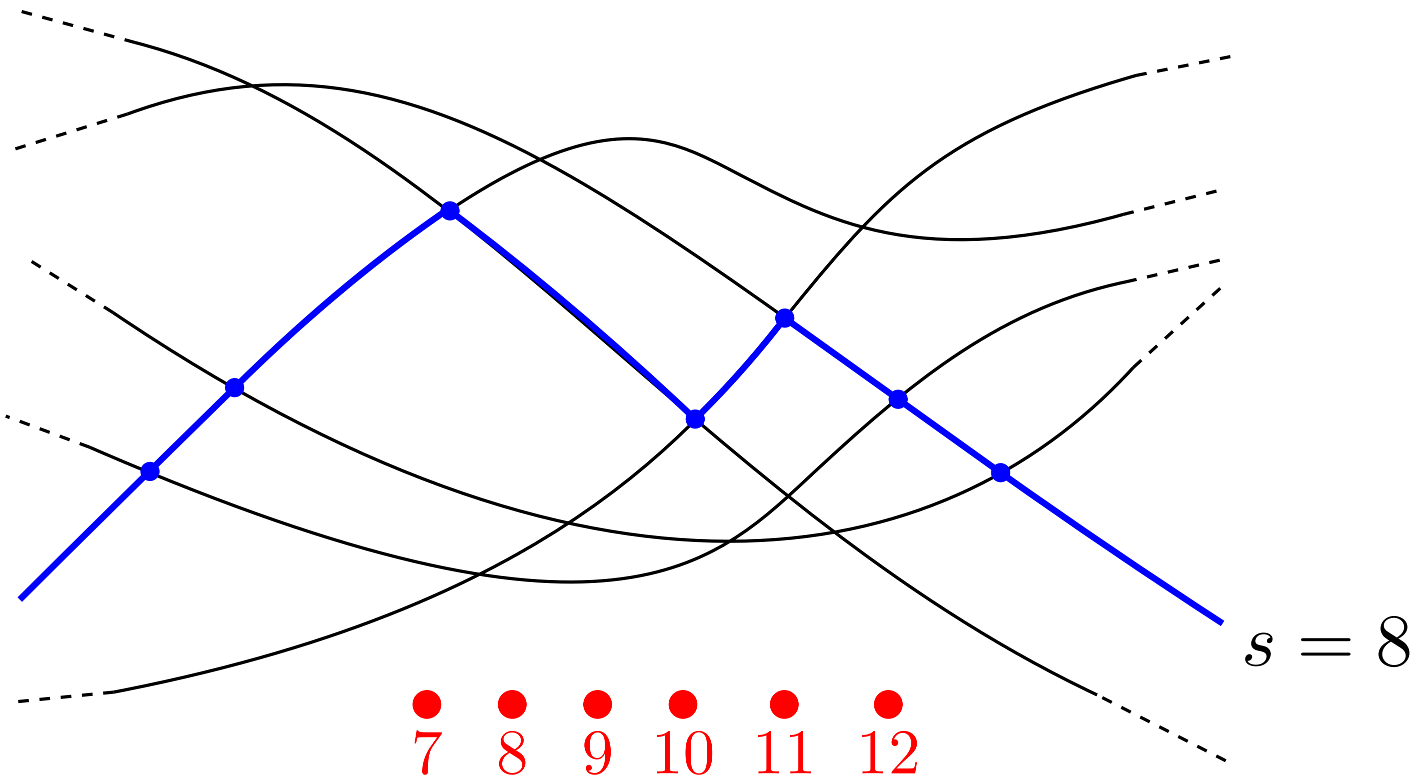


Take a fixed sweep by a sequence of ropes.

Dynamic programming

For each rope:

(s pieces)



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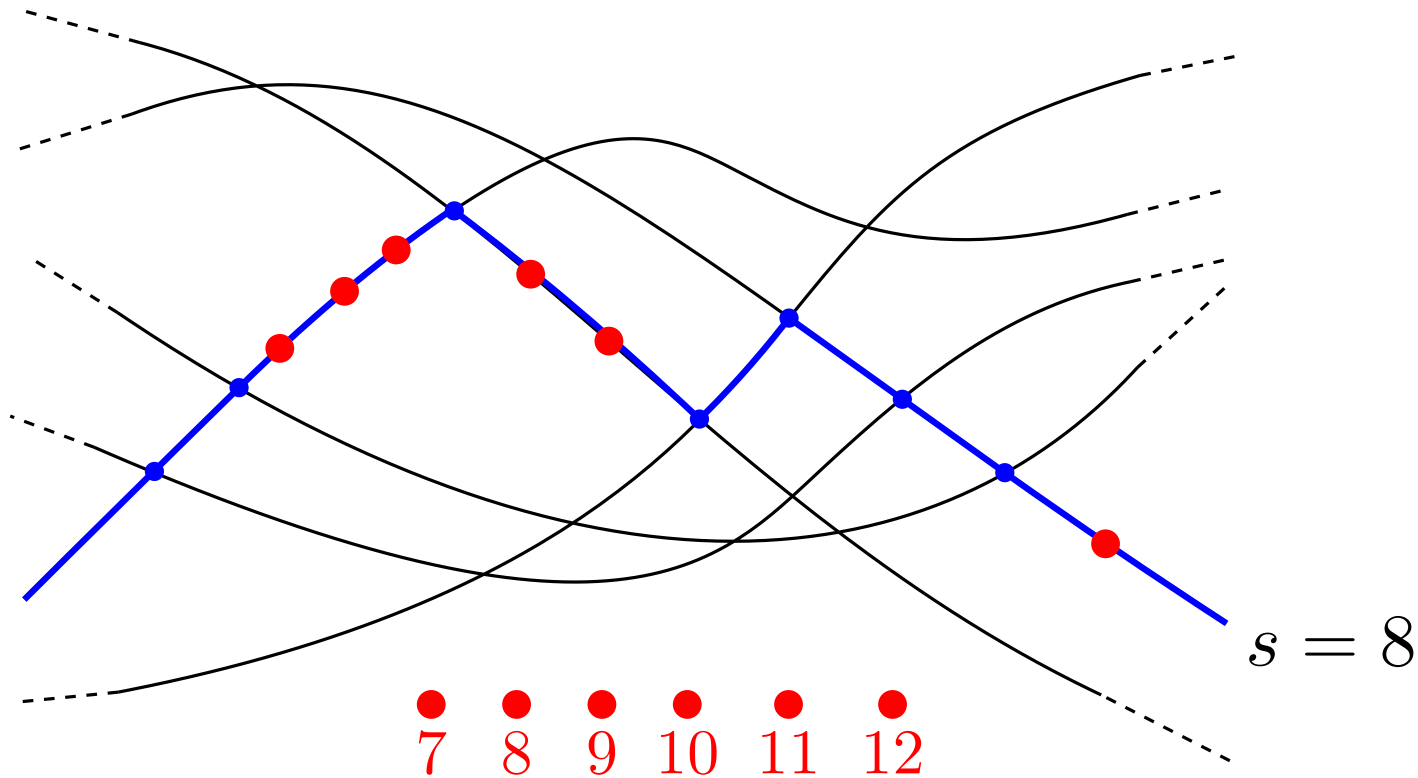

- For every distribution of the ℓ strands to the s pieces
- and for every permutation of the ℓ strands:

Store the number of possibilities to thread the ℓ strands from the bottom face to the rope.

$\rightarrow s(s+1)(s+2)\ldots(s+\ell-1)$ entries

Dynamic programming

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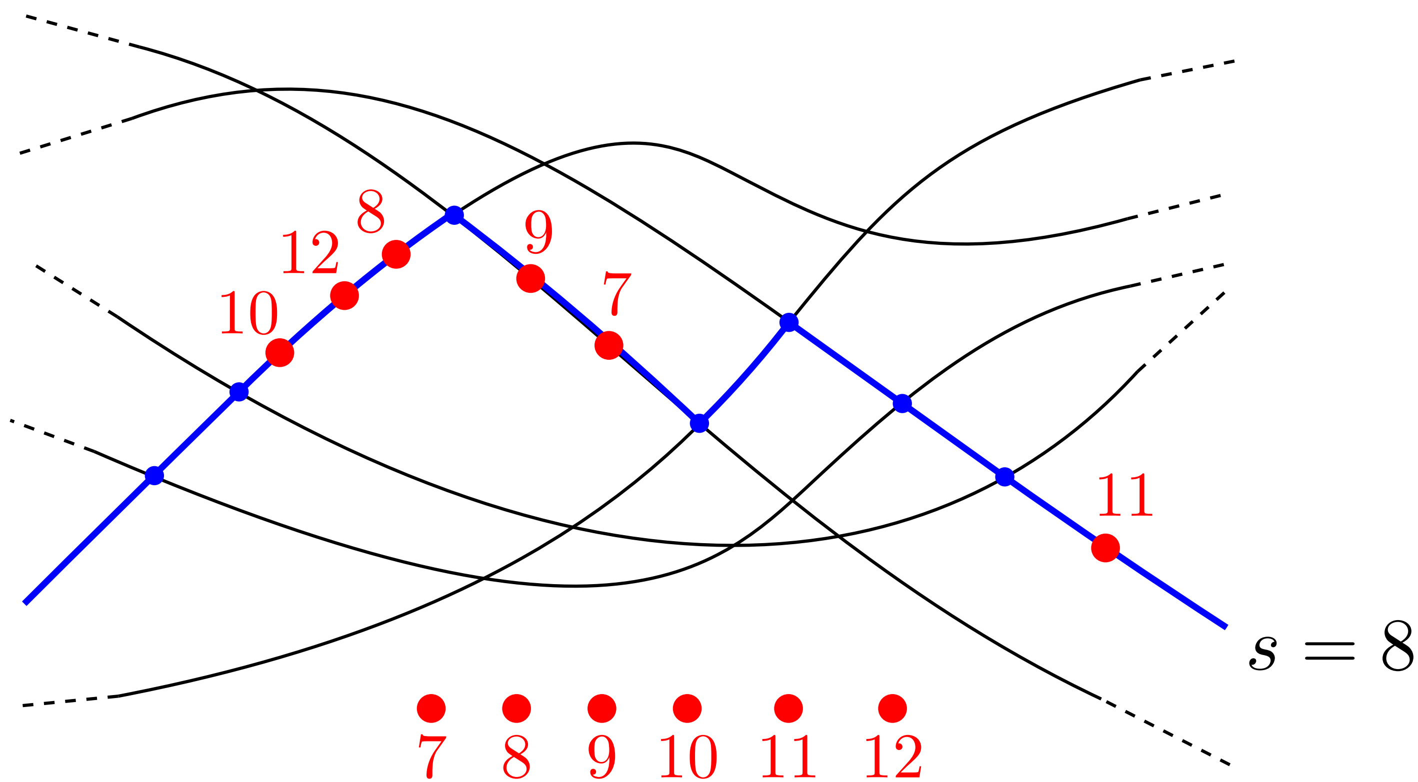
For each rope:

(s pieces)



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Dynamic programming

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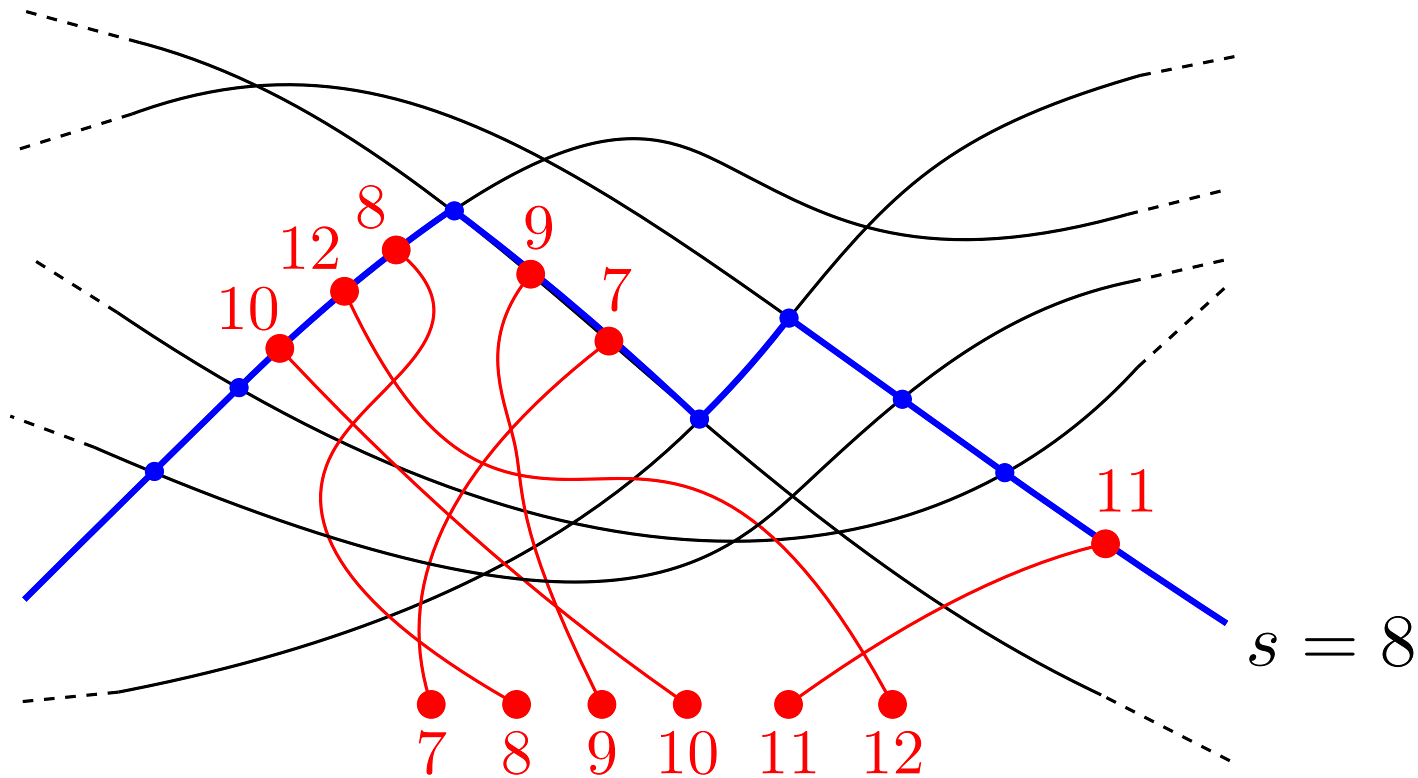

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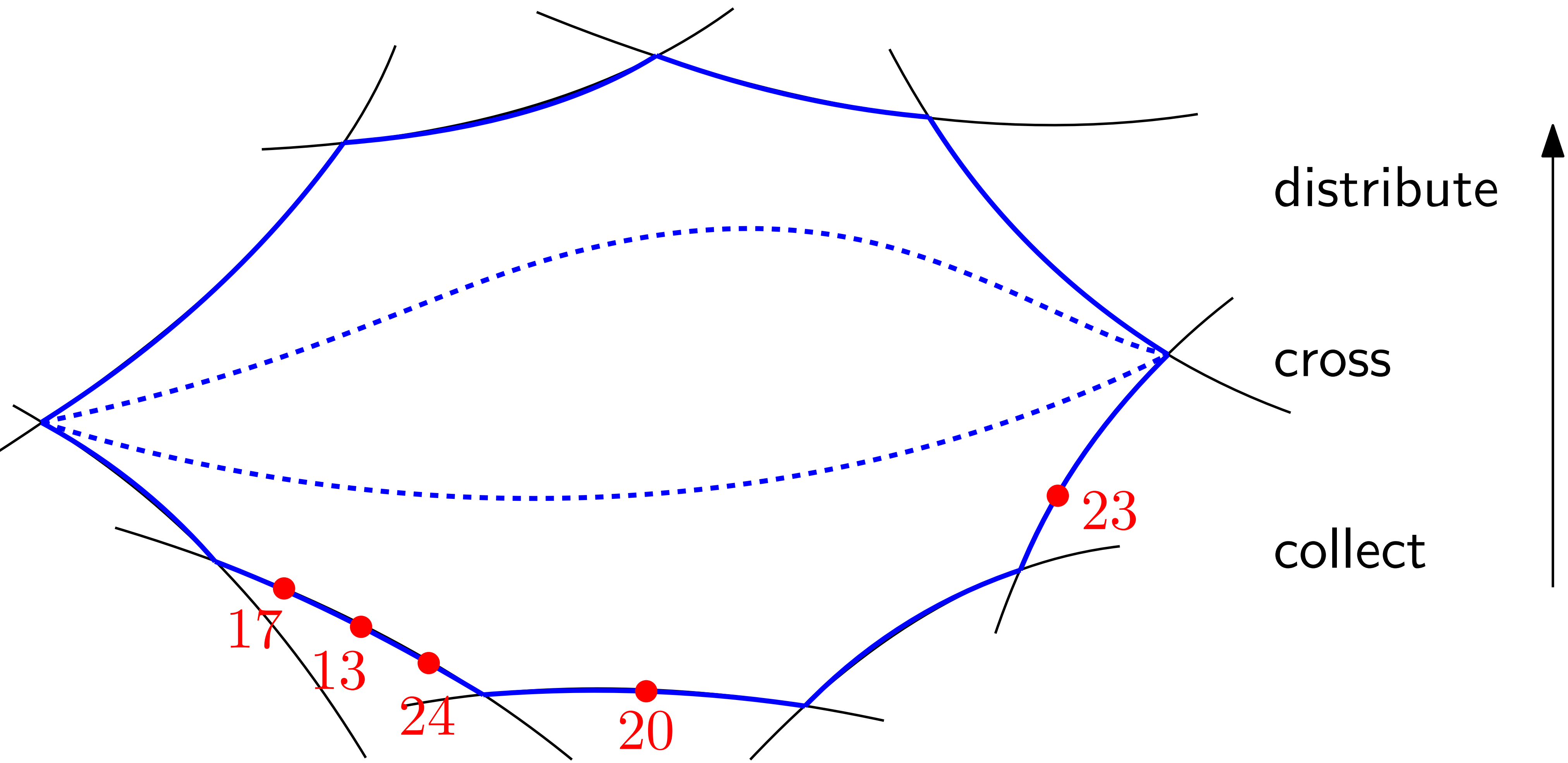
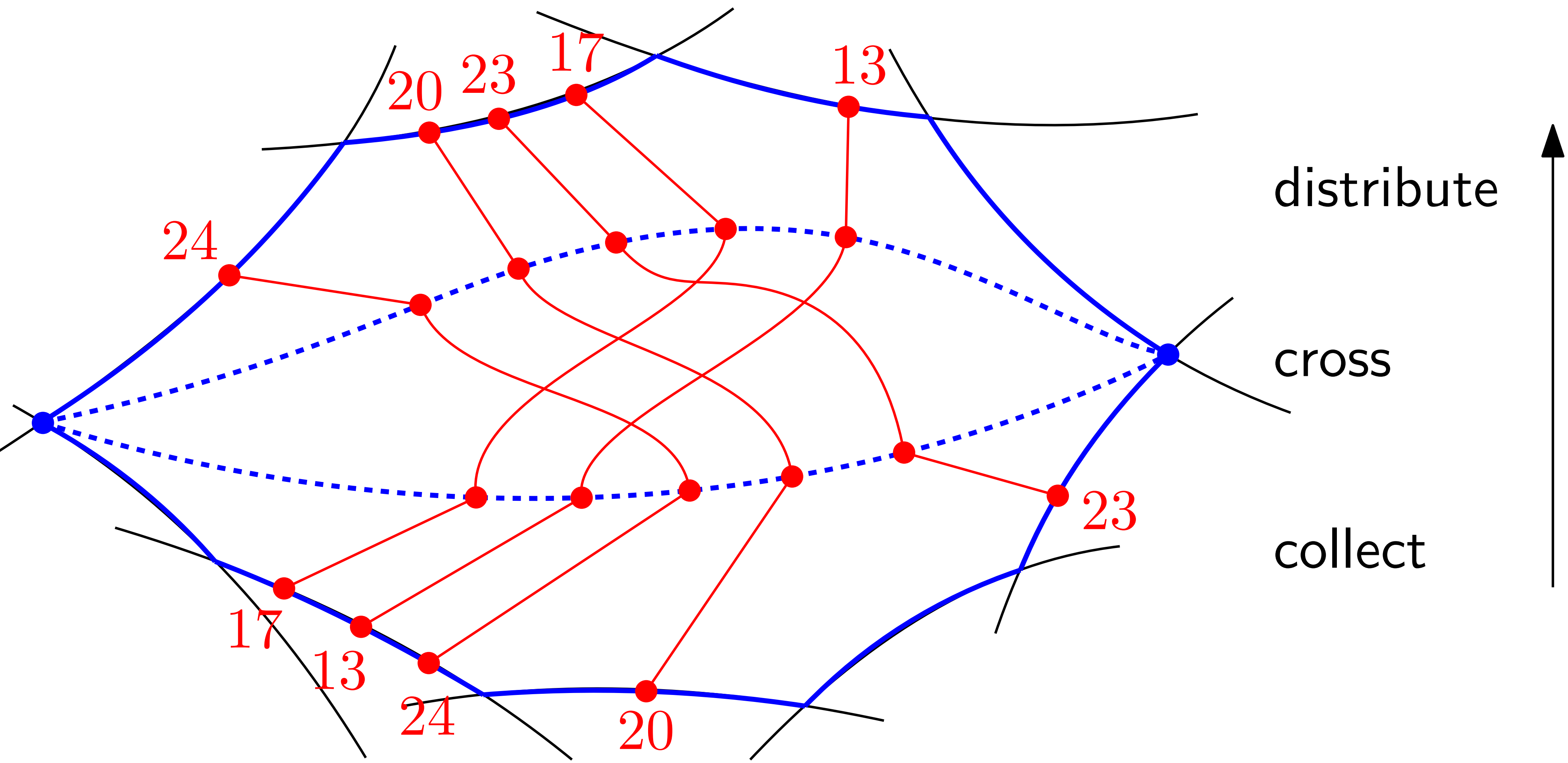
Advancing the rope across a face



What is the contribution to the next rope?

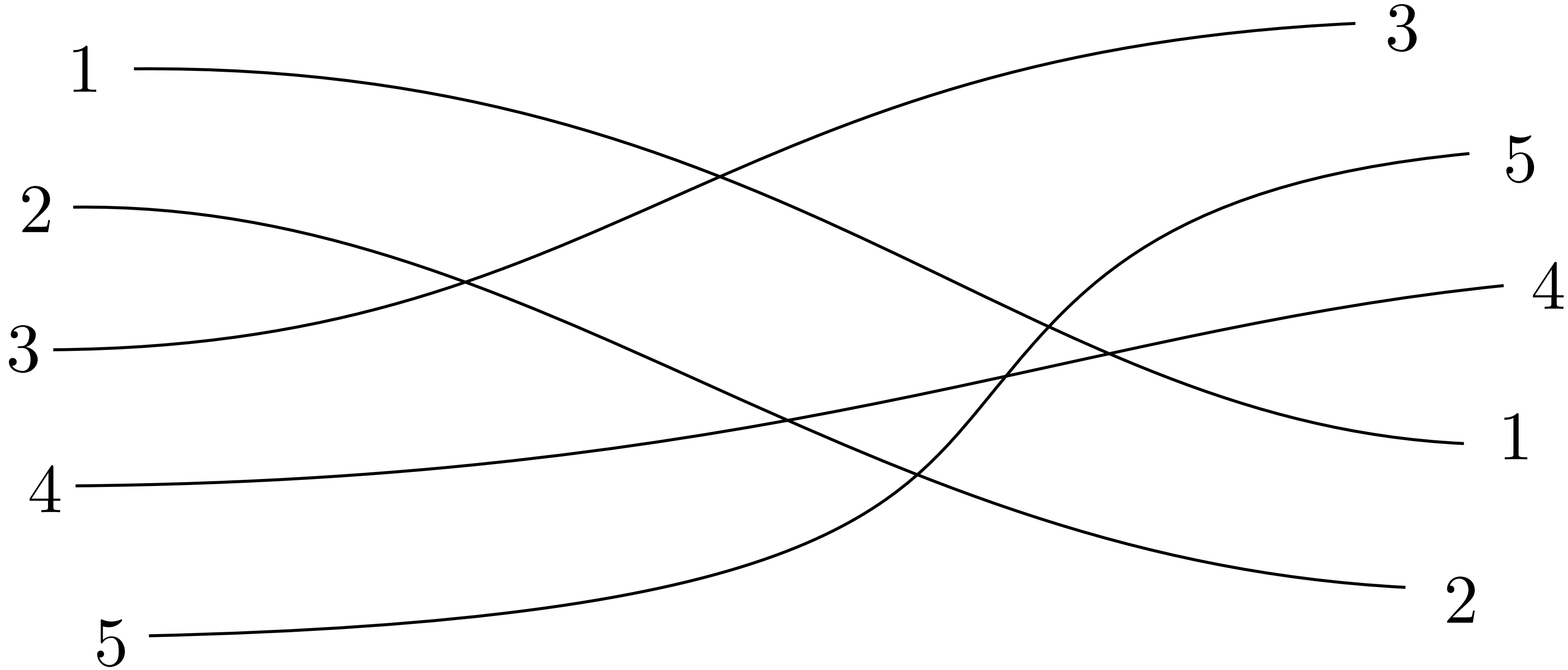
Advancing the rope across a face



What is the contribution to the next rope?

