Matching Shapes with a Reference Point

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Abstract

For two given point sets, we present a very simple (almost trivial) algorithm to translate one set so that the Hausdorff distance between the two sets is not larger than a constant factor times the minimum Hausdorff distance which can be achieved in this way. The algorithm just matches the so-called Steiner points of the two sets.

The focus of our paper is the general study of reference points (like the Steiner point) and their properties with respect to shape matching.

For more general transformations than just translations, our method eliminates several degrees of freedom from the problem and thus yields good matchings with improved time bounds.

1 Introduction

This paper is motivated by a problem that is typical in application areas such as computer vision or pattern recognition, namely, given two figures A, B, to determine how much they "resemble each other".

Here, a "figure" will be a union of finitely many points and line segments in $\mathbb{R}^2$ or triangles in $\mathbb{R}^3$. Note that sets of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ or surfaces in $\mathbb{R}^3$ can be approximated

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arbitrarily closely by these objects. As a measure for "resemblance" we will use the *Hausdorff-metric* δ_H, which is a somehow natural distance measure and gives reasonable results in practice (see [HKR]). It can be defined in arbitrary dimension d for the set $\mathcal{C}^d$ of all compact subsets of $\mathbb{R}^d$ as follows:

Definition 1 *For $A, B \in \mathcal{C}^d$ let*

$$\widetilde{\delta_H}(A,B) := \max_{a \in A} \min_{b \in B} \|a-b\|,$$

where $\|\cdot\|$ is the Euclidean norm. Then the Hausdorff-distance between A and B is defined as

$$\delta_H(A,B) := \max\{\widetilde{\delta_H}(A,B), \widetilde{\delta_H}(B,A)\}.$$

If A and B consist of n and m line segments, respectively, in the plane their Hausdorff-distance $\delta_H(A,B)$ can be computed in time $O((n+m)\log(n+m))$ (cf. [ABB]). However, it is more natural to assume that A and B are not fixed but can be moved by a *translation*, by a *rigid motion* (translation and rotation) or even transformed by a *similarity* (scaling and rigid motion) in order to match them as well as possible and then determine the minimal Hausdorff-distance. So, in general, we have a set $\mathcal{T}$ of allowed transformations and want to determine for given figures A and B:

$$\min_{T \in \mathcal{T}} \delta_H(A, T(B)).$$

Note that, for similarities, it makes a difference if we exchange the sets A and B in this problem.

This problem of finding an optimal matching has been considered for the two-dimensional case in several previous articles: In Alt, Behrends, and Blömer [ABB] an algorithm of running time $O((nm)\log(nm)\log^*(nm))$ is found for the case that $\mathcal{T}$ is the set of translations along one fixed direction; Agarwal, Sharir, and Toledo [AST] describe an $O((nm)^2\log^3(nm))$ algorithm for arbitrary translations (which can be improved to $O((nm)^2\alpha(nm))$ if A and B are finite sets of points, see Huttenlocher and Kedem [HK]) and in Chew et al. [CGHKKK] an $O((nm)^3\log^2(nm))$ algorithm for arbitrary rigid motions. The two latter algorithms use sophisticated and powerful tools like parametric search and therefore do not seem to be applicable in practice.

Here, we follow a different approach which was already used in [ABB]. We do not try to find an *optimal* solution but an *approximation* to the optimal one by simpler algorithms. More precisely, if the optimal matching transformation yields Hausdorff-distance δ our algorithms will find a transformation T such that

$$\delta_H(A, T(B)) \le a\delta$$

for some constant $a > 1$.
We call such a solution an *approximate matching* with *loss factor* a.

The aim of this paper is to work out the general idea of using "reference points" for approximation algorithms. We will then present a reference point that gives better bounds than the one in [ABB] and can be applied to similarities and problems in three dimensions as well.

A preliminary version of this paper appeared in [AAR].