PARTIAL pseudoline arrangements

The ℓ pseudolines may cross or not.



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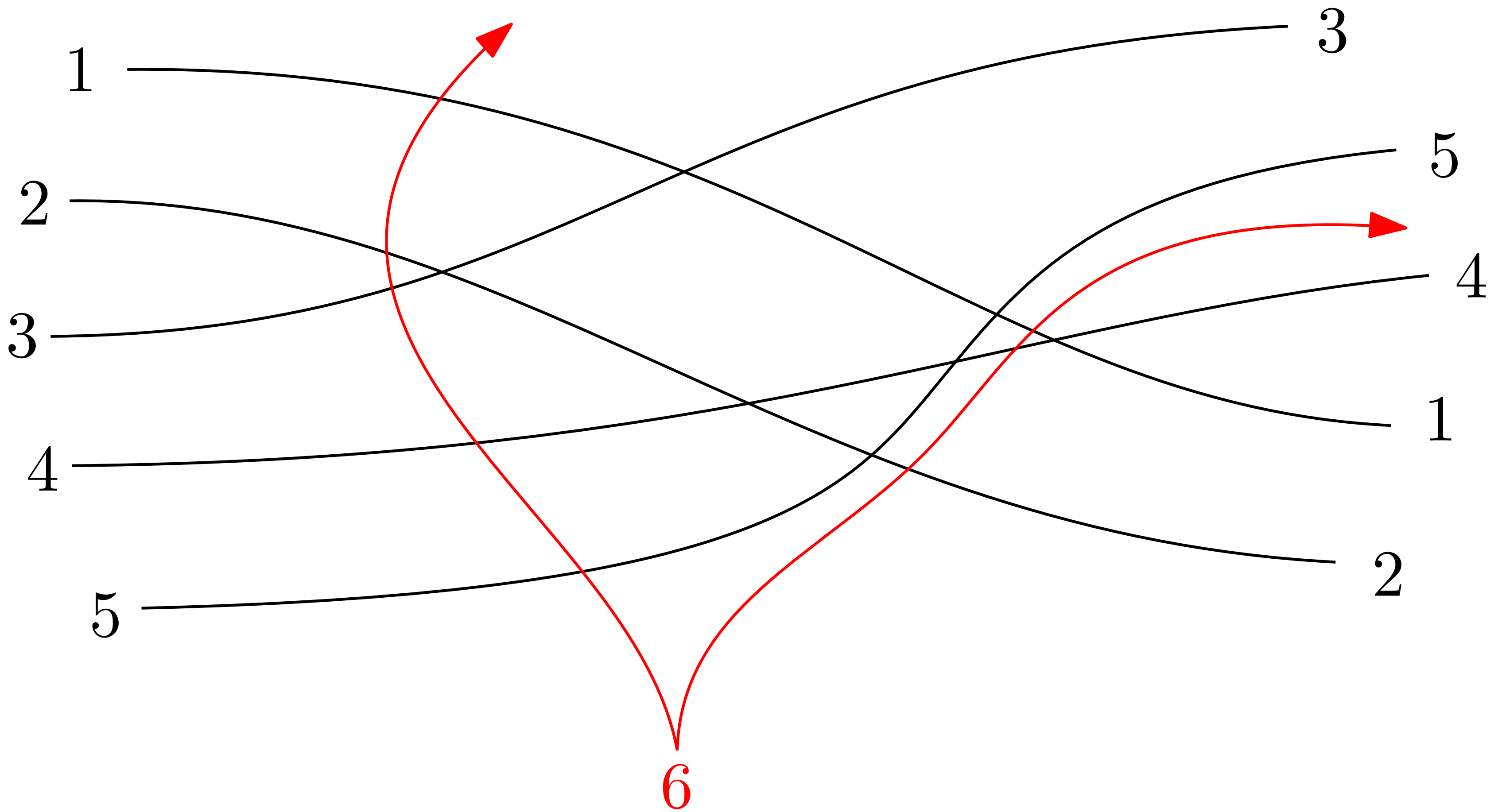

Enumeration is as easy as for full PsA's.

PARTIAL pseudoline arrangements

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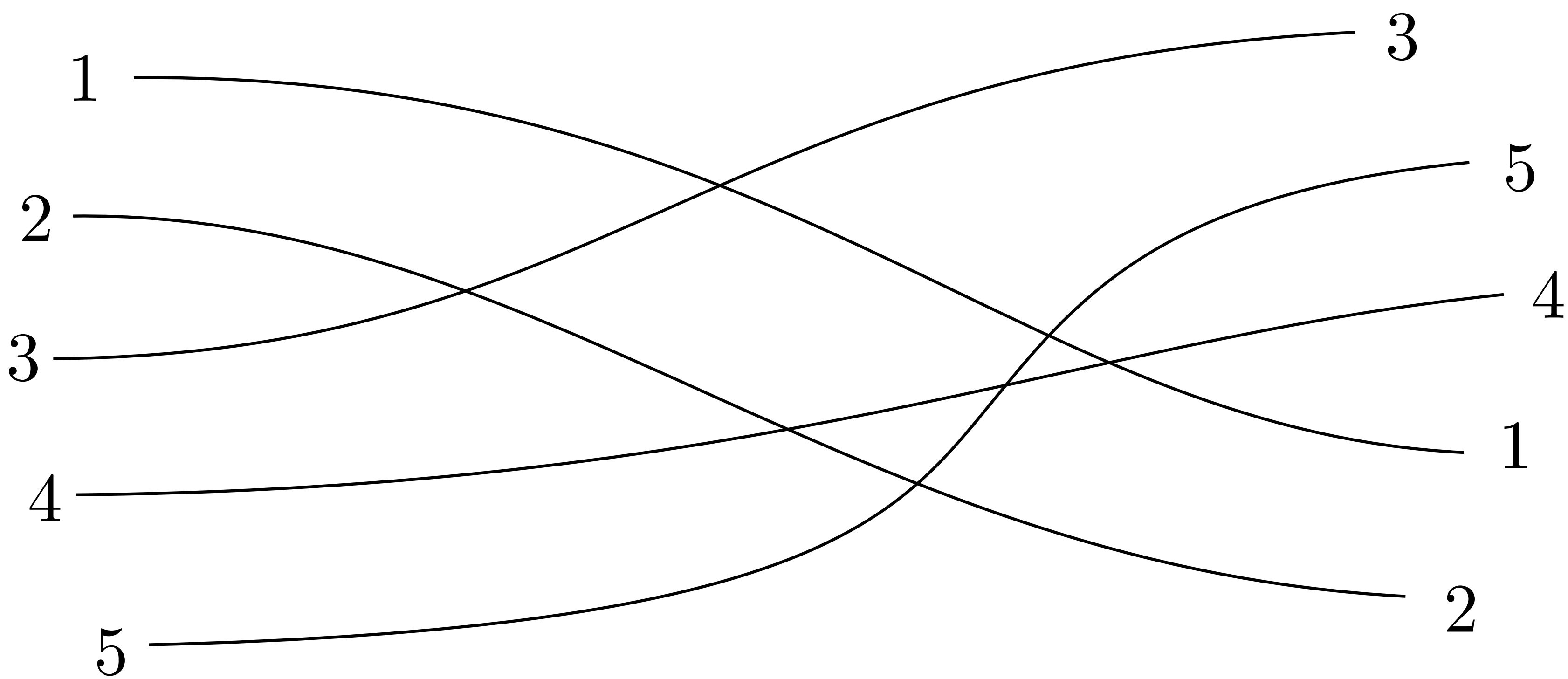

Preprocessing: $\to$ $\ell!$ array

PARTIAL pseudoline arrangements

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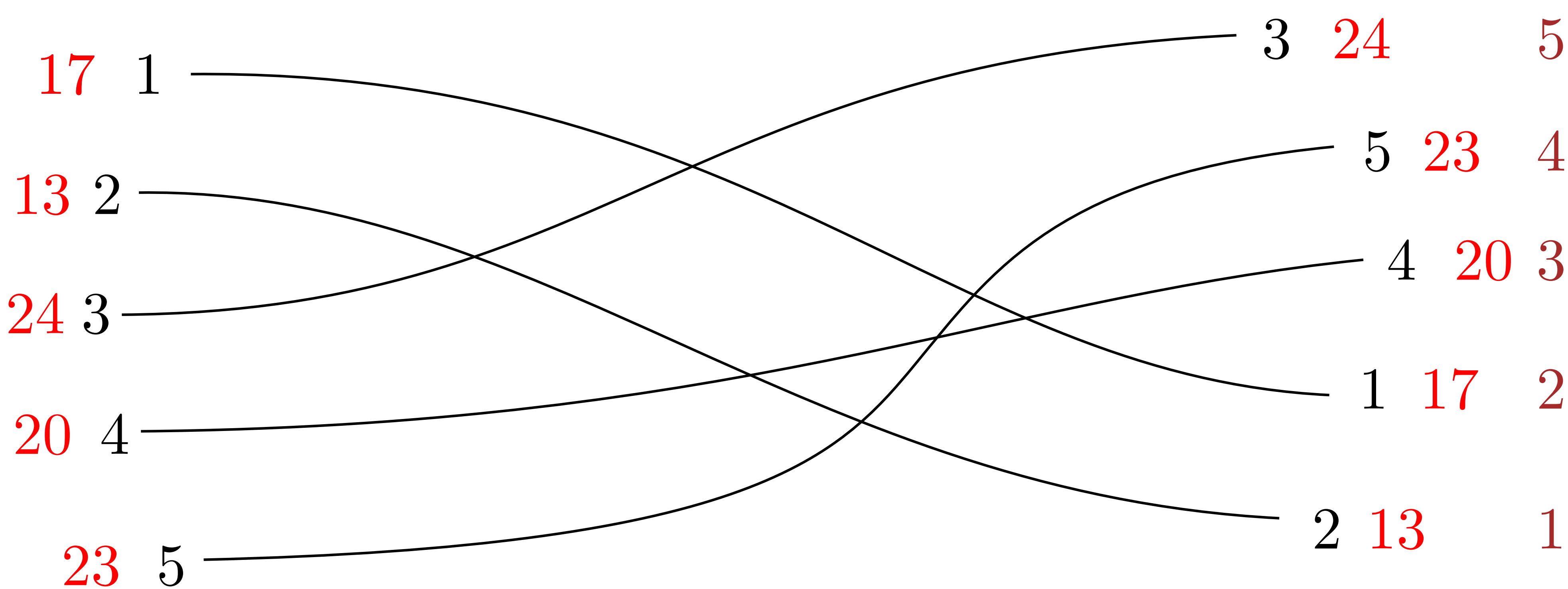
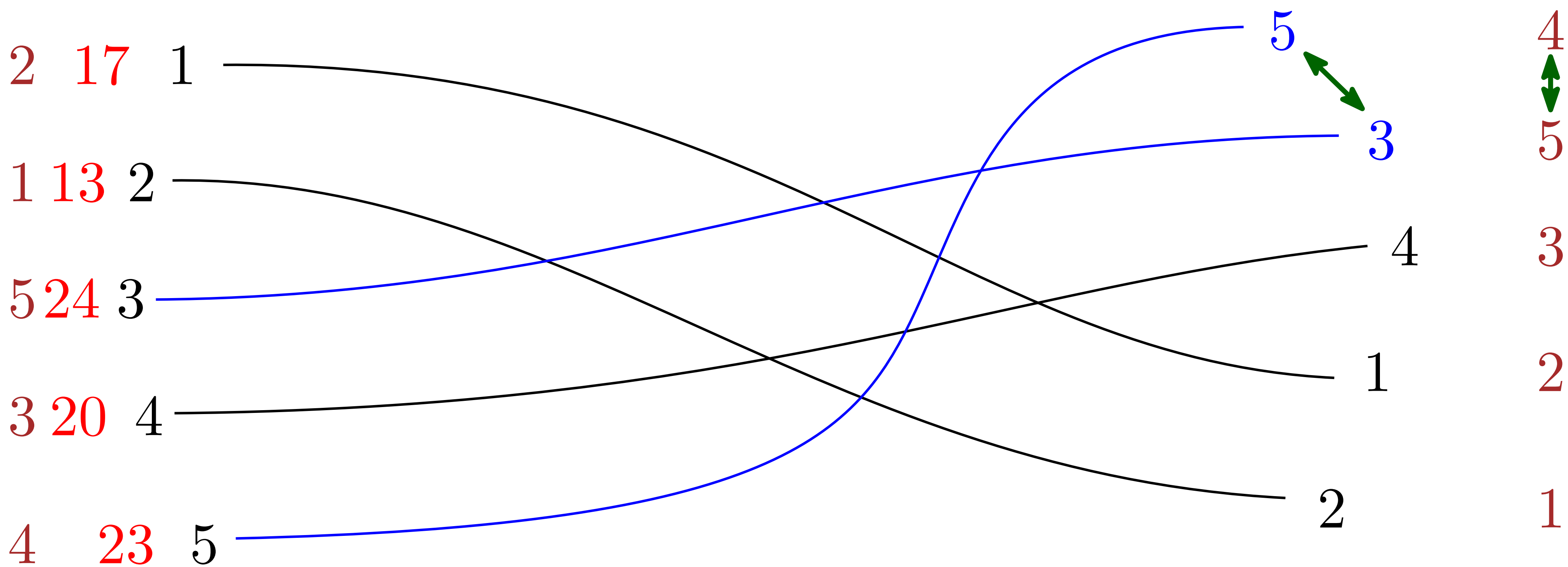


Preprocessing: $\rightarrow$ $\ell!$ array

$\rightarrow$ $\ell! \times \ell!$ table (sparse!)

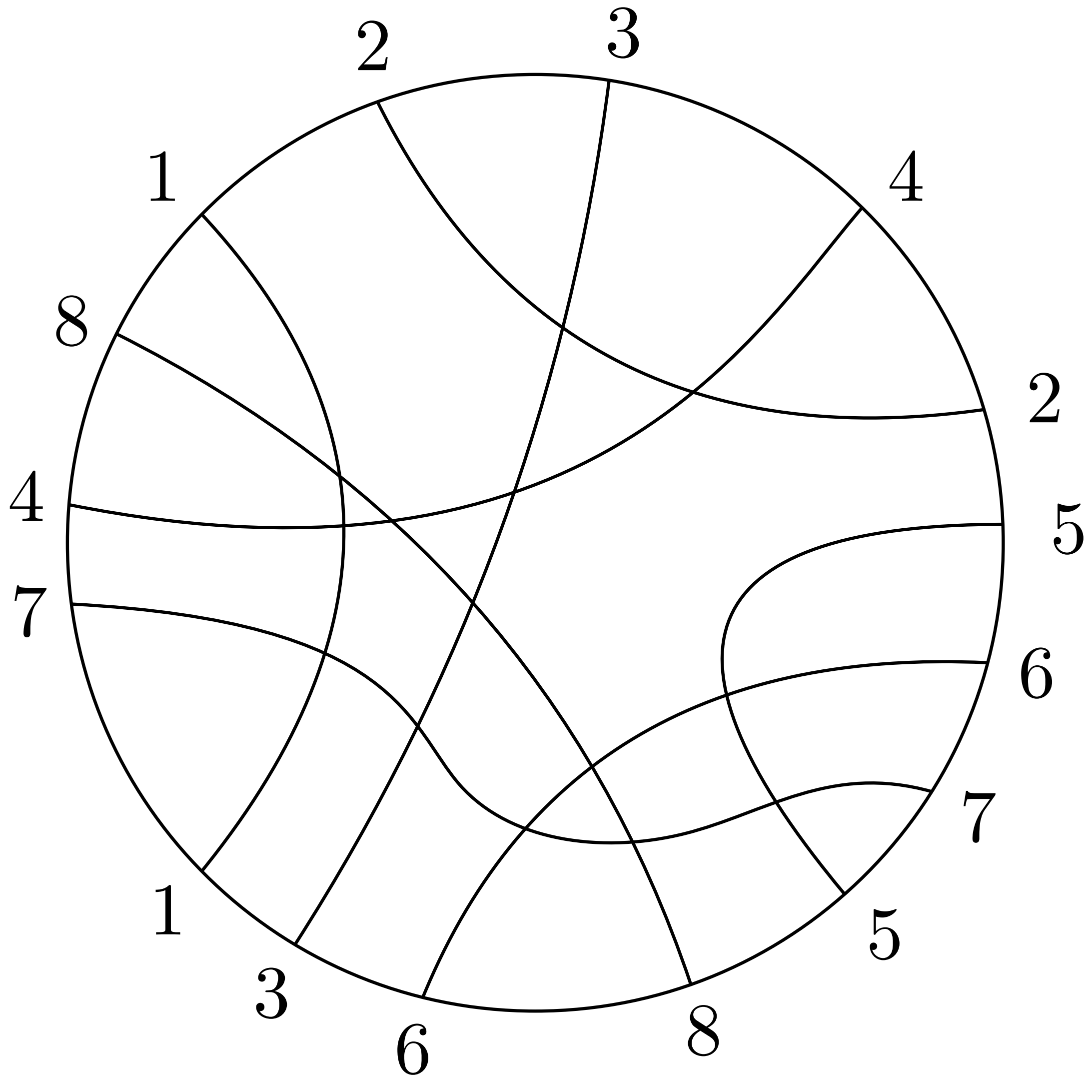
PARTIAL pseudoline arrangements



This is OK as a partial pseudoline arrangement, but not for the input sequence $17, 13, 27, 20, 23 = 2, 1, 5, 3, 4$.

PARTIAL pseudoline arrangements

Distinguish: more general *partial pseudoline arrangements* that are not necessarily x-monotone:



given by a *bipermutation* or (*matching*)

PARTIAL pseudoline arrangements

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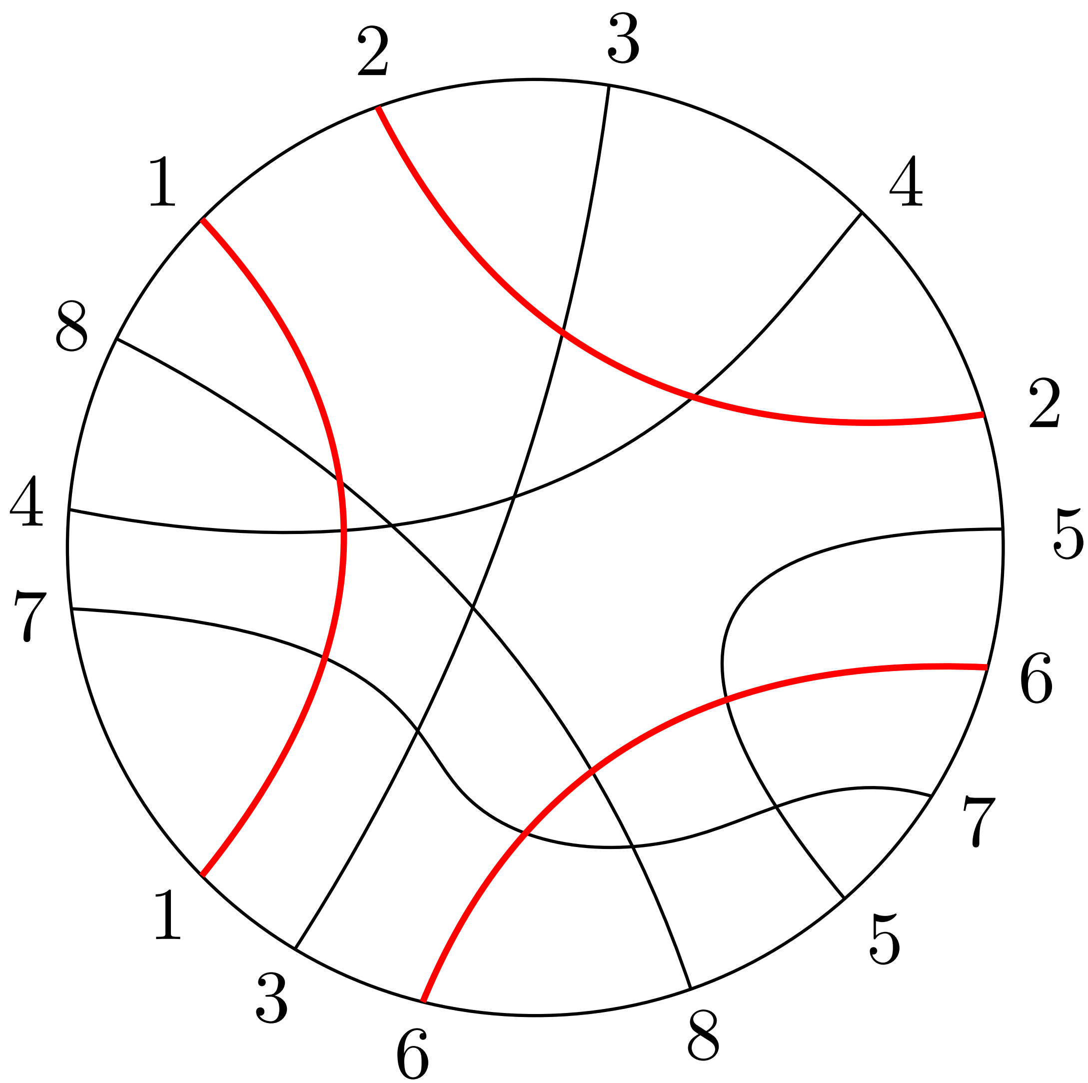

given by a *bipermutation* or (*matching*)

Algorithm summary

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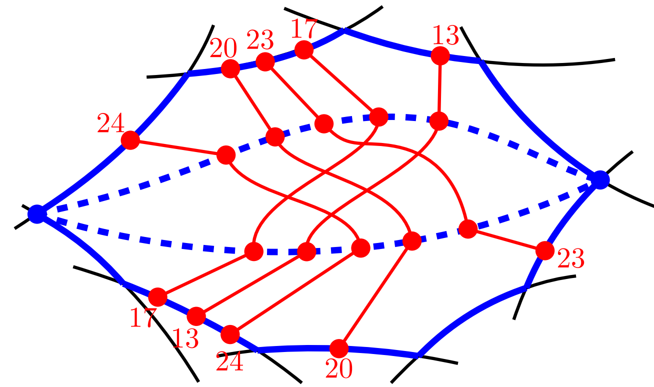
For each PsA of k pseudolines:

- Compute a sweep by ropes
- For each rope:
  - For each distribution and permutation of the ℓ strands:
    - Compute the contributions to the next rope, and accumulate them.