2 Reference Point Methods

Like in [ABB] approximation algorithms use suitable *reference points*, which we define for arbitrary dimension d as follows:

Definition 2 *Let $\mathcal{T}$ be a set of transformations on $\mathbb{R}^d$. A mapping $r: \mathcal{C}^d \to \mathbb{R}^d$ is called a* reference point *with respect to $\mathcal{T}$ iff*

(a) r is equivariant *with respect to $\mathcal{T}$, i.e., for all $A, B \in \mathcal{C}^d$ and $T \in \mathcal{T}$ we have*

$$r(T(A)) = T(r(A))$$

and

(b) there exists some constant $c \geq 0$ such that if for all $A, B \in \mathcal{C}^d$,

$$\|r(A) - r(B)\| \leq c \cdot \delta_H(A, B).$$

In other words, r is a Lipschitz-continuous mapping between the metric spaces $(\mathcal{C}^d, \delta_H)$ and $(\mathbb{R}^d, \|\cdot\|)$ with Lipschitz constant c. We call c the quality *of the reference point r.*

Based on the existence of a reference point for $\mathcal{T}$ we obtain the following algorithms for approximately optimal matchings where $\mathcal{T}$ is the set of translations, rigid motions, and similarity transformations, respectively:

Algorithm T

1. Compute $r(A)$ and $r(B)$ and translate B by $r(A) - r(B)$ (so that $r(B)$ is mapped onto $r(A)$). Let B' be the image of B.
2. Output B' as the approximately optimal solution (together with the Hausdorff-distance $\delta_H(A, B')$).

Algorithm R

1. as in Algorithm T.
2. Find an optimal matching of A and B' under rotations of B' around $r(A)$.
3. Output the solution B'' and the Hausdorff-distance $\delta_H(A, B'')$.

Algorithm S

1. as in Algorithm T.
2. determine the diameters $d(A)$ and $d(B)$ and scale B' by $\alpha := d(A)/d(B)$ around the center $r(A)$.
3. as Step 2 in algorithm R with the scaled image of B'.
4. as Step 3 in algorithm R.

As the algorithms R and S are formulated, they look only for *proper* rigid motions and similarities, respectively. Reflections can be included by simply running the algorithm a second time with a reflected copy of A.

These algorithms are simpler than the ones for finding the optimal solutions, since after Step 1 the matchings are restricted to ones leaving the reference point invariant. In d dimensions this eliminates d degrees of freedom. The qualities and running times of these algorithms are as follows:

Theorem 3 *Suppose that a reference point of quality c for the sets of transformations $\mathcal{T}$ in the Algorithms T, R, and S can be determined in linear time. In the case of similarity transformations also assume that $r(A)$ always lies within the convex hull conv(A).*

(a) *Algorithm T finds an approximately optimal matching for translations with loss factor $a = c + 1$.*

(b) *Algorithm R finds an approximately optimal matching for rigid motions with loss factor $a = c + 1$.*

(c) *Algorithm S finds an approximately optimal matching for similarity transformations with loss factor $a = c + 3$.*

In the plane, the running times for two sets of n and m points and line segments are $O((n + m)\log(n + m))$ for Algorithm T and $O(nm\log(nm)\log^*(nm))$ for Algorithms R and S. In space, where A and B are sets of triangles, the running times become $O(mn)$ for Algorithm T and $O((nm)^3 H(n, m))$ for Algorithms R and S. Here $H(n, m)$ is the time to compute the Hausdorff distance.

Notice that an upper bound of $O((n^2m + nm^2)log^3(nm))$ for $H(n, m)$ is known, see Alt and Godau [AG].

For the proof of the theorem we need the following lemmas, which can be shown by elementary geometrical considerations.

Lemma 4 *Let $B \subset \mathbb{R}^d$ be a compact set with diameter $d(B)$, and let p be a point in its convex hull conv(B). Let τ_1, τ_2 be homotheties (scalings) with center p and ratios (scaling factors) α_1 and α_2, respectively. Then*

$$\delta_H(\tau_1(B), \tau_2(B)) \leq |(\alpha_1 - \alpha_2)d(B)|.$$

Lemma 5 *If $A, B \subset \mathbb{R}^d$ are compact sets with diameters $d(A)$ and $d(B)$, respectively, then*

$$|d(A) - d(B)| \leq 2\delta_H(A, B).$$

Proof: This follows from the fact that B is contained in the δ-neighborhood of A and vice versa. □

Proof of Theorem 3: We prove only (c) which implicitly contains the proofs for (a) and (b). Consider an optimal similarity transformation S_{opt}. It can be written as $S_{\mathrm{opt}} = \tau_{\mathrm{opt}} \circ T_{\mathrm{opt}}$, where T_{opt} is a rigid motion and τ_{opt} is a homothety with ratio α_{opt}. Let δ be the optimal Hausdorff-distance $\delta = \delta_H(A, S_{\mathrm{opt}}(B))$. Then

$$\|r(A) - r(S_{\mathrm{opt}}(B))\| \leq c\delta. \tag{1}$$

Let t be the translation by $r(A) - r(S_{\mathrm{opt}}(B))$; then $\widetilde{S} = t \circ S_{\mathrm{opt}}$ is a similarity transformation mapping $r(B)$ onto $r(A)$ and