Network model

PSA $\equiv$ source-to-sink path

levels $\hat{\approx}$ ropes



distribute

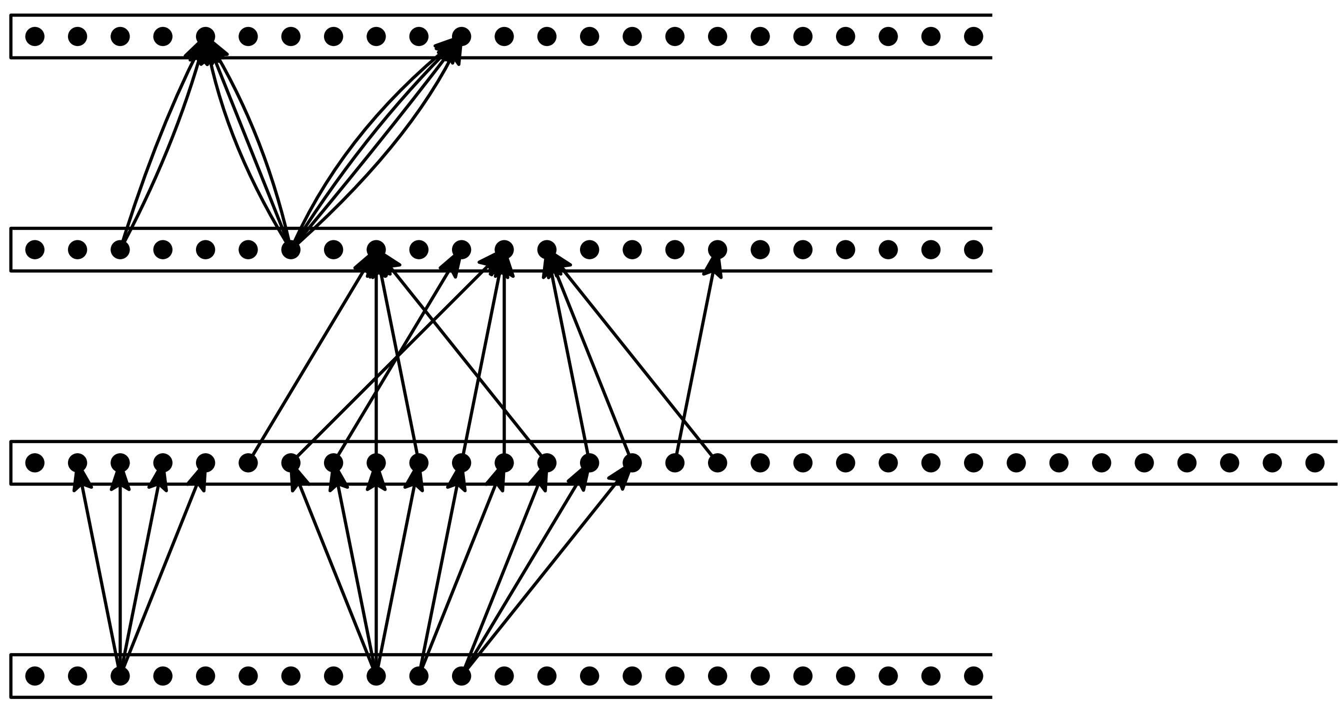
cross

collect

cross

collect

distribute



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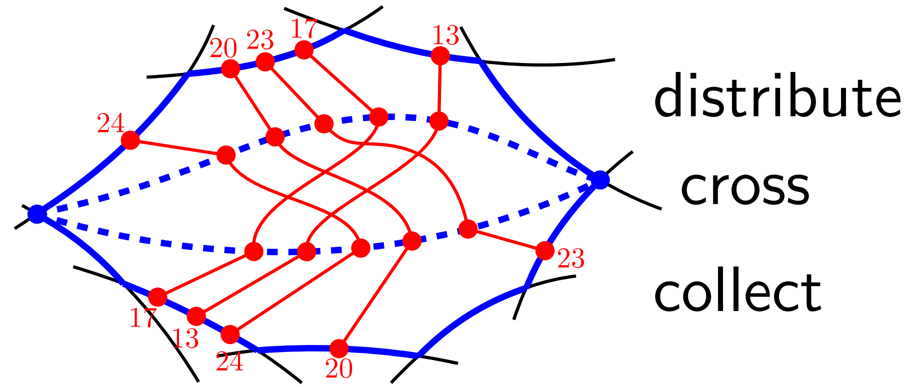
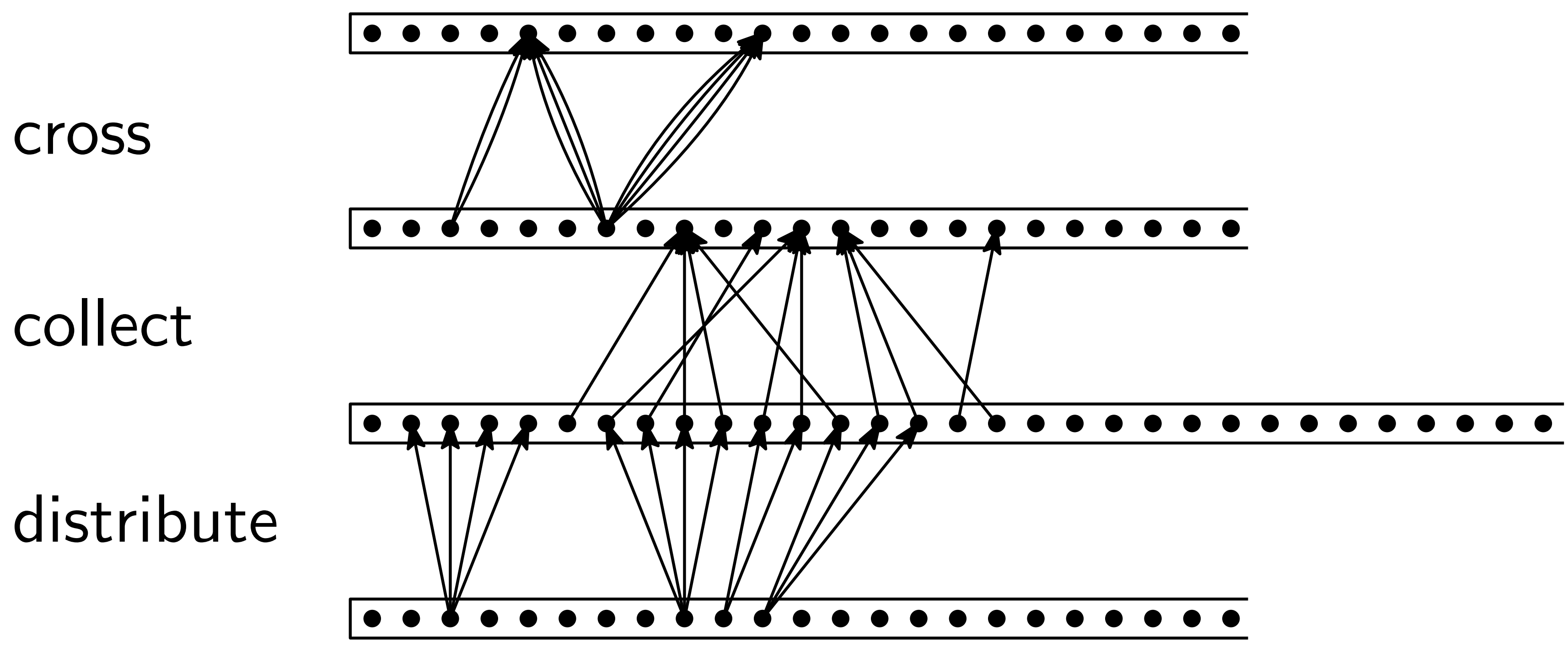
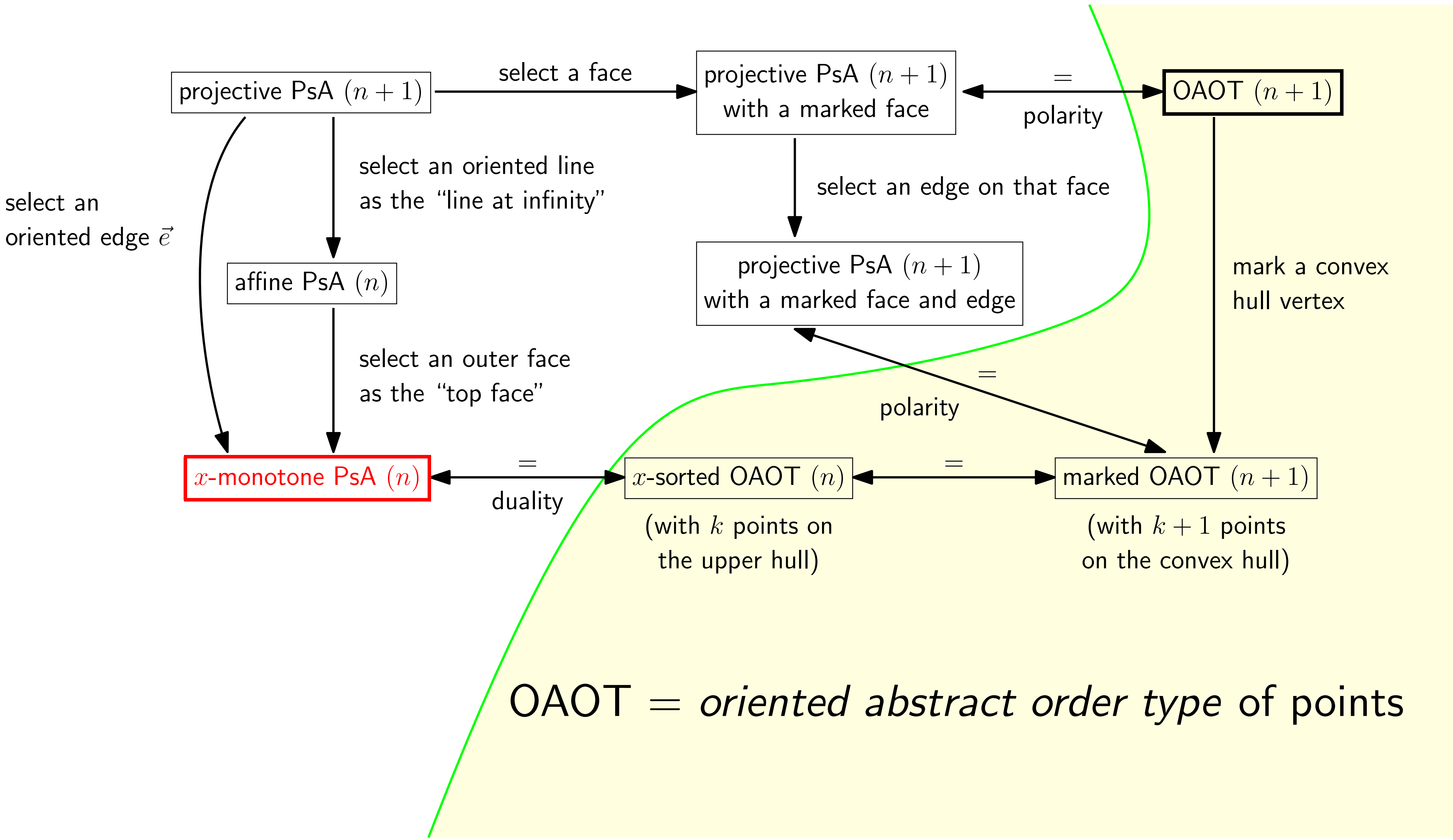




Some implementation details

- PYTHON, with `scipy` for large arrays of 32/64-bit integers
- modular arithmetic with 3 moduli: 2^{64}, two 30-bit numbers
- $n = 16 = k + \ell = 7 + 9$. Large memory! Max. "rope" $s = 7$. 256 GBytes is enough; 128 GBytes sometimes failed.
- easy to parallelize: 24,698 independent tasks
- total CPU time: about 5.5 months, using various workstations of different speeds
- CPU time for $n = 15 = 6 + 9$ (exploiting symmetry): 6 h. By contrast: PYTHON without `scipy` took 50 CPU days. (using a greedy rope)
- There is also a version in C (using `CWEB`) for the task of *enumerating* PsA's $\rightarrow$ OAOT of 13 points [OEIS A006247]

What else to enumerate



OAOT = *oriented abstract order type* of points

The required rope length

- Every arrangement requires $\geq n+1$ pieces (for $n \geq 3$).
- can always do with $\leq 2n-2$ pieces. (greedy sweep)
- Some arrangements require $\lfloor \frac{7n}{4} \rfloor - 1$ pieces.

  (This is the true maximum for $n \leq 9$.)
- NP-hard? (homotopy height, cutwidth)

  [Biedl, Chambers, Kostitsyna, Rote, 2020/2022/2024?, unpublished]

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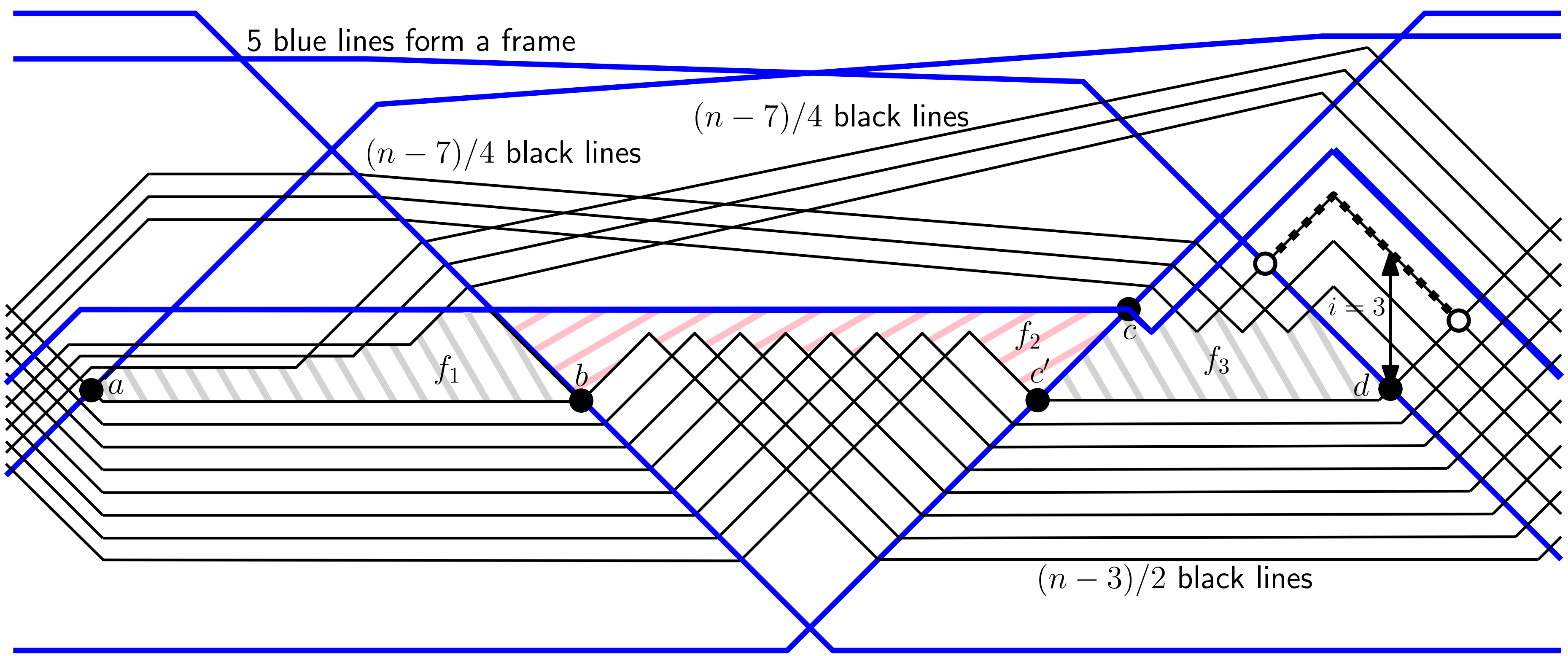
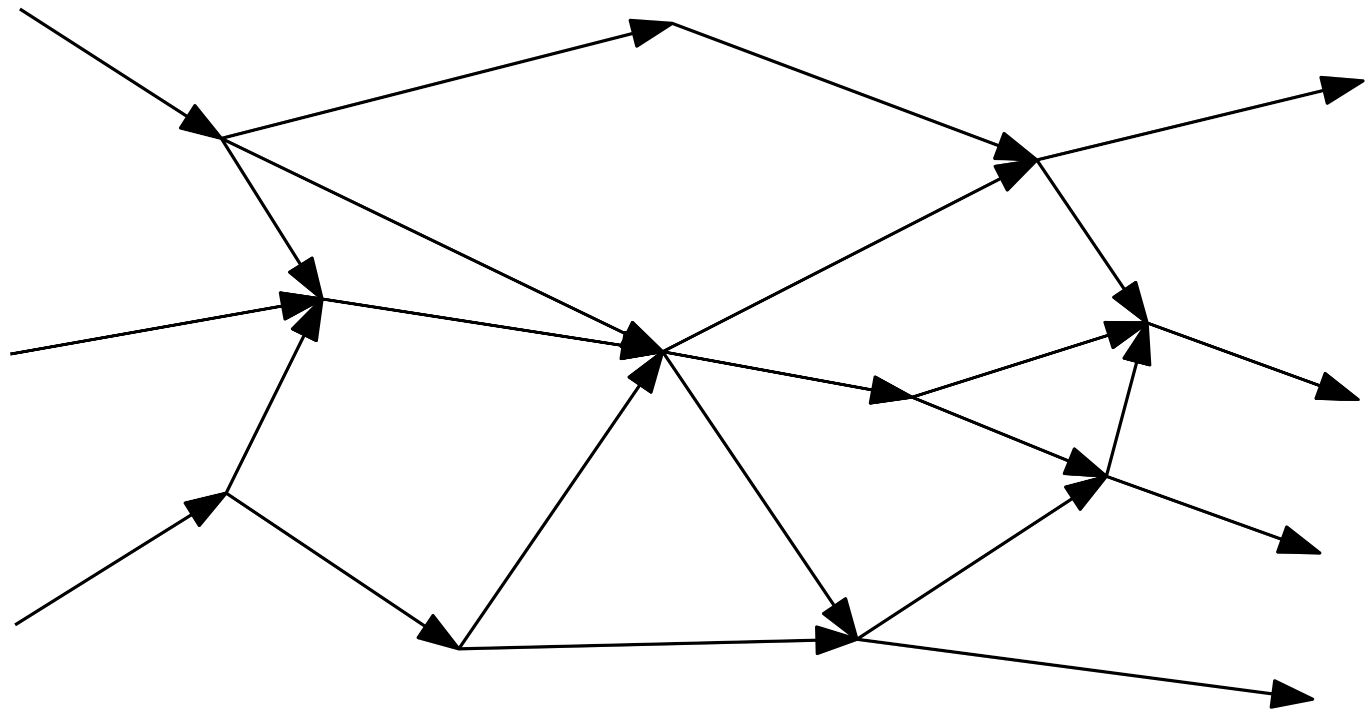


The required rope length

This is really about *bipolar orientations* (s-t-planar DAGs):



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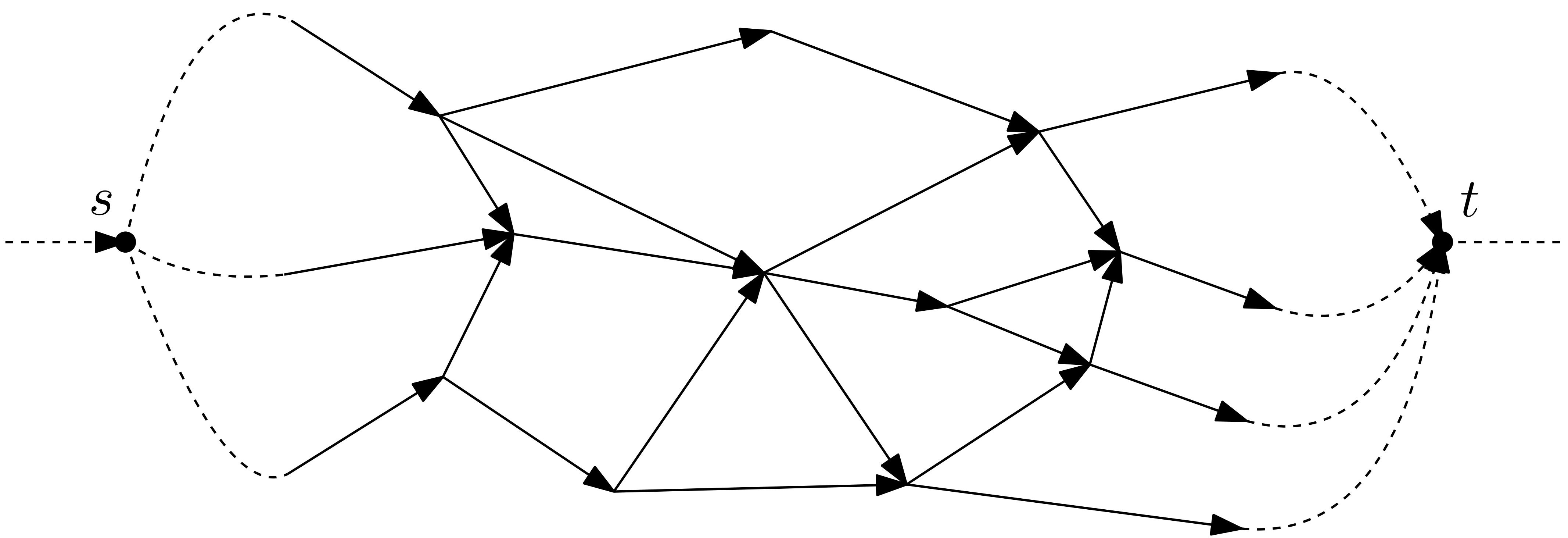

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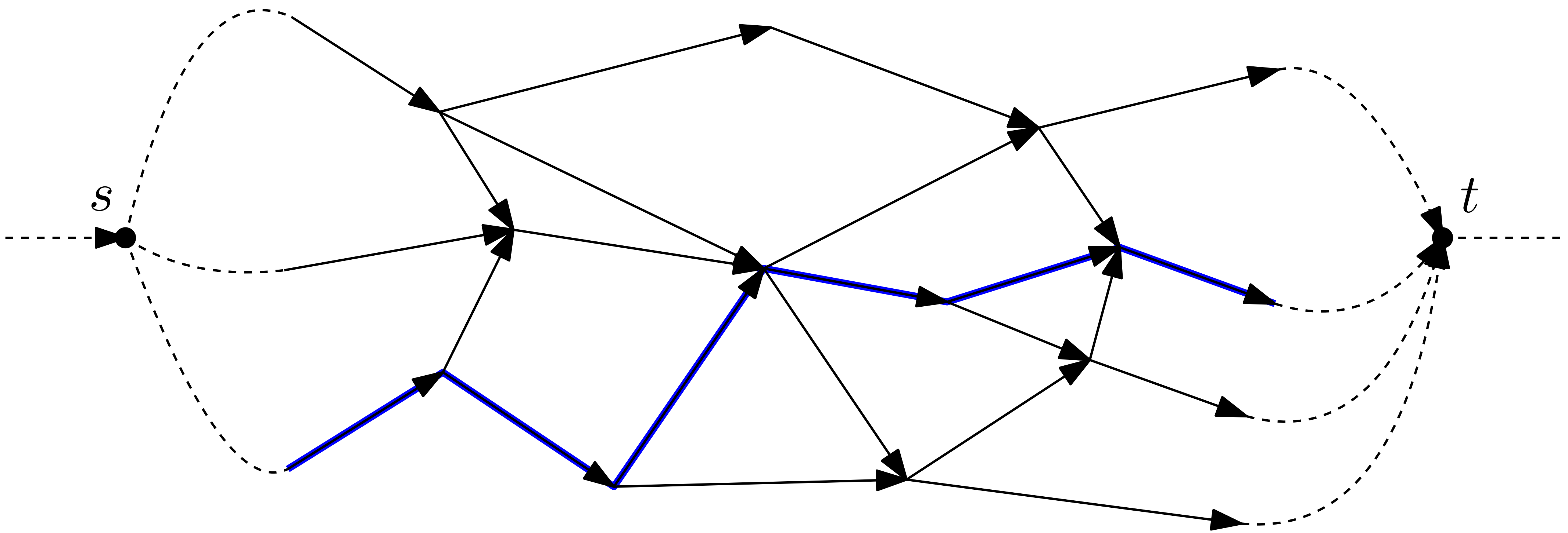

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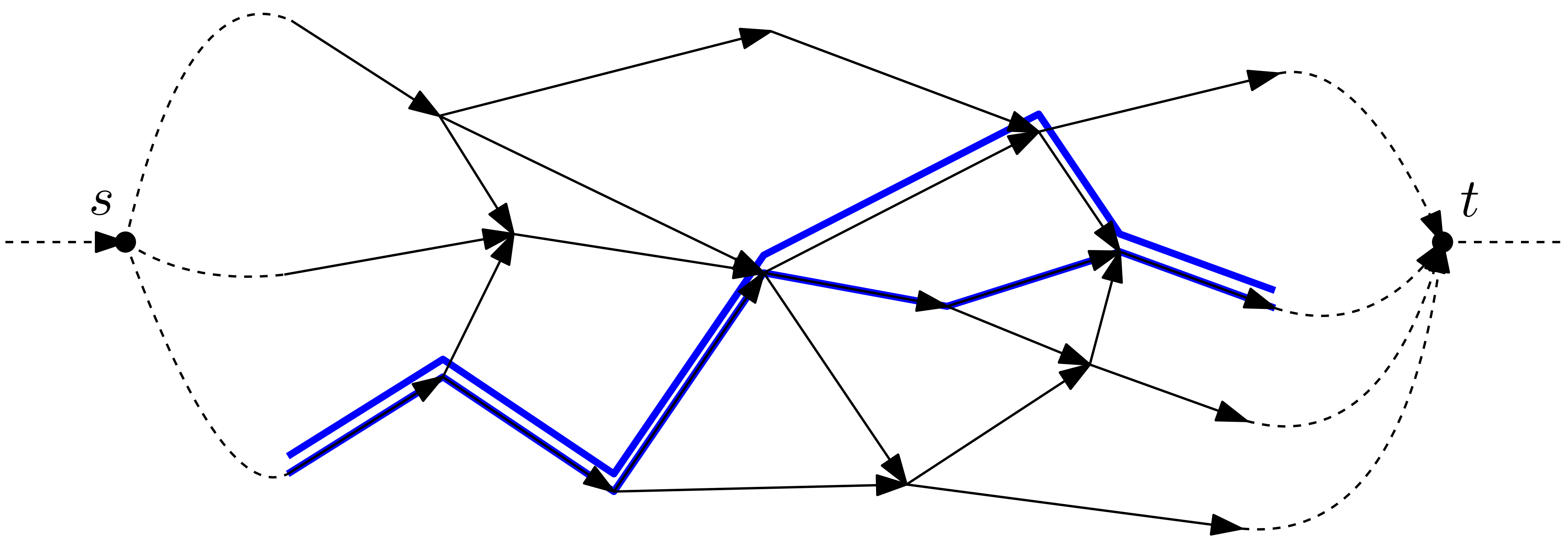

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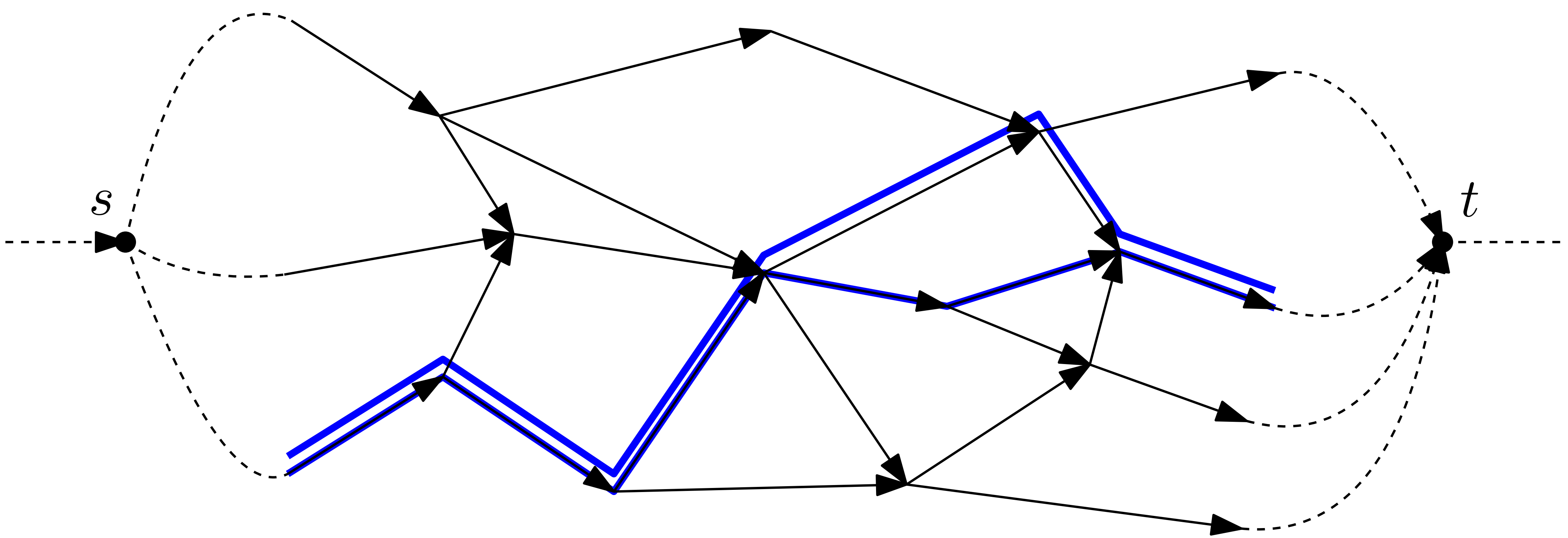
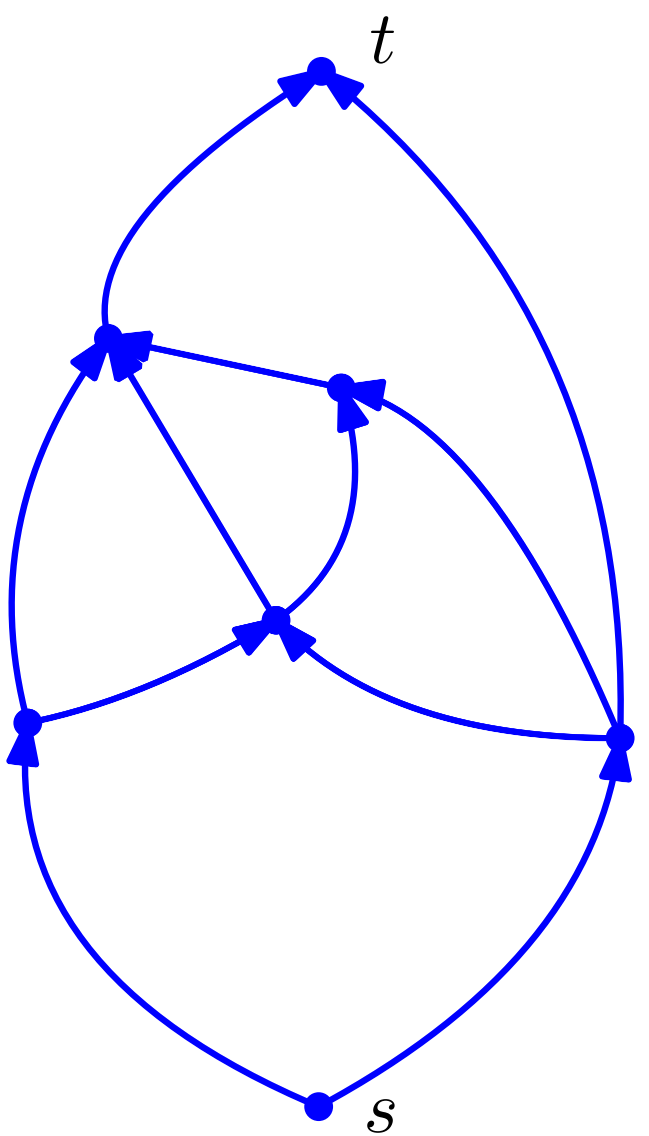


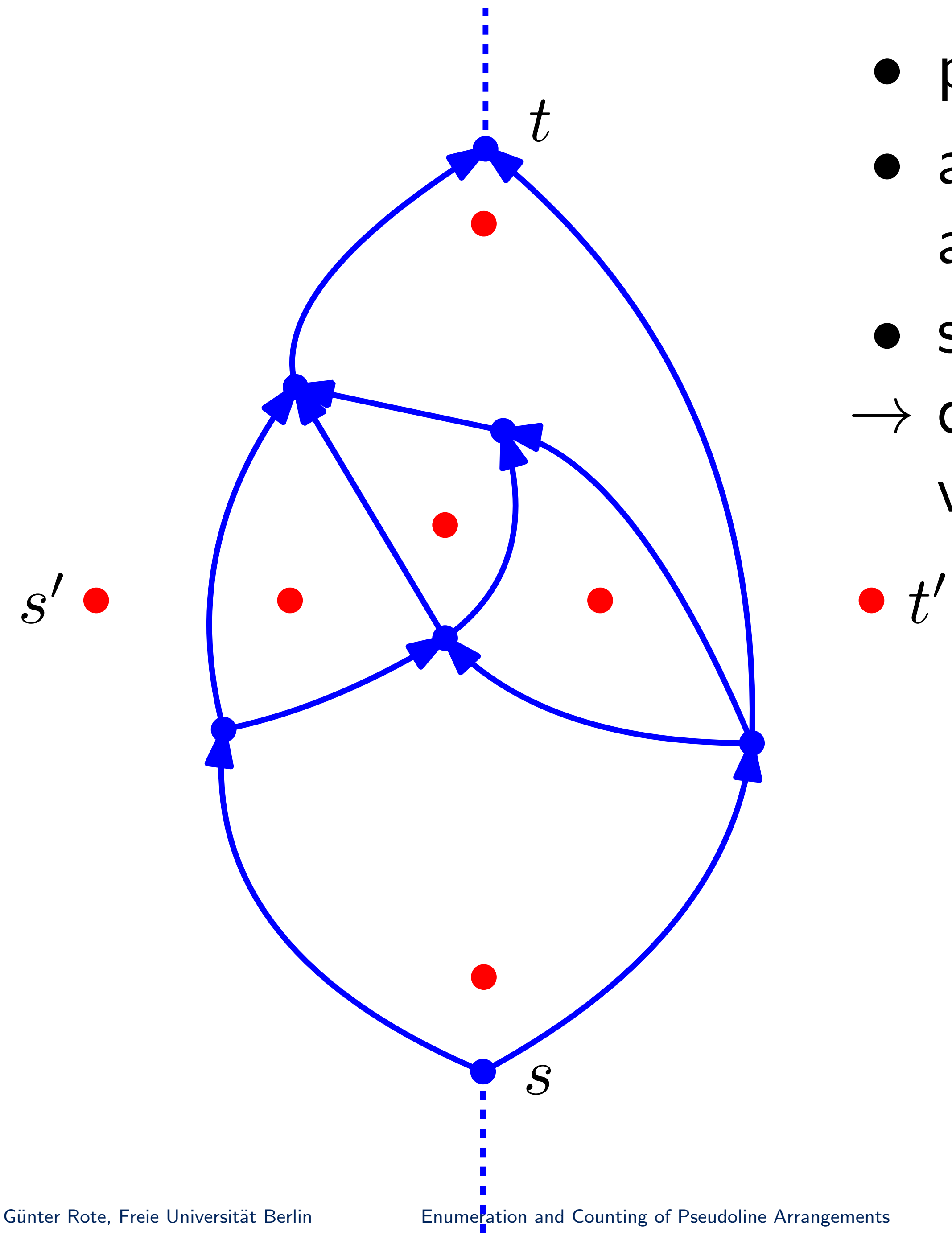
“leftmost-first” greedy sweep
$\rightarrow$ coordinated simultaneous primal-dual sweep

Bipolar orientations (s-t-planar graphs)



- plane directed acyclic graph
- a single source s and
  a single sink t

Bipolar orientations (s-t-planar graphs)



- plane directed acyclic graph
- a single source s and a single sink t
- split the outer face:

$\rightarrow$ dual graph with a *left* outer vertex s' and a *right* vertex t'

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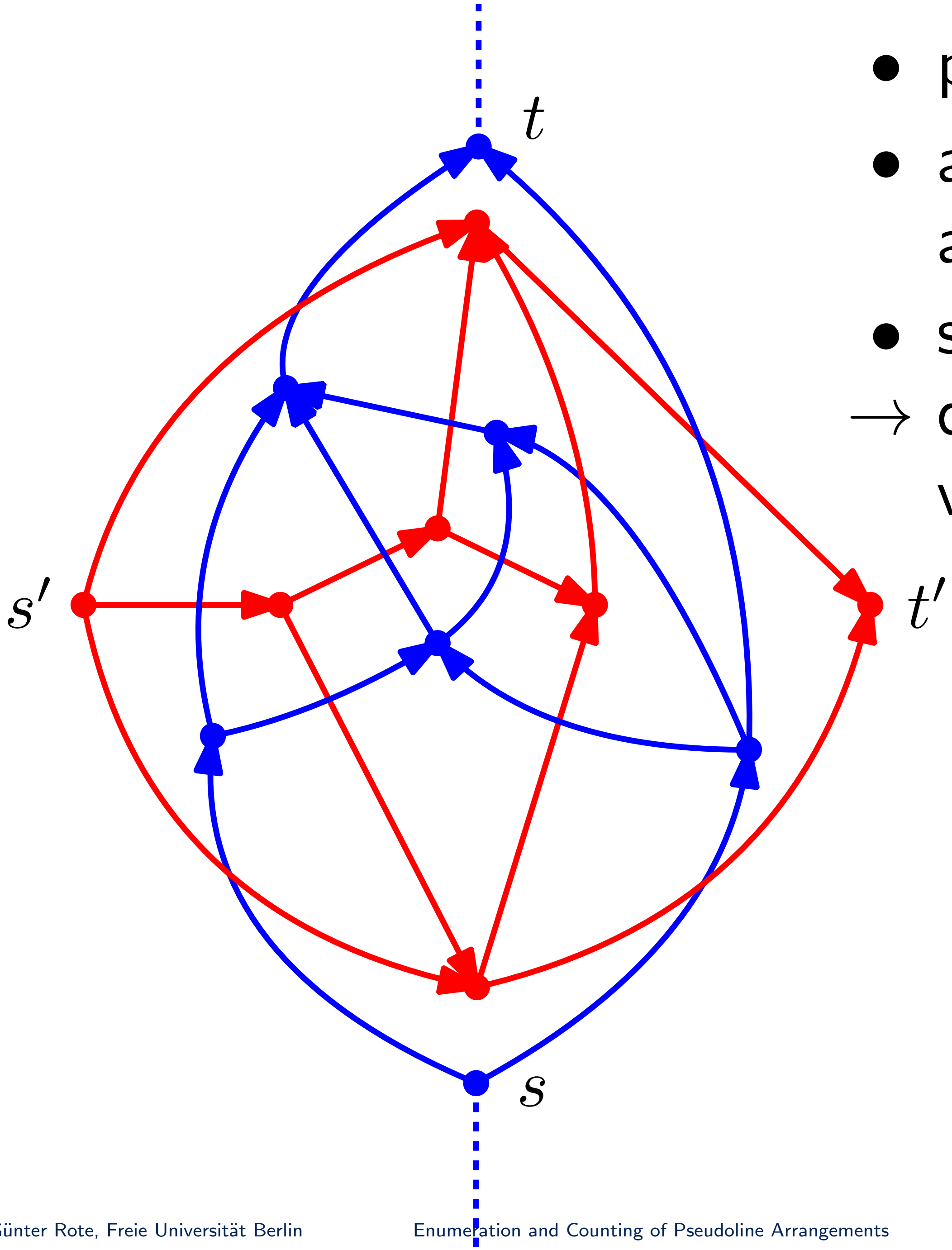

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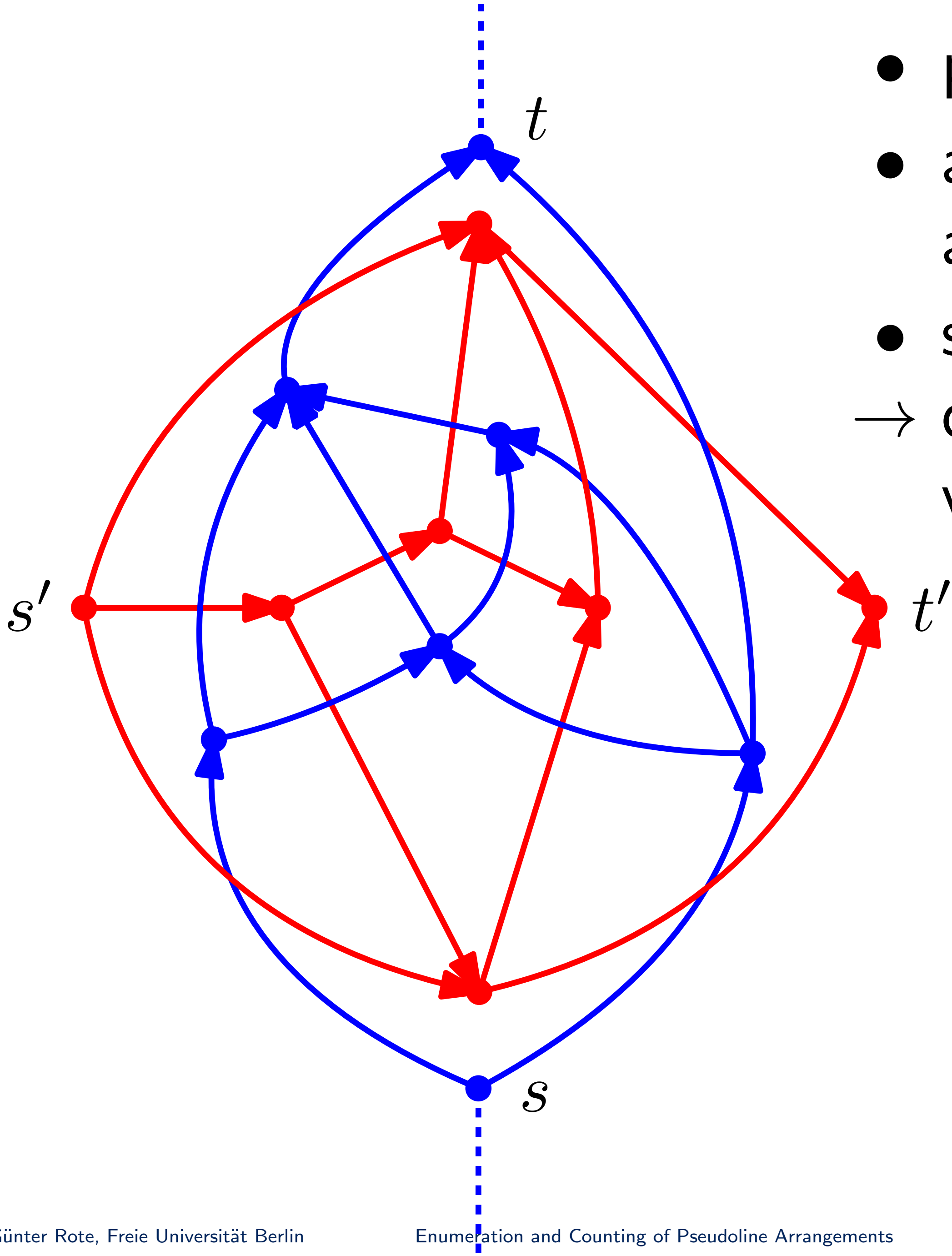
Bipolar orientations (s-t-planar graphs)



- plane directed acyclic graph
- a single source s and a single sink t
- split the outer face:

$\rightarrow$ dual graph with a *left* outer vertex s' and a *right* vertex t'

- The dual graph is also a bipolar orientation. (may be a multigraph)

Bipolar orientations (s-t-planar graphs)

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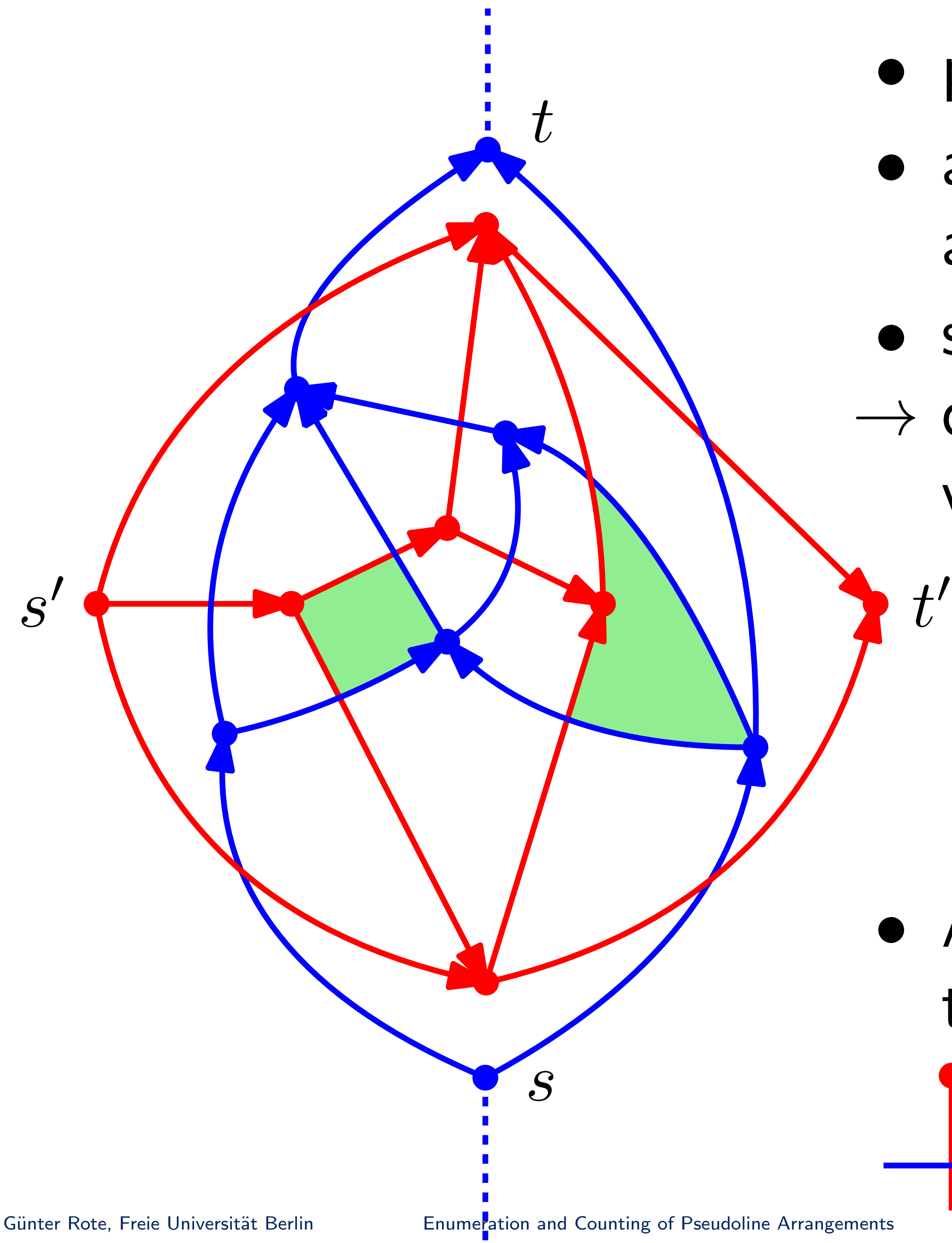