$$\delta_H(A, \widetilde{S}(B)) \leq (c + 1)\delta. \tag{2}$$

Write $\widetilde{S}$ as $\widetilde{S} = \widetilde{\tau} \circ \widetilde{T}$, where $\widetilde{T}$ is a rigid motion mapping $r(B)$ onto $r(A)$ and $\widetilde{\tau}$ is a homothety with center $r(A)$ and ratio is α_{opt}. Let $\alpha = d(A)/d(B)$ as in Algorithm S, τ the homothety with center $r(A)$ and ratio α, and $S = \tau \circ \widetilde{T}$. Then

$$\delta_H(A, S(B)) \leq \delta_H(A, \widetilde{S}(B)) + \delta_H(\widetilde{S}(B), S(B)). \tag{3}$$

Now

$$\begin{aligned} \delta_H(\widetilde{S}(B), S(B)) &= \delta_H\left(\widetilde{\tau}(\widetilde{T}(B)),\ \tau(\widetilde{T}(B))\right) \\ &\leq \left|(\alpha_{\text{opt}} - \alpha) d(\widetilde{T}(B))\right|, \text{ by Lemma 4} \\ &= \left|(\alpha_{\text{opt}} - \alpha) d(B)\right| = \left|\alpha_{\text{opt}} d(B) - d(A)\right| \\ &= \left|d(S_{\text{opt}}(B)) - d(A)\right|, \text{ since } \alpha_{\text{opt}} \text{ was the ratio of } S_{\text{opt}} \\ &\leq 2 \cdot \delta_H(S_{\text{opt}}(B), A) = 2\delta, \text{ by Lemma 5.} \end{aligned} \tag{4}$$

From (2), (3), and (4) we have

$$\delta_H(A, S(B)) \leq (c+3)\delta \tag{5}$$

for some similarity transformation S composed of a rigid motion that maps $r(B)$ onto $r(A)$ and a homothety with center $r(A)$ and ratio α. Since Algorithm S finds the optimum among these similarity transformations the bound (5) holds for it, as well.

For the time bound we observe that Step 1 can be done in linear time. In order to determine the diameters of A and B, we observe that they are equal to the diameters of their convex hulls. So we first compute the convex hulls in time $O(n \log n + m \log m)$. In two dimensions then the diameters can be computed in linear time by rotating calipers, in three dimensions in time $O(n^2 + m^2)$ by considering the distances between all pairs of vertices, edges, or faces.

Step 3, finding the optimal matching under rotations, can be done in time $O(nm \log(nm) \log^*(nm))$ in the plane, as has been shown by Alt, Behrends, and Blömer [ABB] using Davenport-Schinzel sequences. It explicitly computes the Hausdorff-distance, so Step 4 is for free. In 3-space, we have rotations in $\mathbb{R}^3$ around a fixed center, which is an optimal matching problem with 2 degrees of freedom. It can be solved in time $O((nm)^3 H(n,m))$ by methods of [ABB]. $\square$

3 The Steiner Point

The previous section would be useless if it were not possible to find suitable reference points. In [ABB] it was observed that in the two-dimensional case the point $r(A) = (x_{\max}, y_{\max})$, where $x_{\max}$ and $y_{\max}$ are the maximal x- and y-coordinates of points in A, is a reference point of quality $\sqrt{2}$ for translations.

For rigid motions the situation is not as easy. We will first list a few points that come to mind but turn out not to be reference points. In fact, for arbitrary small $\delta > 0$, we can construct figures A_δ, B_δ for which $\delta_H(A_\delta, B_\delta) \leq \delta$, but $\|r(A_\delta) - r(B_\delta)\|$ is not in $O(\delta)$ or does not even converge to 0 for $\delta \to 0$.

point $r(A)$	$\|r(A_\delta) - r(B_\delta)\|$
a) centroid of the vertices of the convex hull	$\Omega(1)$
b) centroid of the convex hull	$\Omega(1)$
c) center of the smallest enclosing circle	$\Omega(\sqrt{\delta})$
d) center of the smallest enclosing ellipse	$\Omega(1)$
e) center of the smallest enclosing rectangle	$\Omega(1)$

Counterexamples a), b), and c) are from Behrends [B]. Figure 1 shows possible sets A_δ, B_δ for cases b) and d). The center of the smallest enclosing rectangle (case e)) is not even well defined for rectangular triangles (see Figure 2). There are two possible smallest enclosing rectangles whose centers are $r_1 = (a/2, b/2)$ and $r_2 = (a/2, b/2) - (ab^2, a^2b) \,/\, 2(a^2 + b^2)$, respectively.