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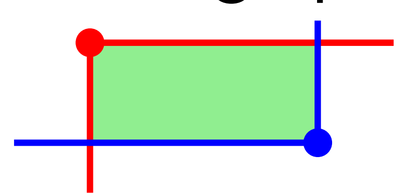
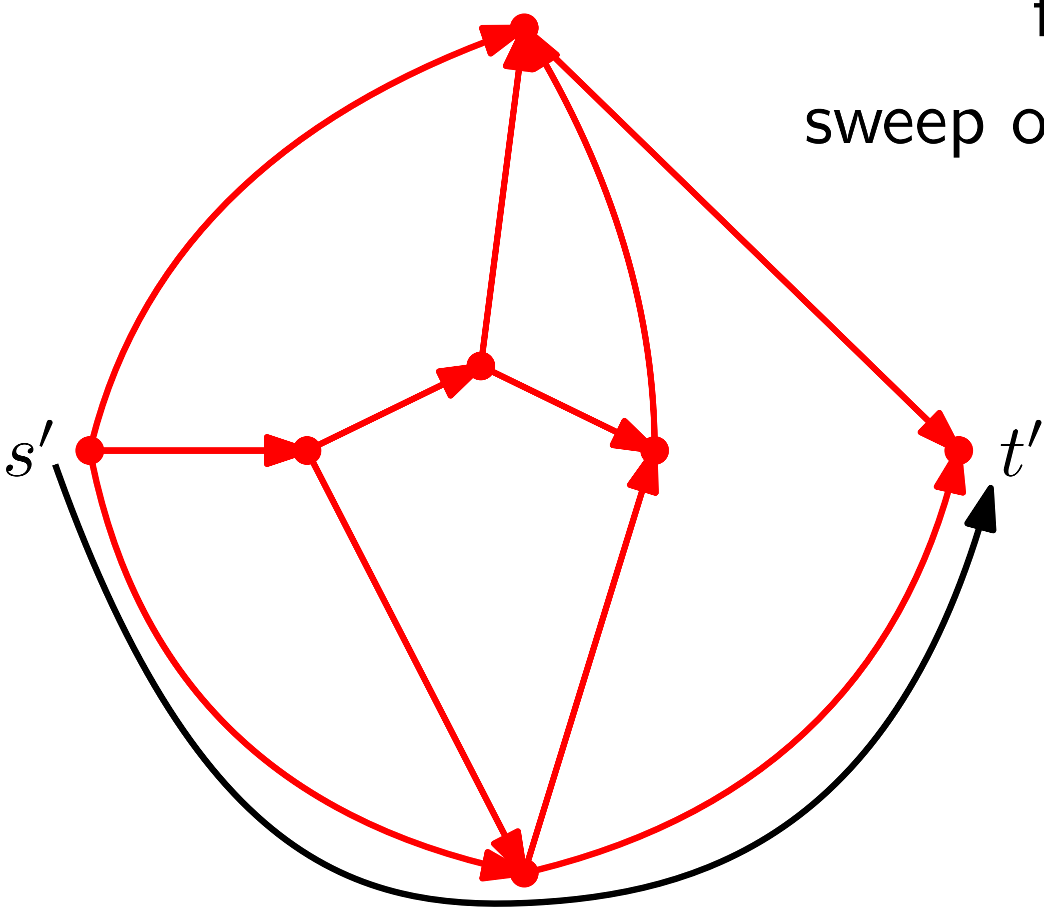
- All faces in the overlay of the two graphs are quadrilaterals:



Coordinated primal-dual sweep

- sweep the dual graph with an s'–t' rope from bottom to top

sweep over the *leftmost* possible face



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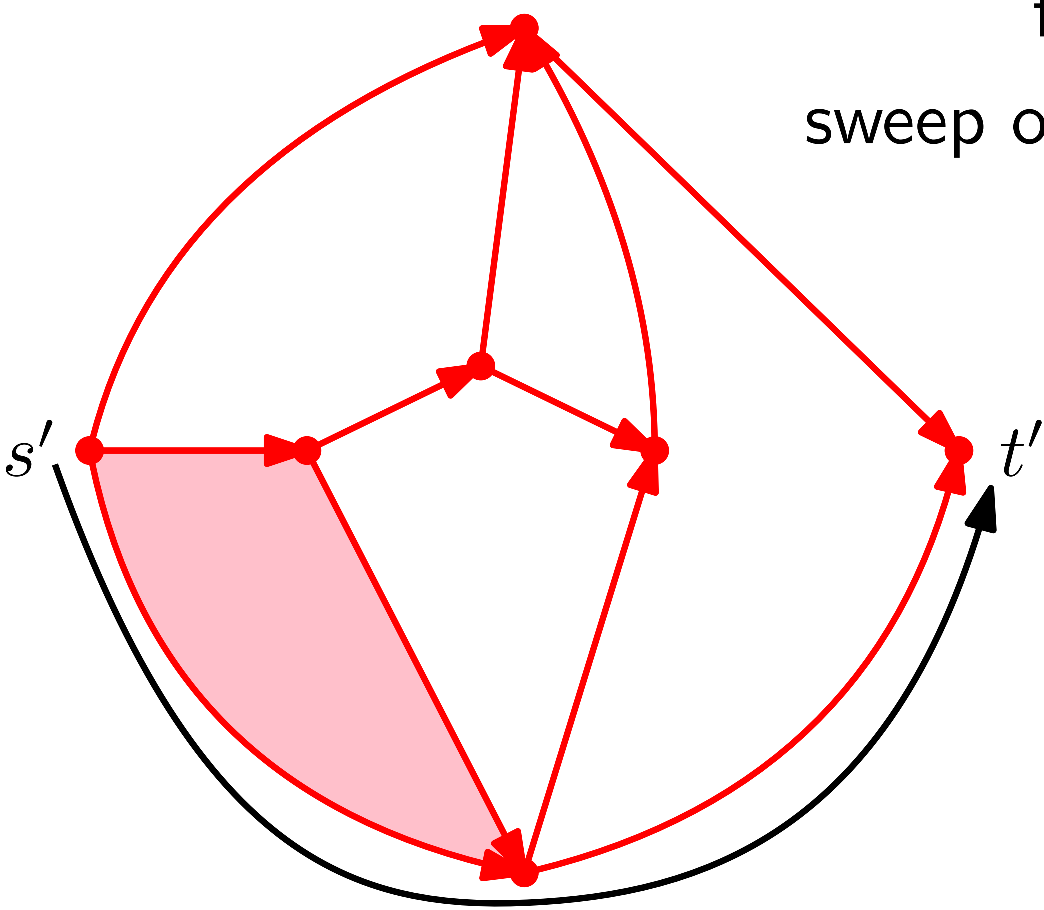

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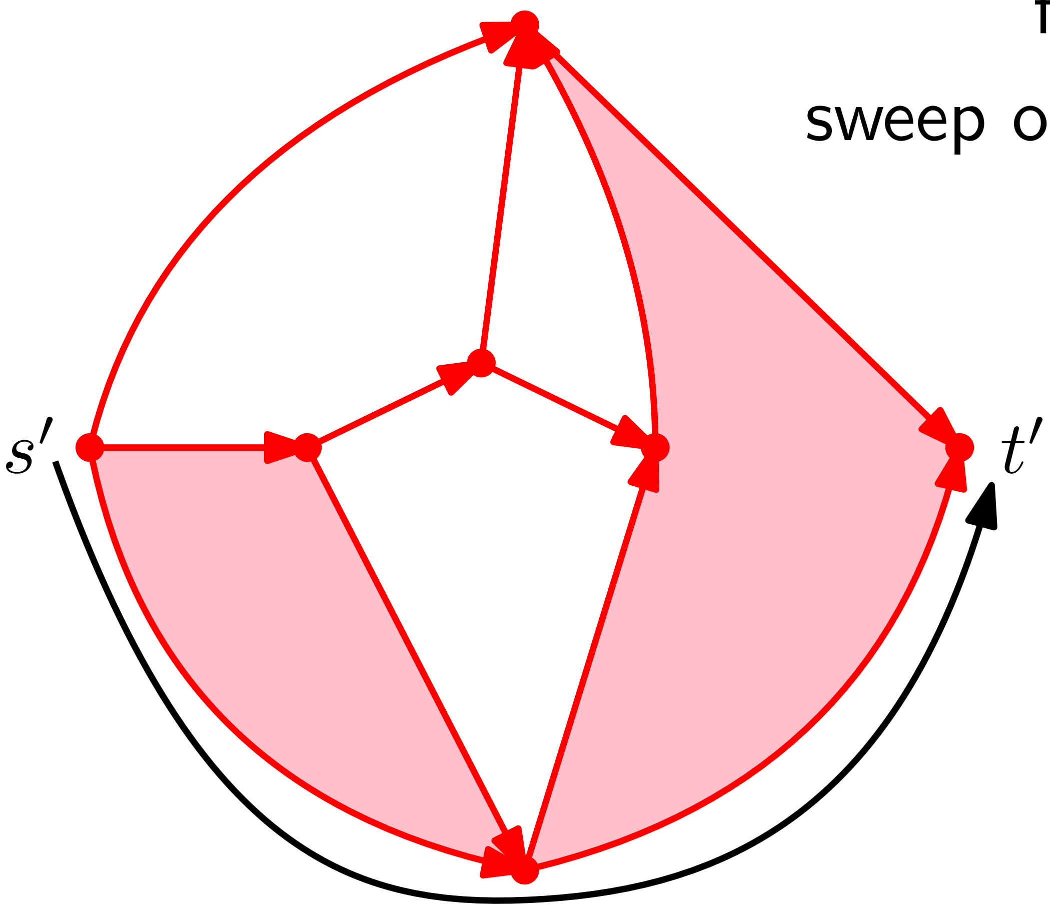

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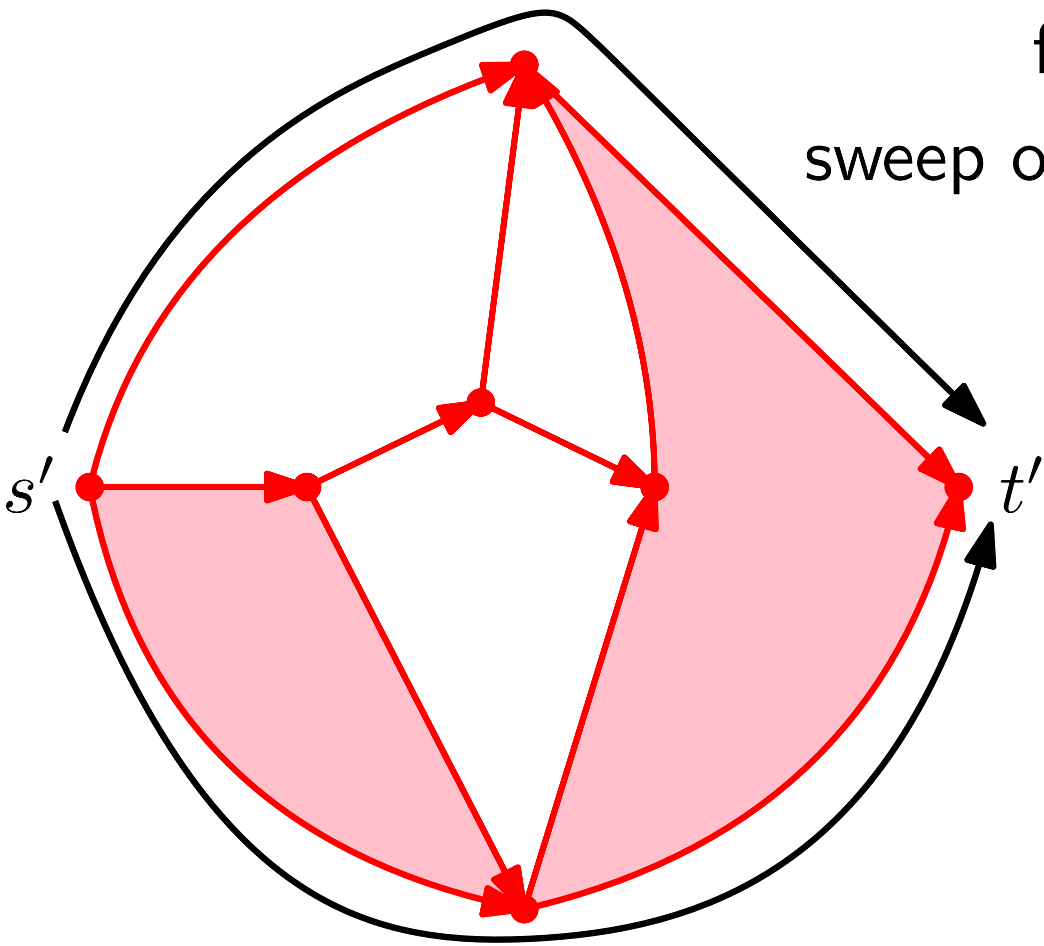

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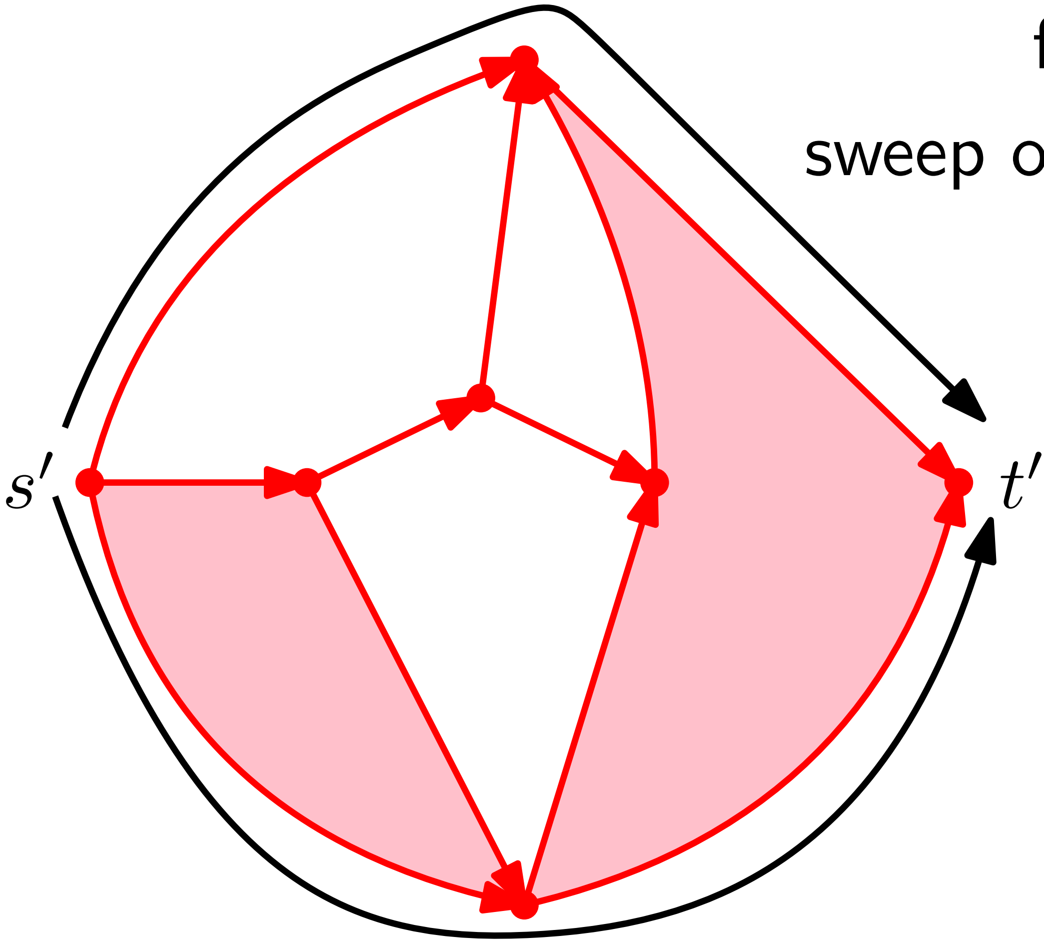
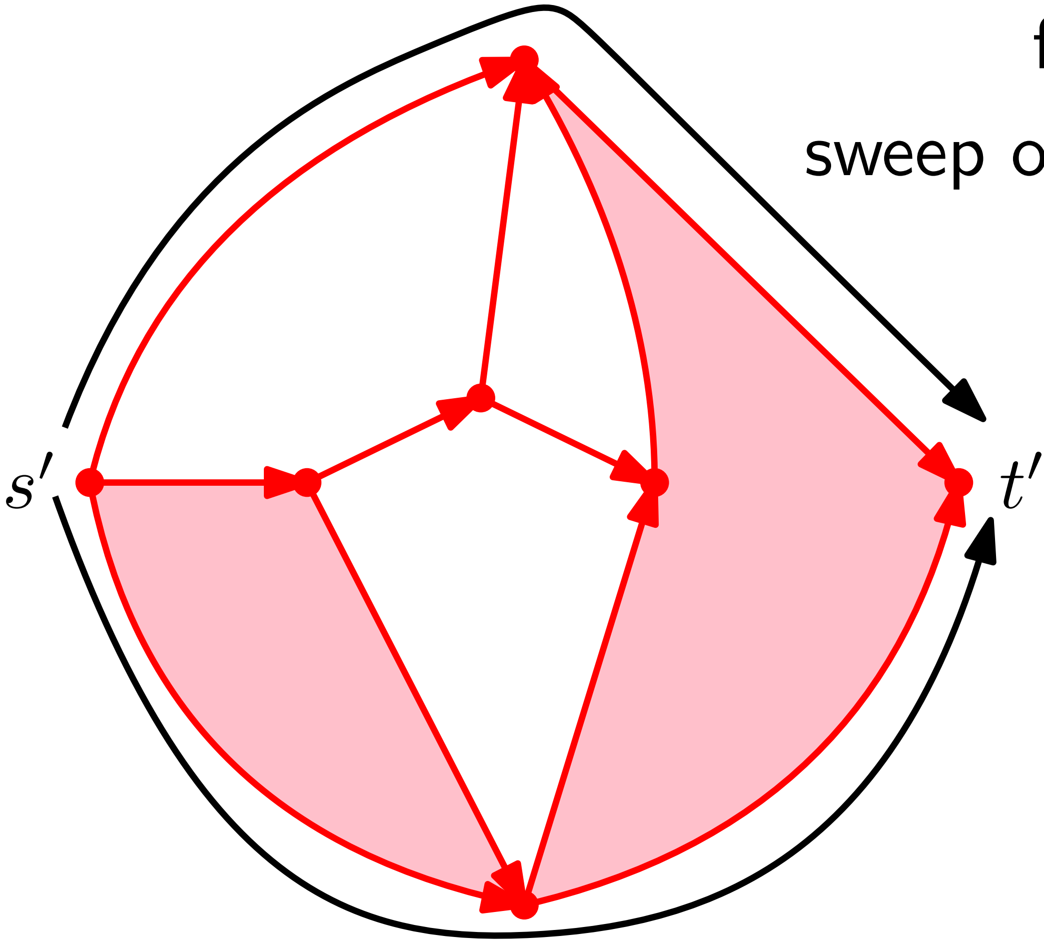


- sweep the dual graph with an s'–t' rope from bottom to top

sweep over the *leftmost* possible face

Sweep is always possible!

Coordinated primal-dual sweep



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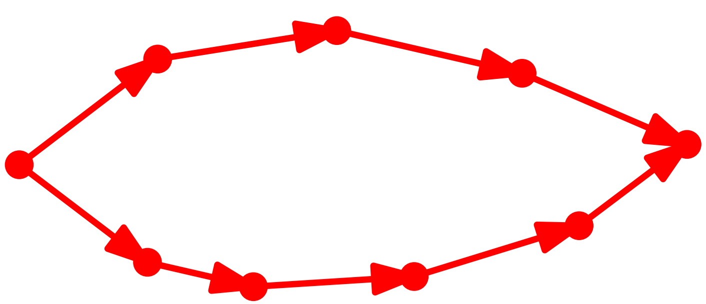
General form of a face



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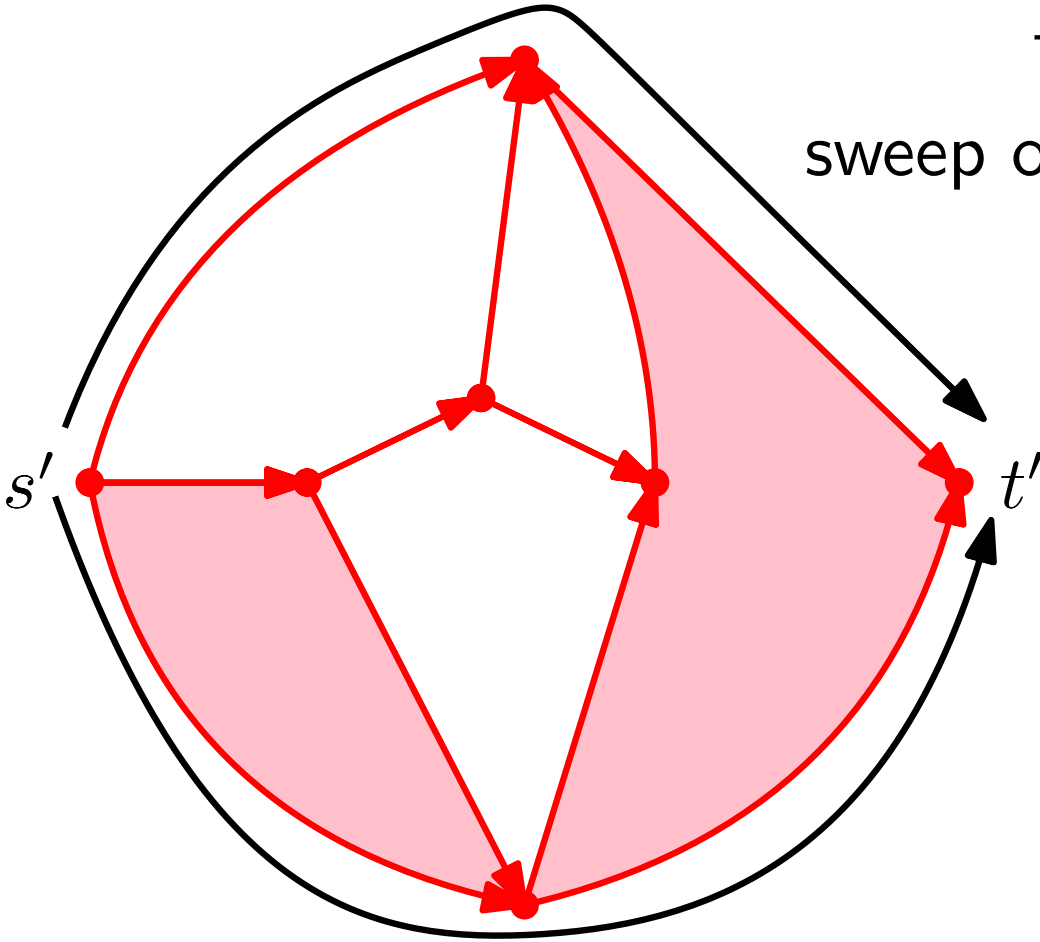

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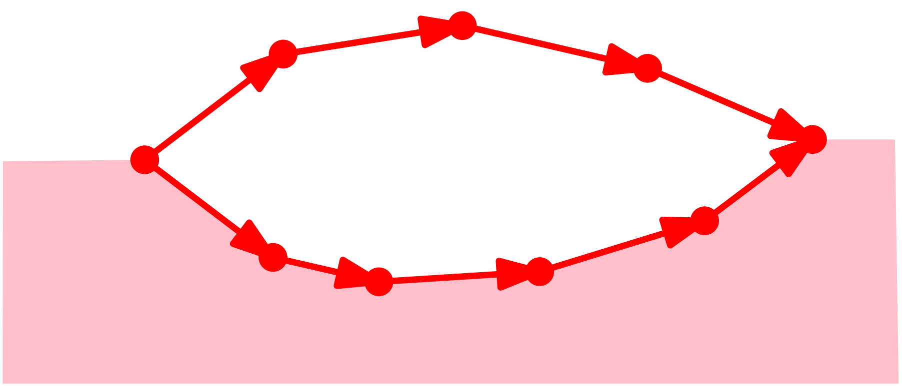

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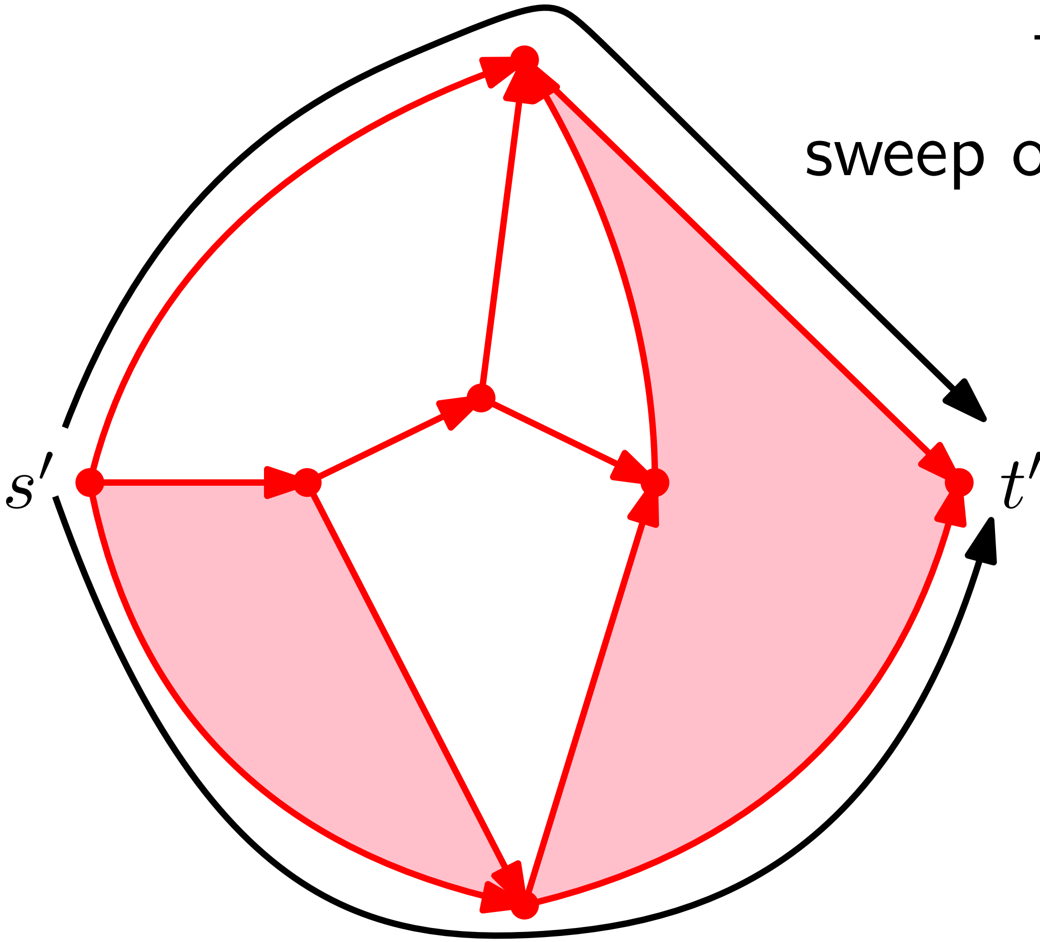

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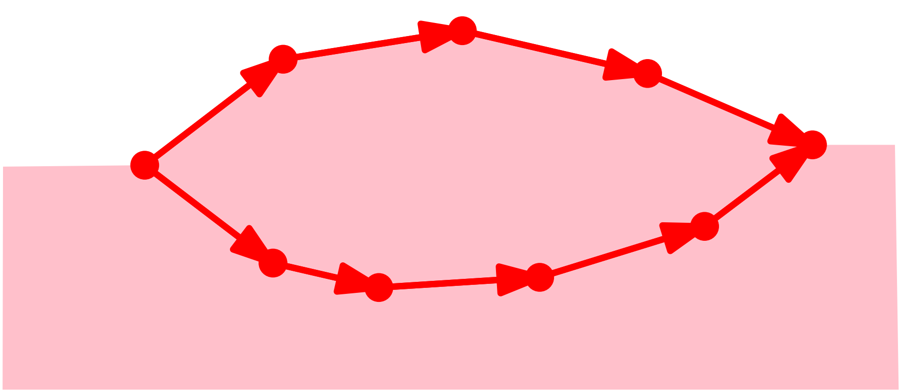

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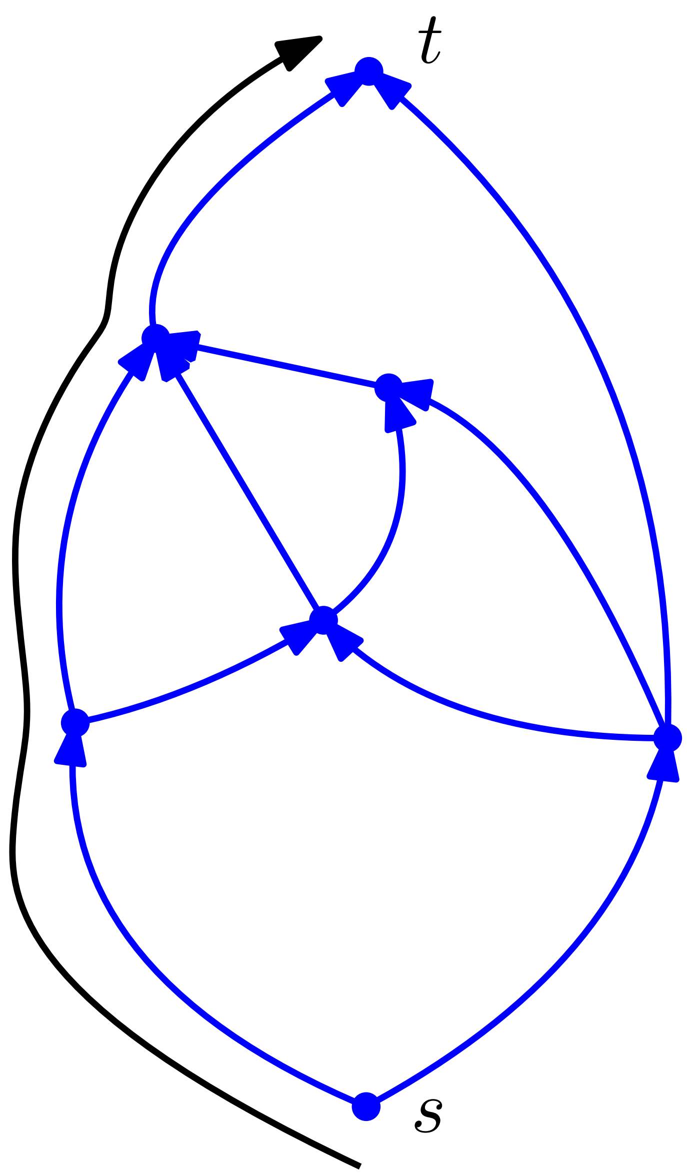
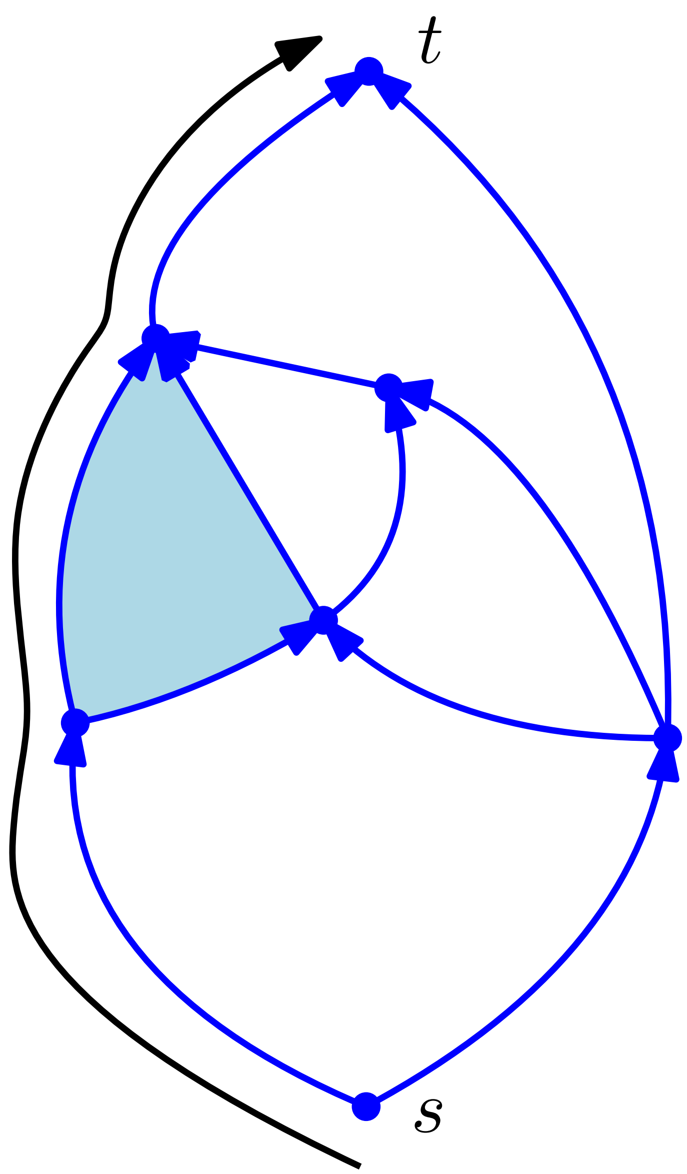


- sweep the primal graph with an s–t rope from left to right

Coordinated primal-dual sweep



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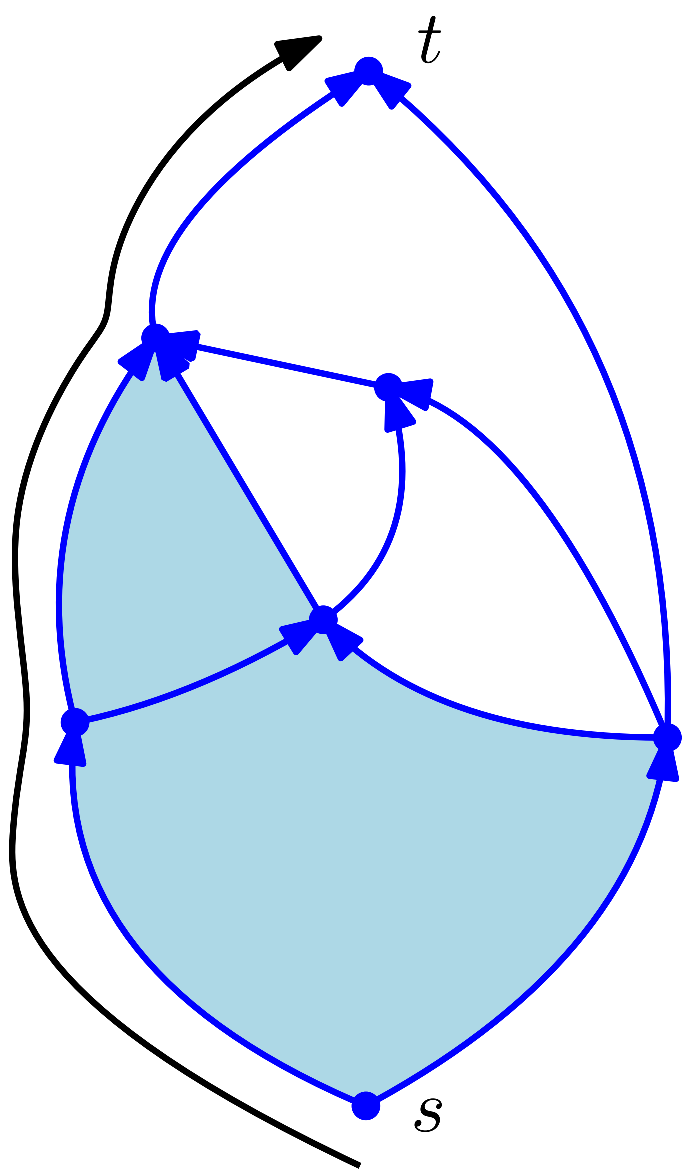

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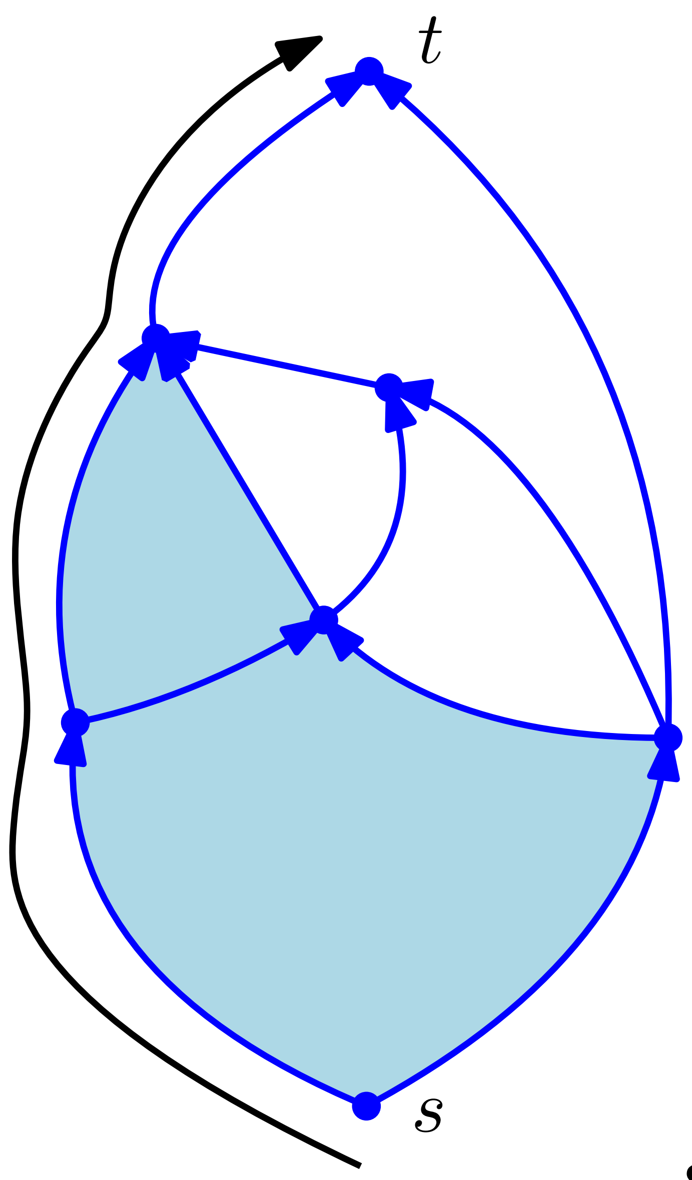
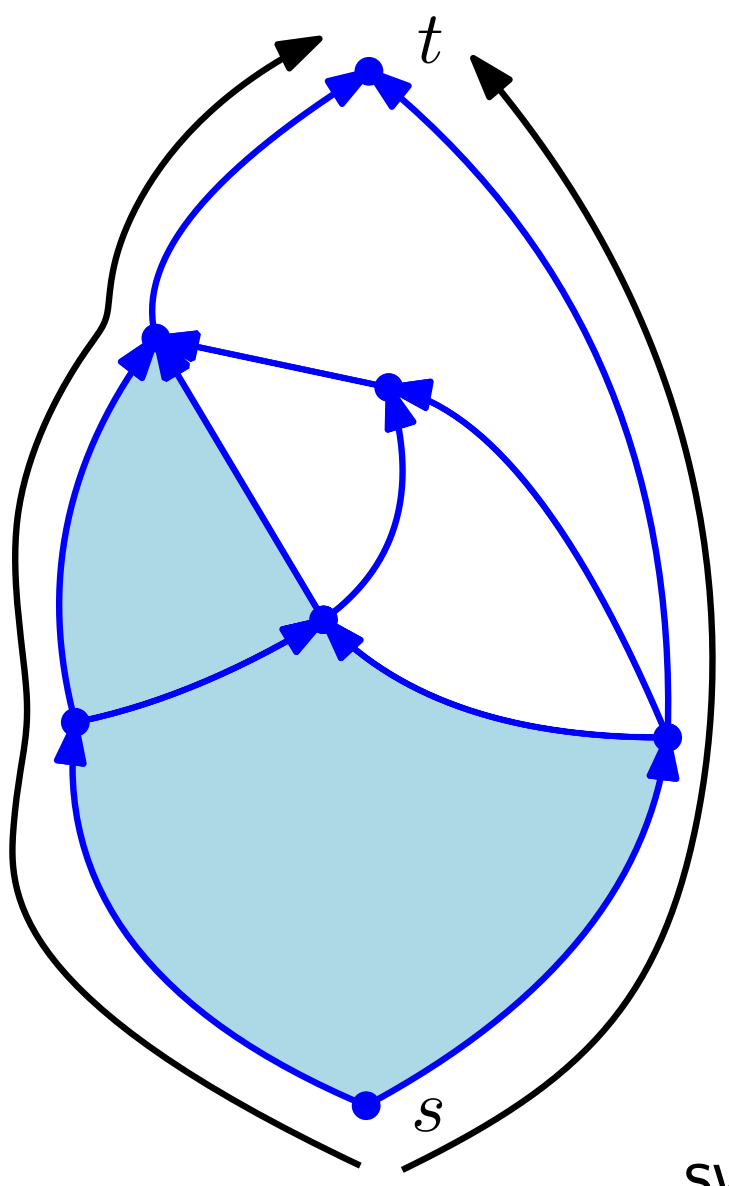


- sweep the primal graph with an s–t rope from left to right

sweep over the *lowest* possible face

Coordinated primal-dual sweep



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Animation

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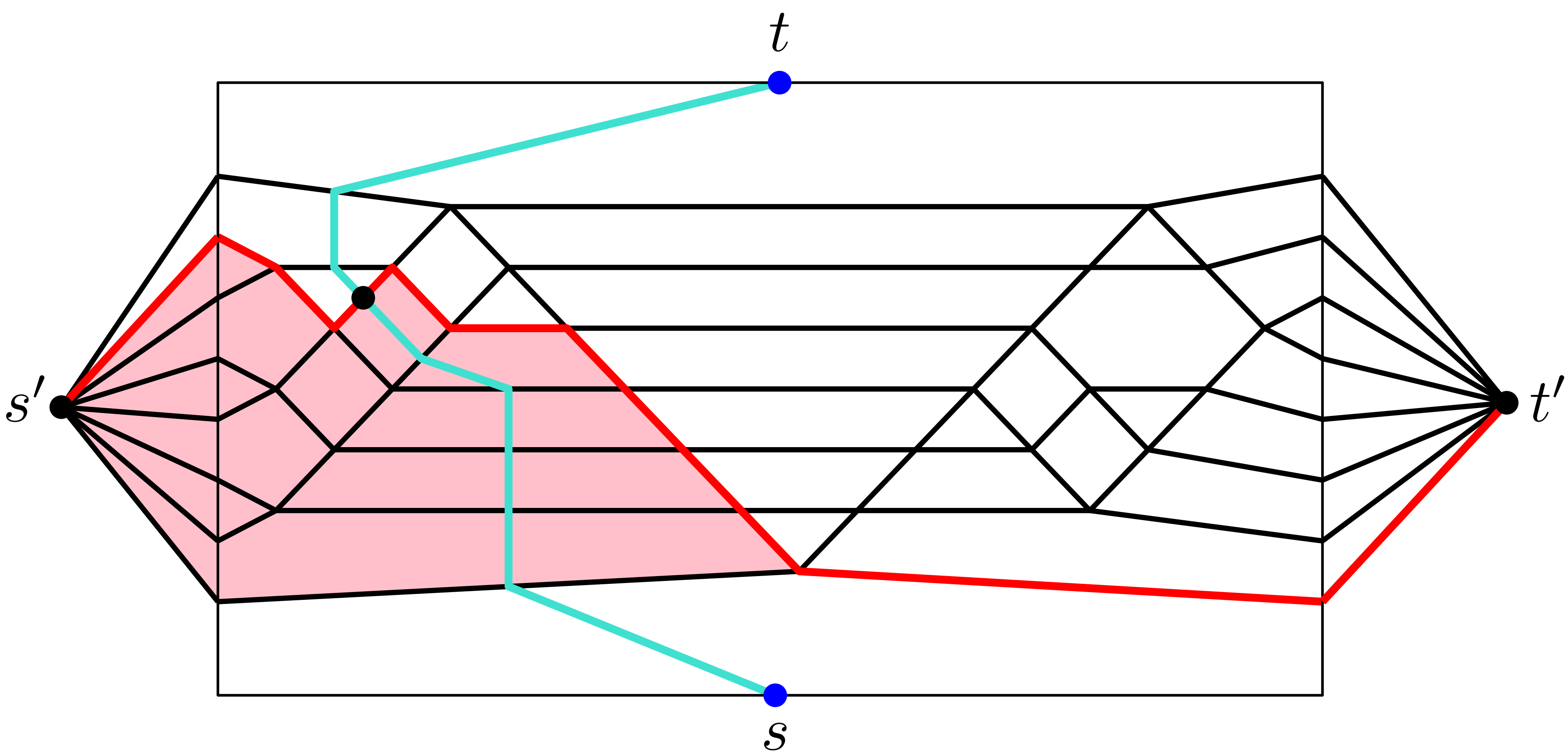
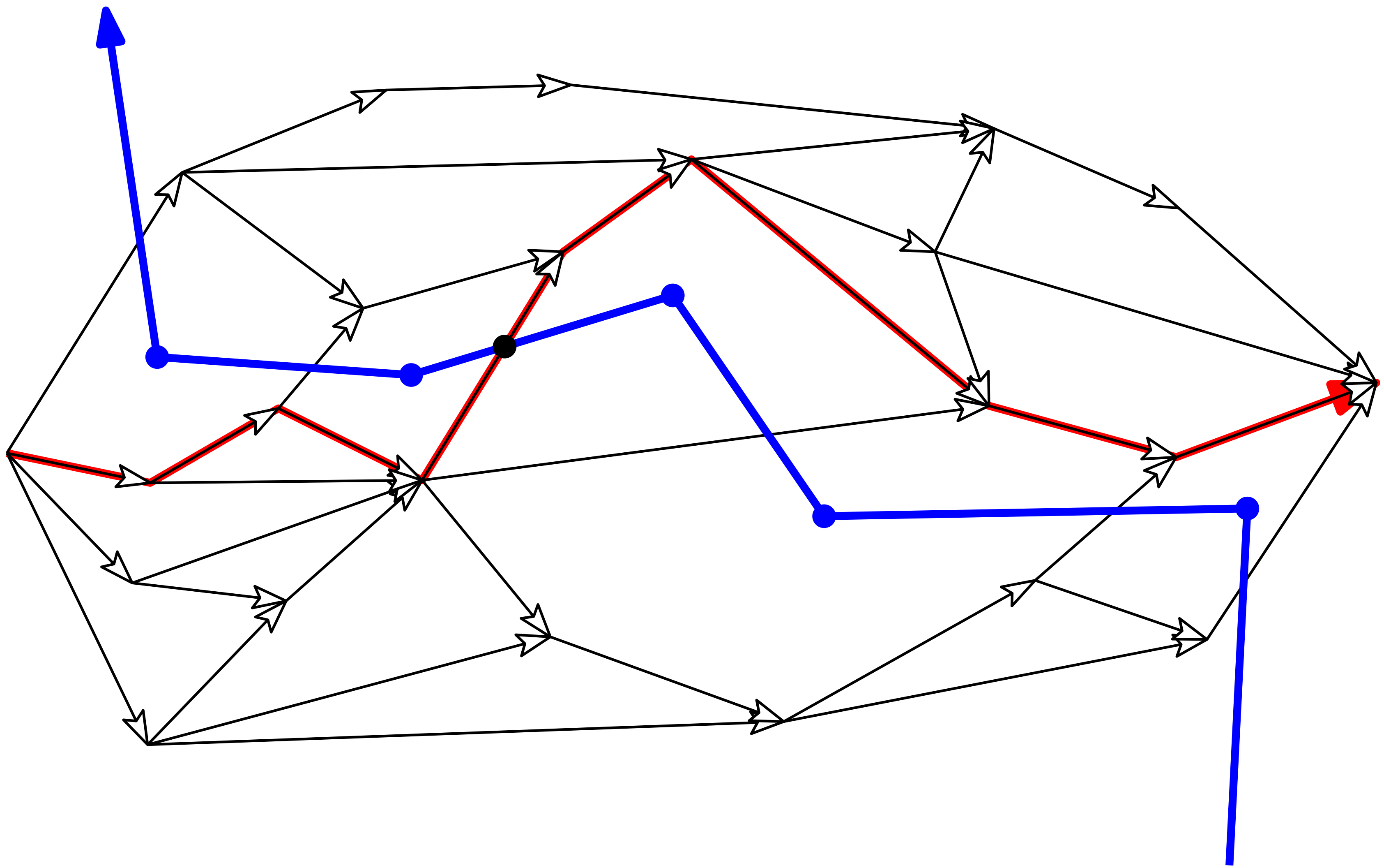


dual rope in the dual (multi-)graph

primal rope (The primal graph is not shown.)

page.mi.fu-berlin.de/rote/Papers/slides/Wuerzburg-2020-Simultaneous-sweep-Animation.pdf

A snapshot



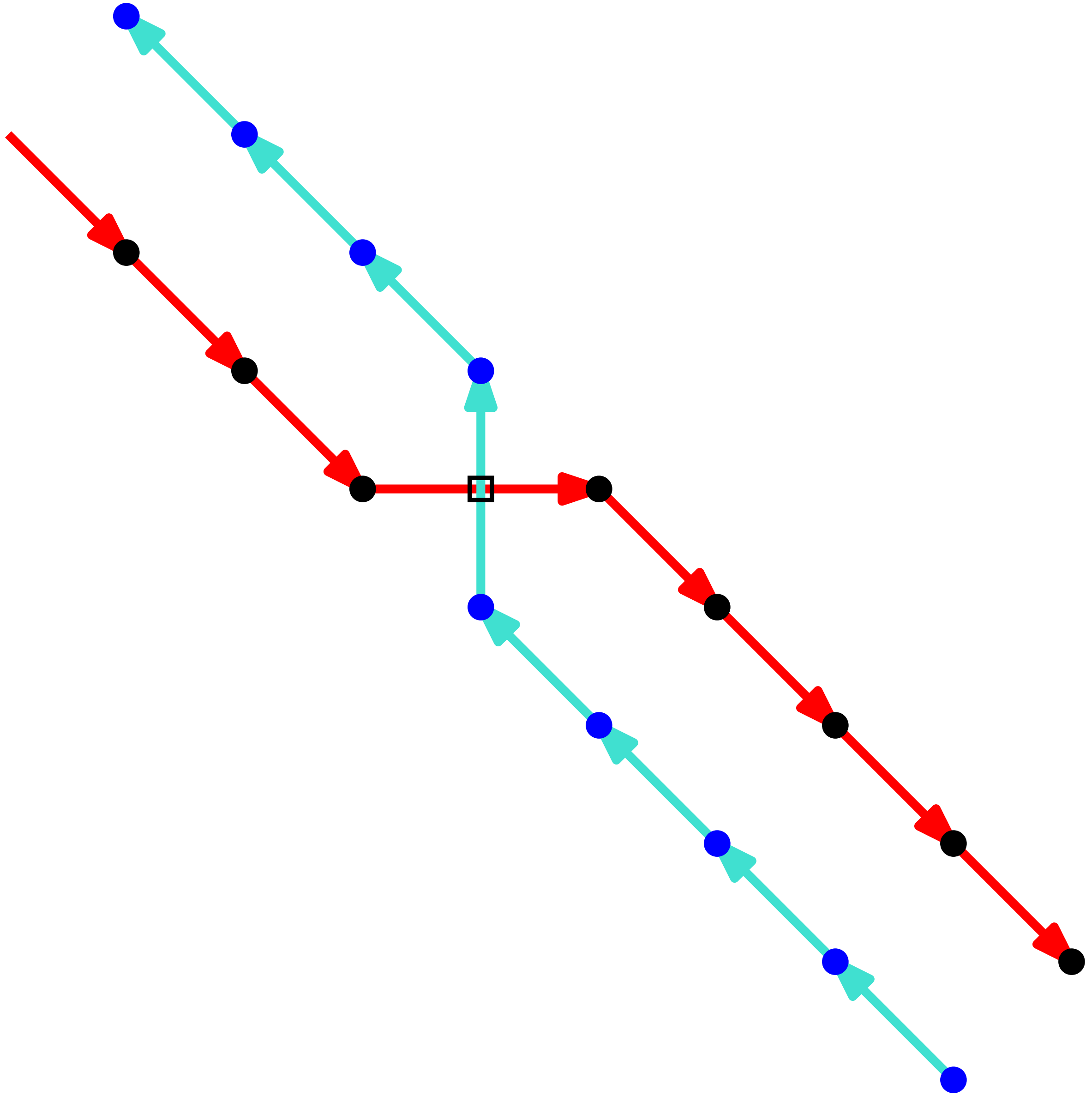
Coordinated sweep

There is a (unique) coordinated primal-dual sweep with the following properties:

- The primal rope always crosses the dual rope exactly once.
- The primal and the dual rope stay "close" to each other.
- Exactly one rope can advance, depending on the situation at the crossing.
- Every primal-dual edge pair is visited exactly once.
- Each individual sweep is a leftmost/bottommost sweep.

Coordinated sweep

general situation:



Coordinated sweep

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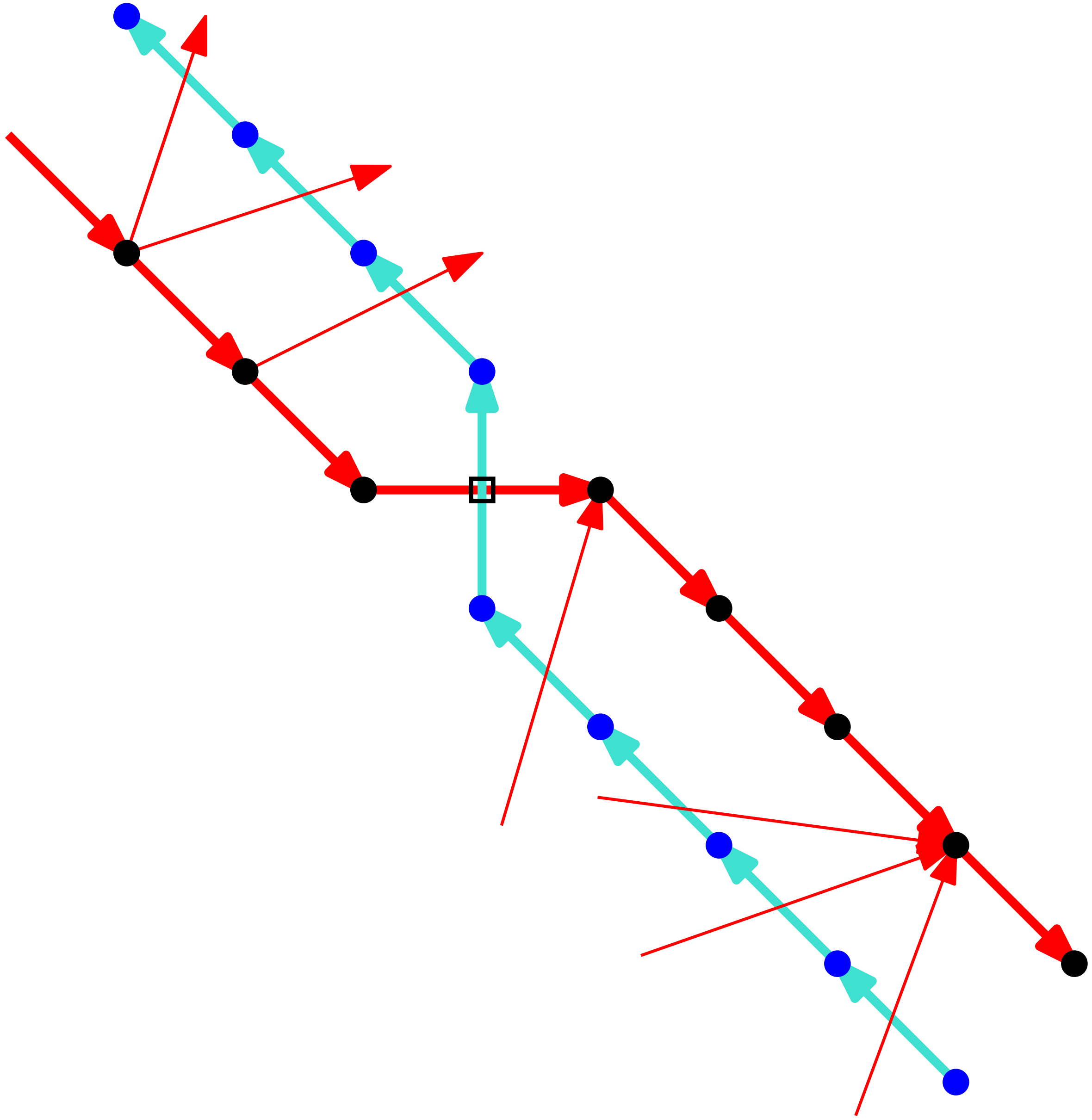

Coordinated sweep

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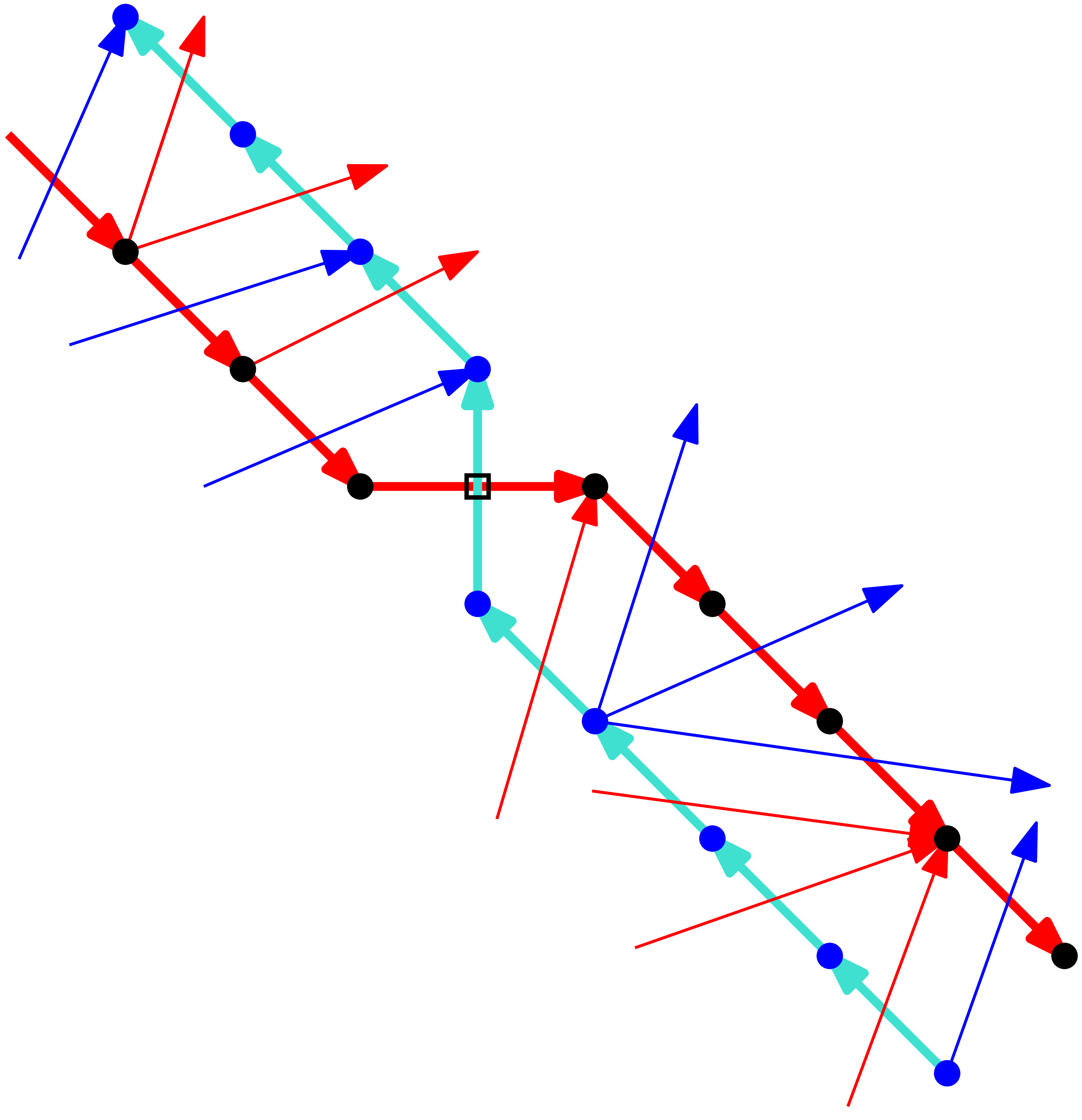

Coordinated sweep

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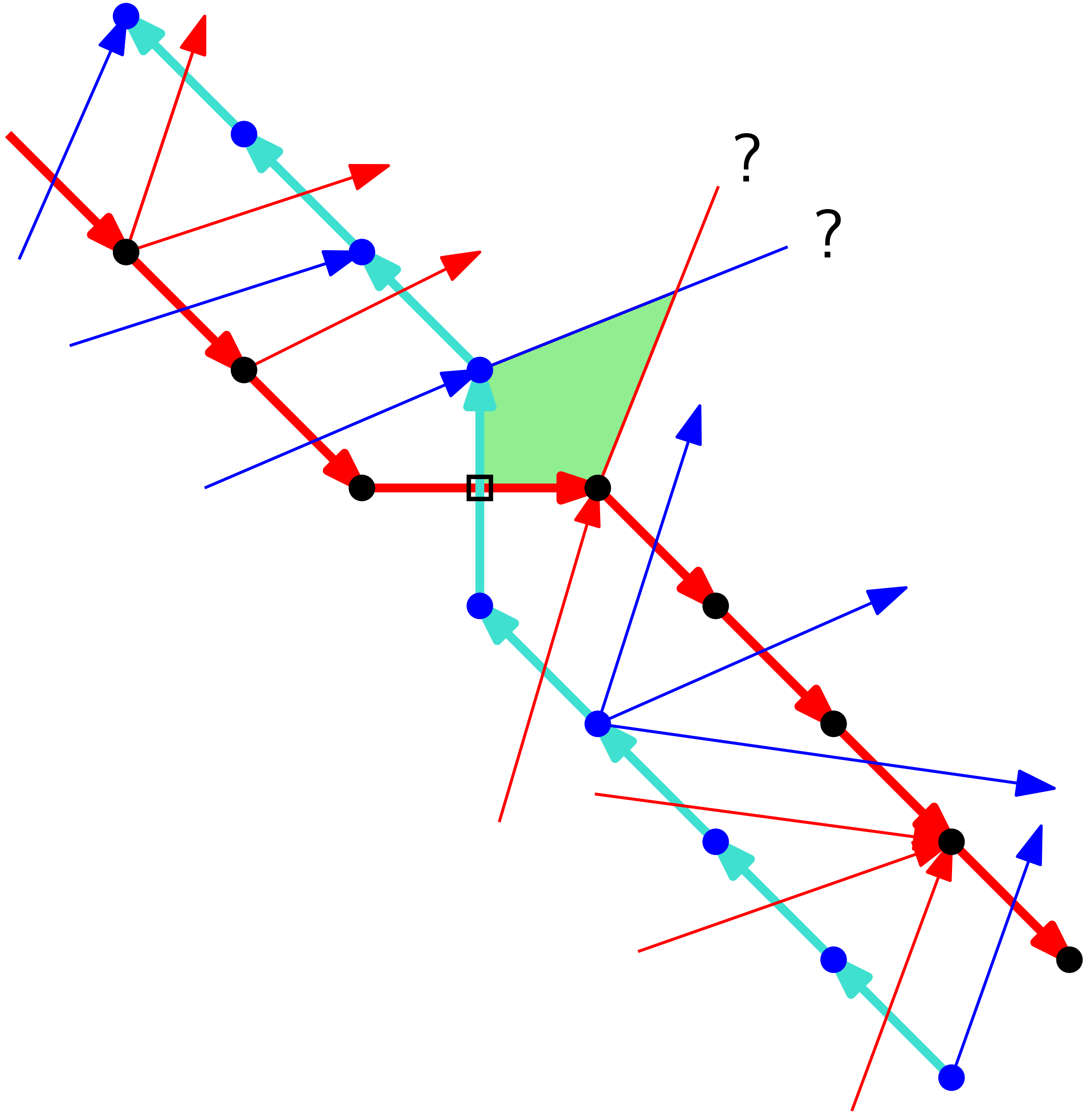

Coordinated sweep

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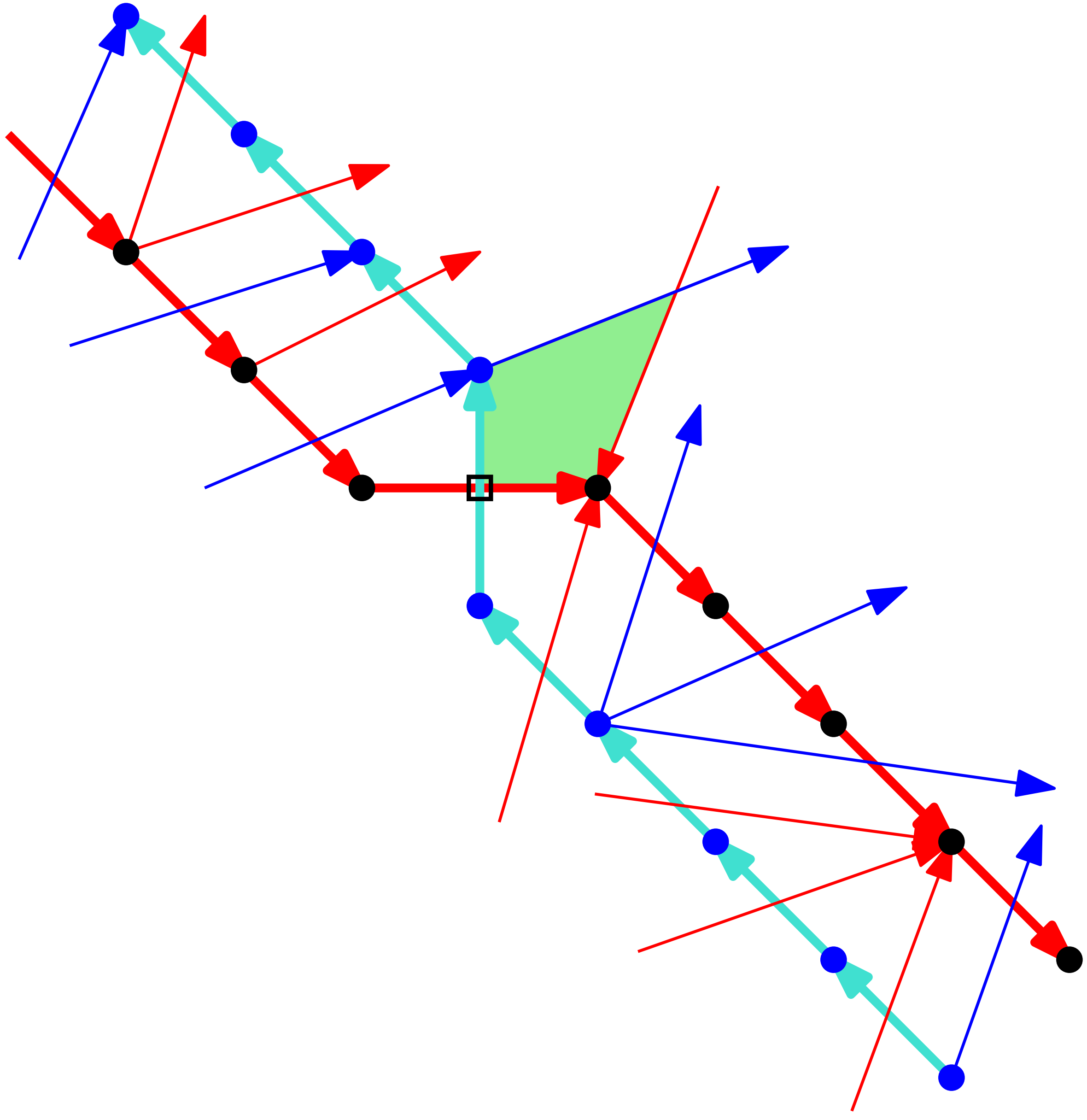

Coordinated sweep

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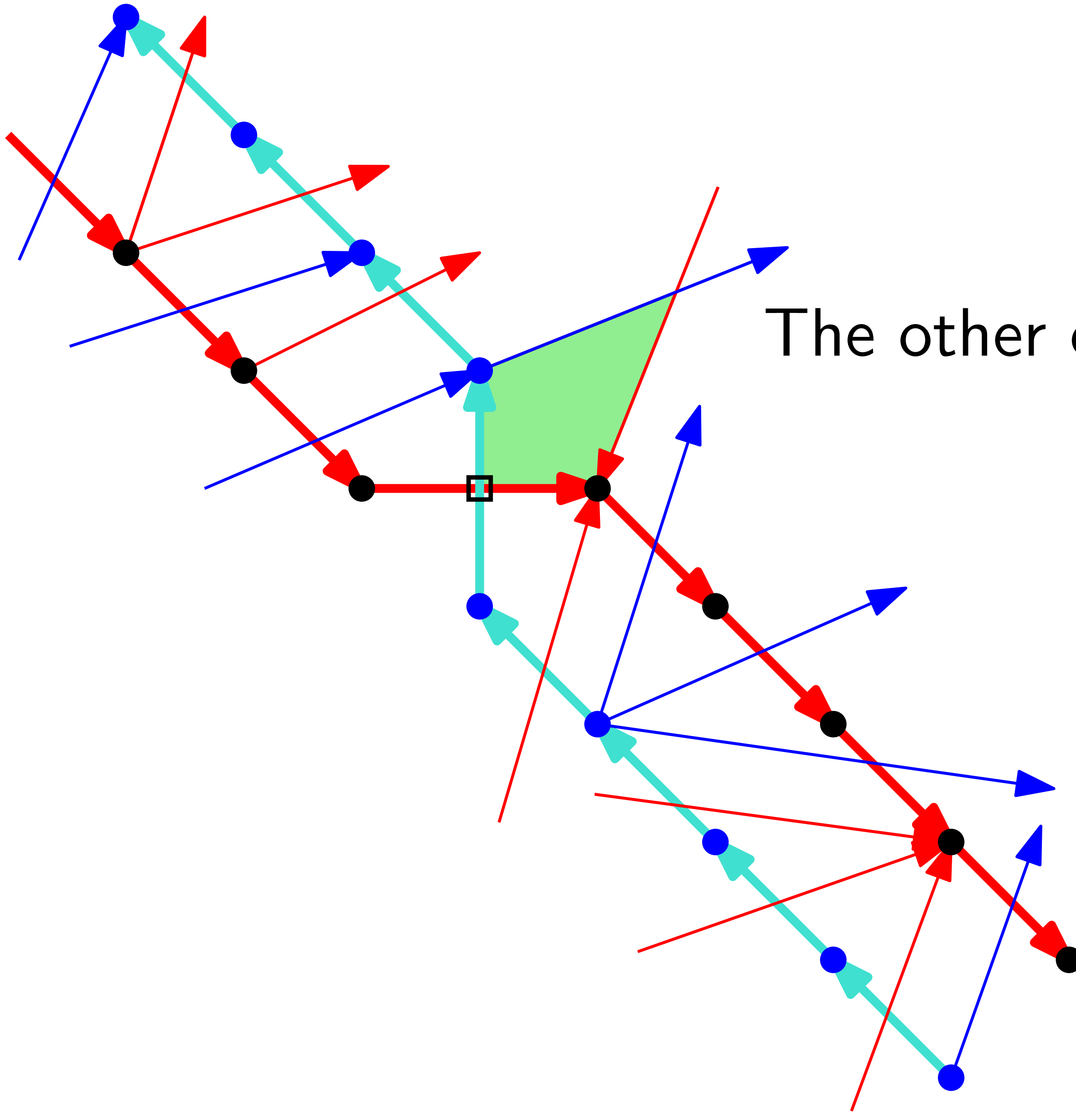
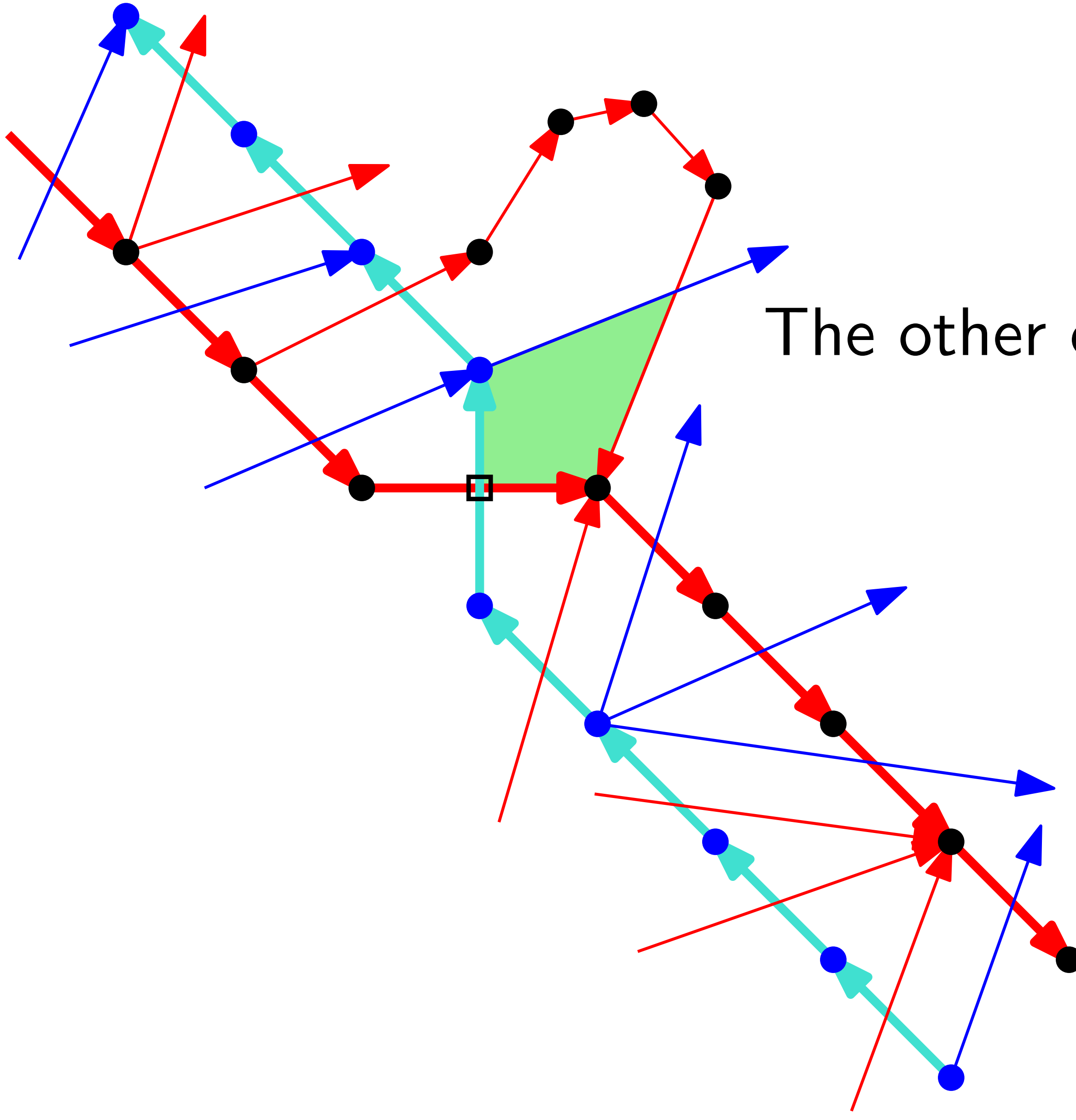
general situation:



The other case is symmetric.

Coordinated sweep

general situation:



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Coordinated sweep

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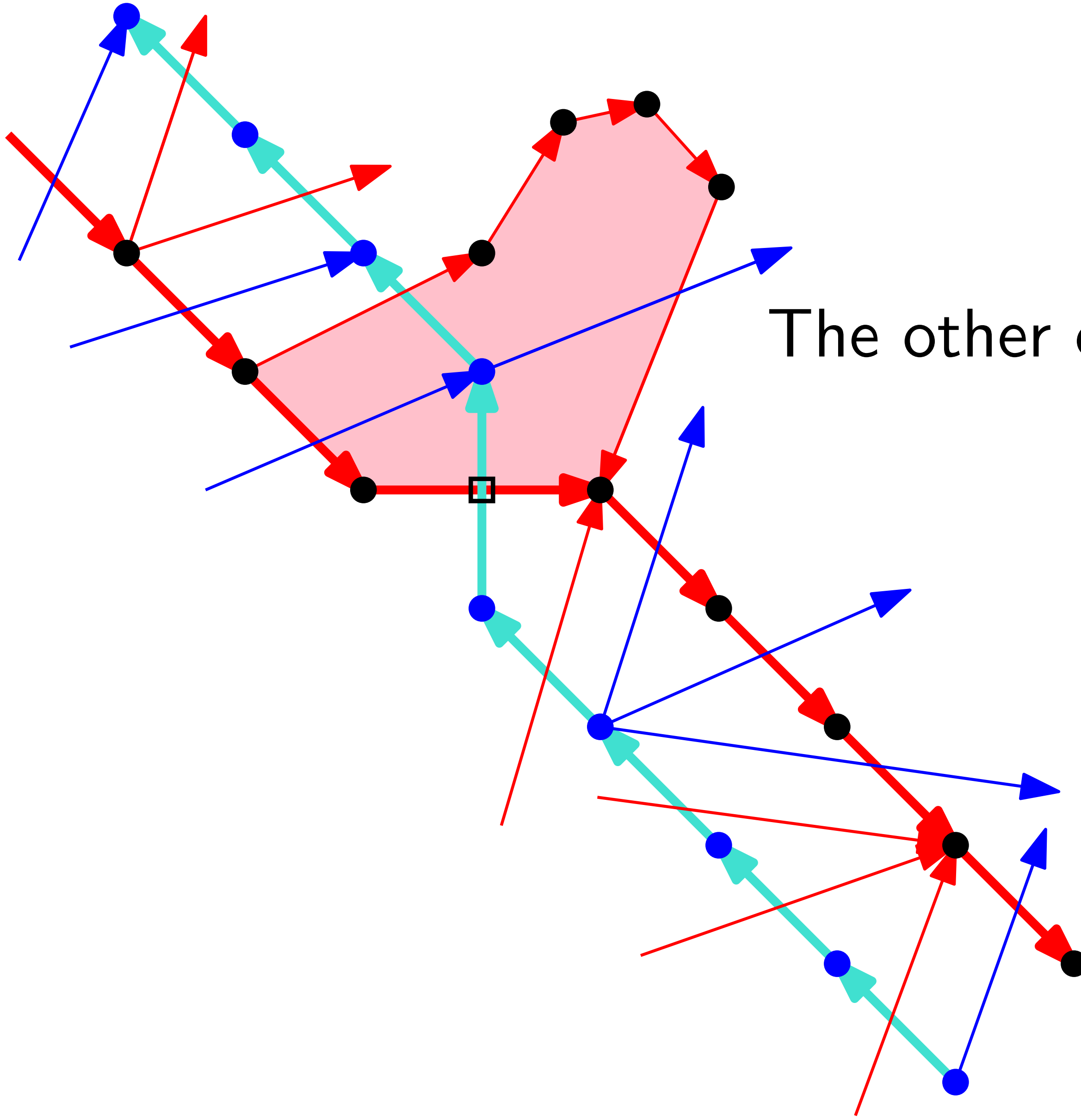

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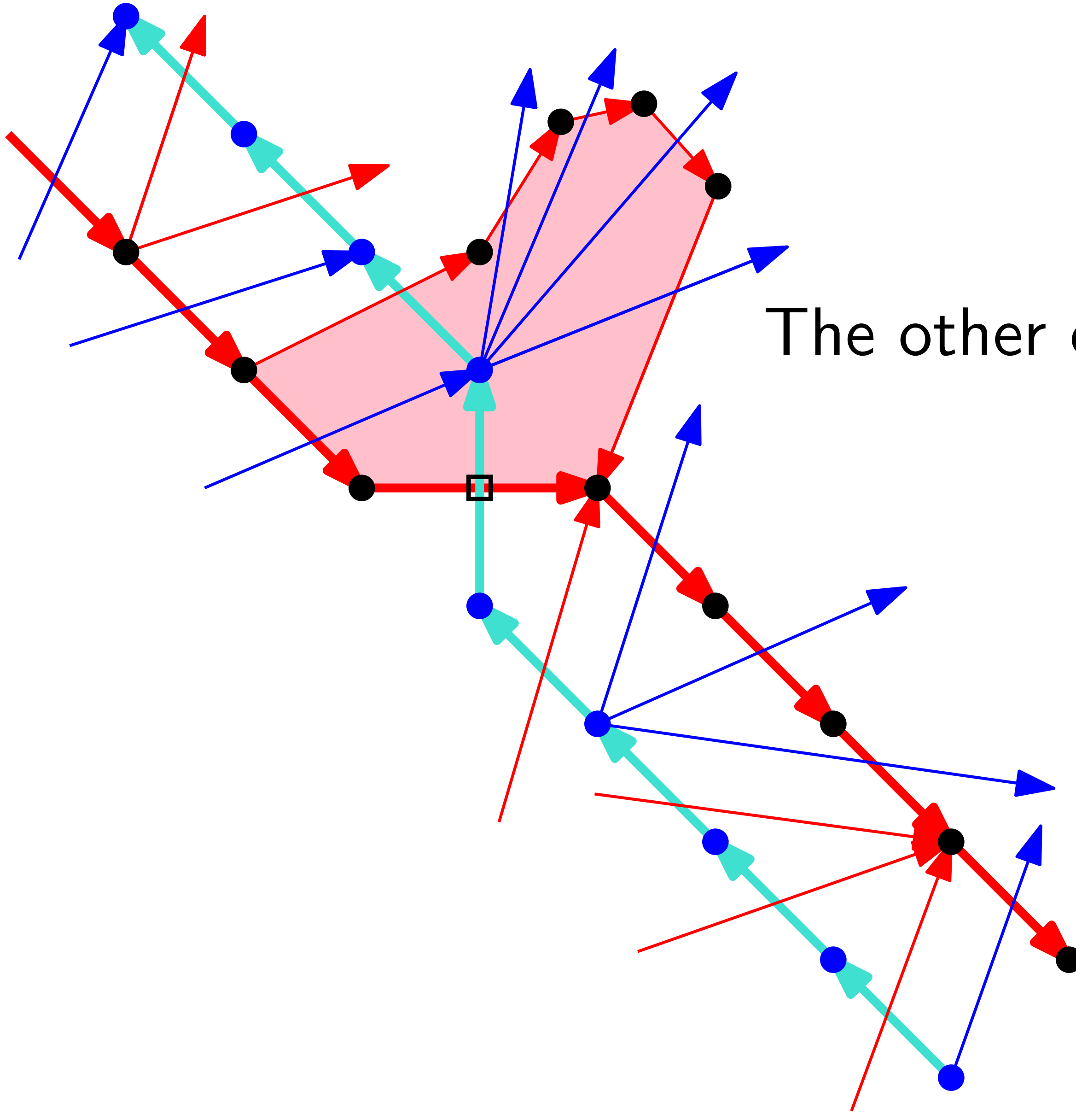


The other case is symmetric.

Coordinated sweep

Questions:

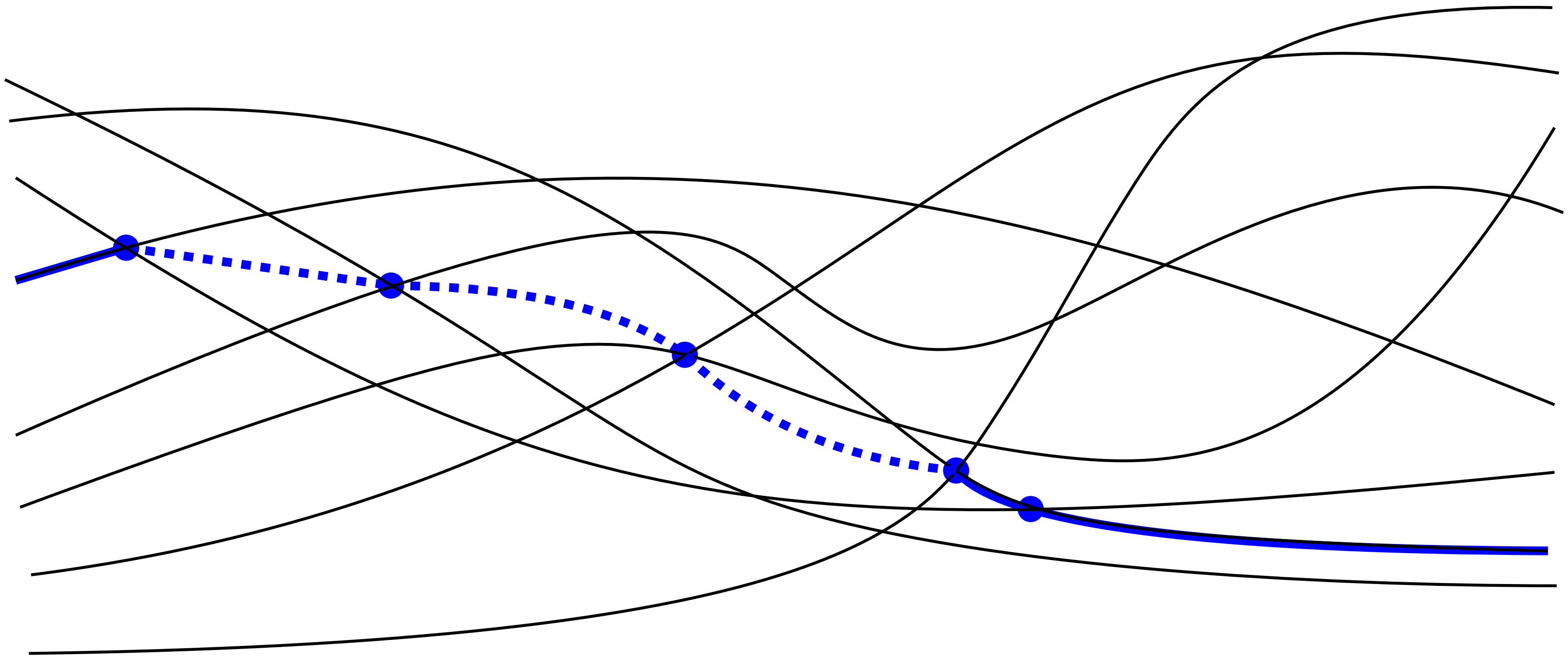
- Consider the (primal/dual) rope length:

  In terms of which parameters can it be bounded?

- Consider a primal sweep in which several independent faces can be swept simultaneously:

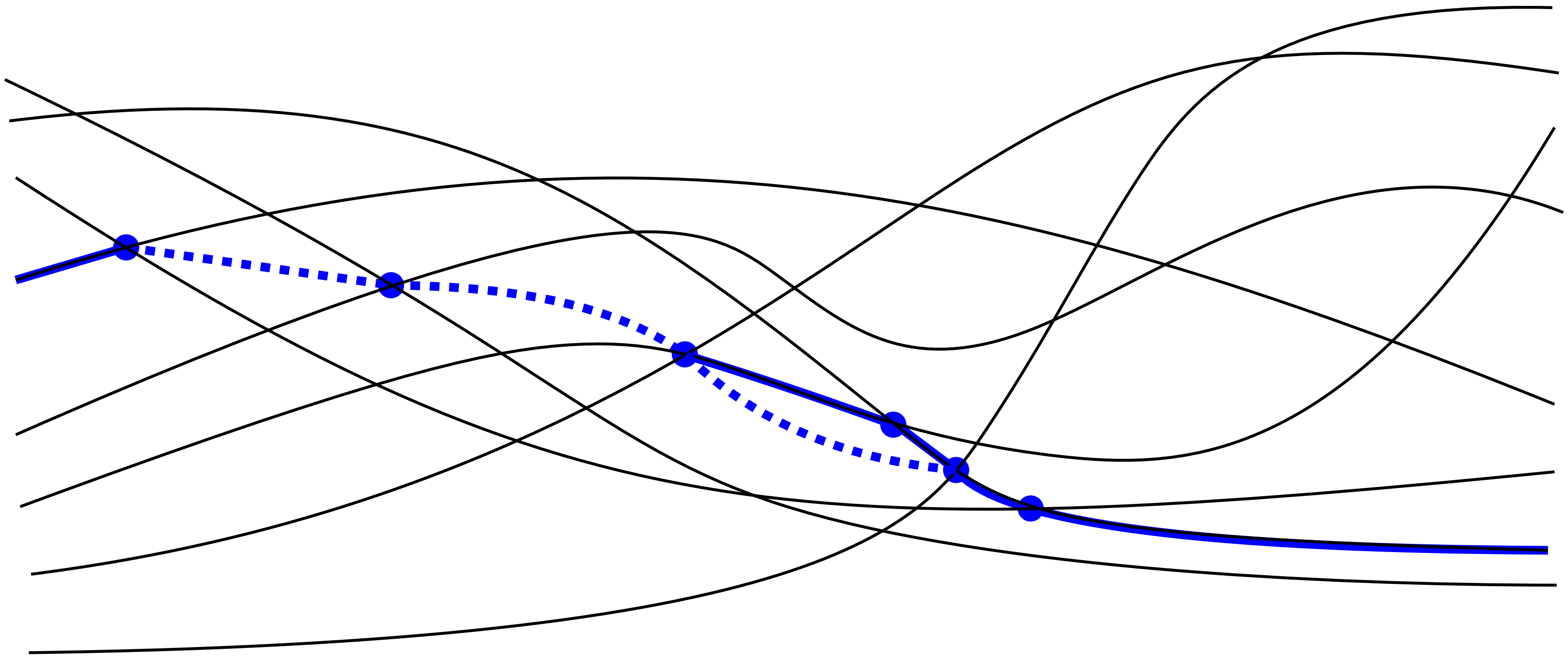
  Can this reduce the required rope length?

What really matters in practice



- Several *distribute* steps are done simultaneously, followed by *collect*s
- *cross* steps are done individually

What really matters in practice



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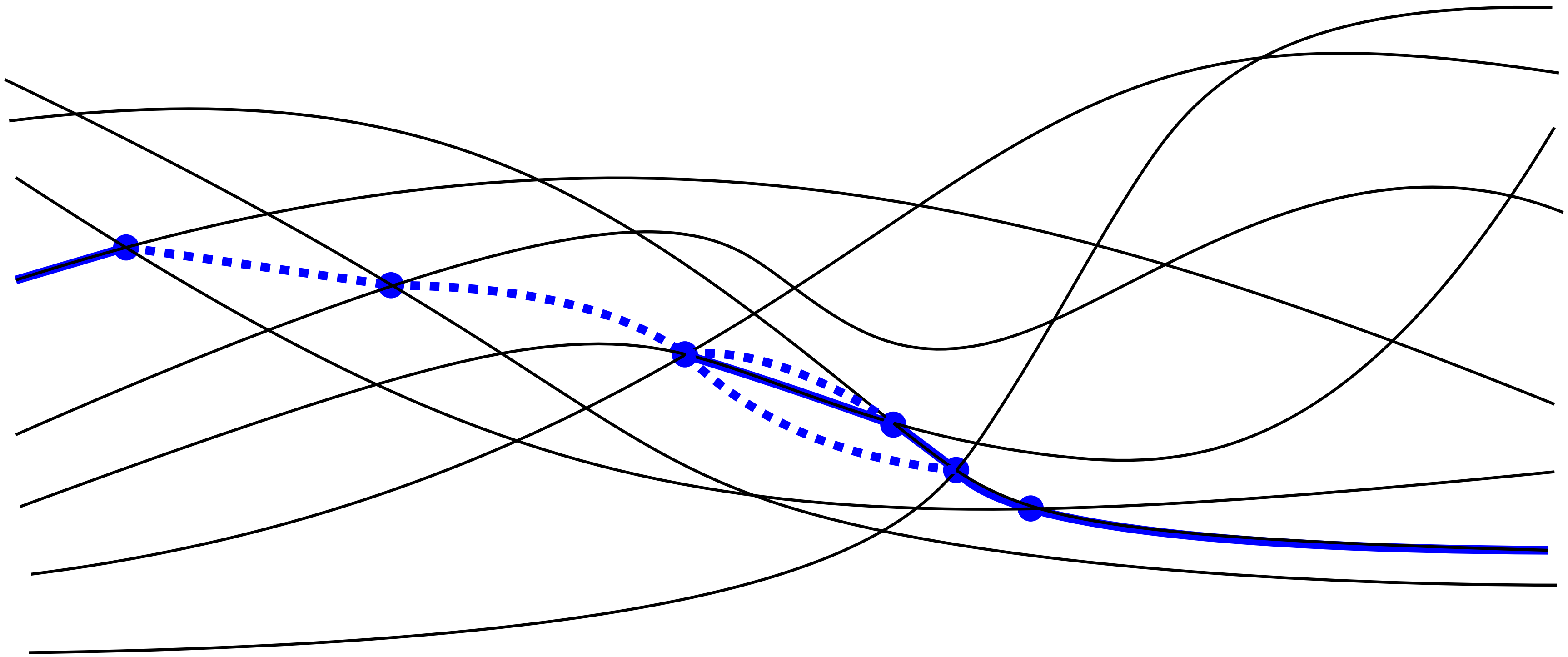
What really matters in practice



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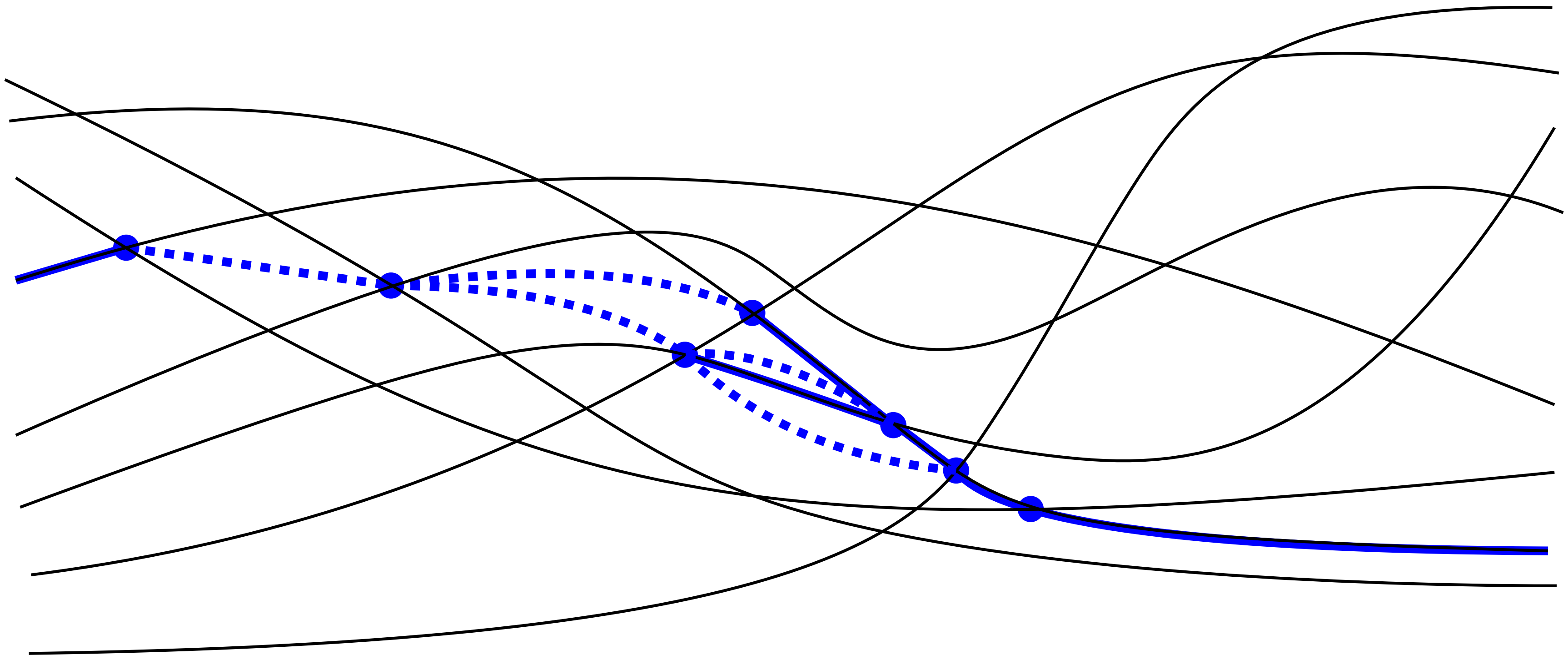

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