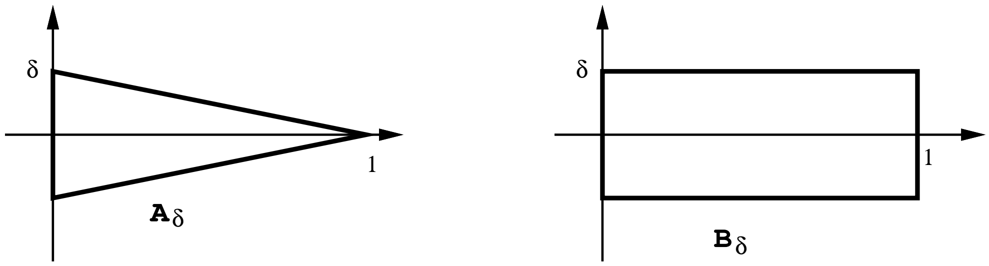

Figure 1: $r(A_\delta) = (1/3, 0)$, $r(B_\delta) = (1/2, 0)$, where r is either the center of the smallest enclosing ellipse or the centroid of the figure. This can be seen by applying an affine transformation which maps A_δ to an equilateral triangle or B_δ to a square.

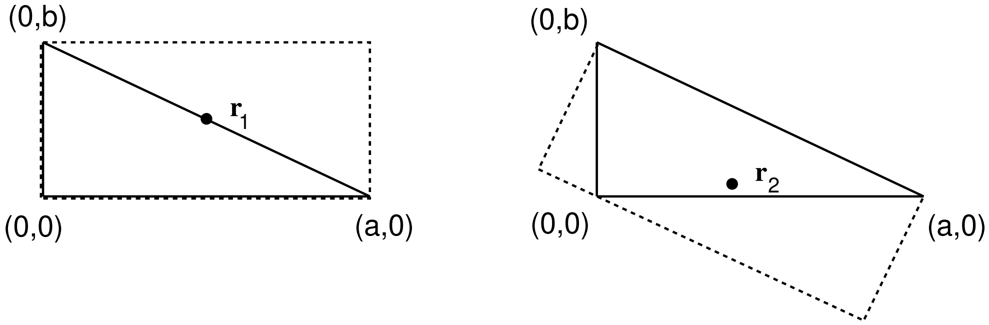

Figure 2: Two smallest enclosing rectangles.

Alt, Behrends, and Blömer [ABB] also gave a positive example of a reference point for rigid motions in two dimensions: the centroid of the boundary of the convex hull. It was shown that this reference point is of quality at most $4\pi + 4 \approx 16.57$. Here, we will give a reference point which works even for similarity transformations, is easy to compute, can be generalized to higher dimensions, and whose quality is $4/\pi \approx 1.27$.

First we observe that we can without loss of generality restrict our attention to *convex* figures. In fact, in [ABB] it was shown that for any two compact sets A, B:

$$\delta_H(conv(A), conv(B)) \leq \delta_H(A, B).$$

From that it follows easily that a reference point for the convex hull of a compact set A is a reference point for A, as well.

Our candidate for a reference point is the so-called *Steiner point*, which has been investigated intensively in the field of convex geometry [G, Sh, Sch].

Definition 6 *We denote by B^d the d-dimensional unit ball and by S^{d-1} its boundary, the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$.*

Let A be a convex body (convex and compact subset) in $\mathbb{R}^d$. The support function *$h_A : \mathbb{R}^d \to \mathbb{R}$ of A is given by*

$$h_A(\boldsymbol{u}) = \max_{\boldsymbol{a} \in A} \langle \boldsymbol{a}, \boldsymbol{u} \rangle$$

(see Figure 3).

The Steiner point *$s(A)$ of A is defined as*

$$s(A) = \frac{d}{\mathrm{Vol}(S^{d-1})} \int_{S^{d-1}} h_A(\boldsymbol{u})\boldsymbol{u} \, d\omega(\boldsymbol{u})$$

where $d\omega(\boldsymbol{u})$ is the surface element of S^{d-1}.

For a non-convex compact set $A \in \mathbb{R}^d$, we define the support function and hence the Steiner point in the same way. They coincide with the support function and the Steiner point of the convex hull conv(A).

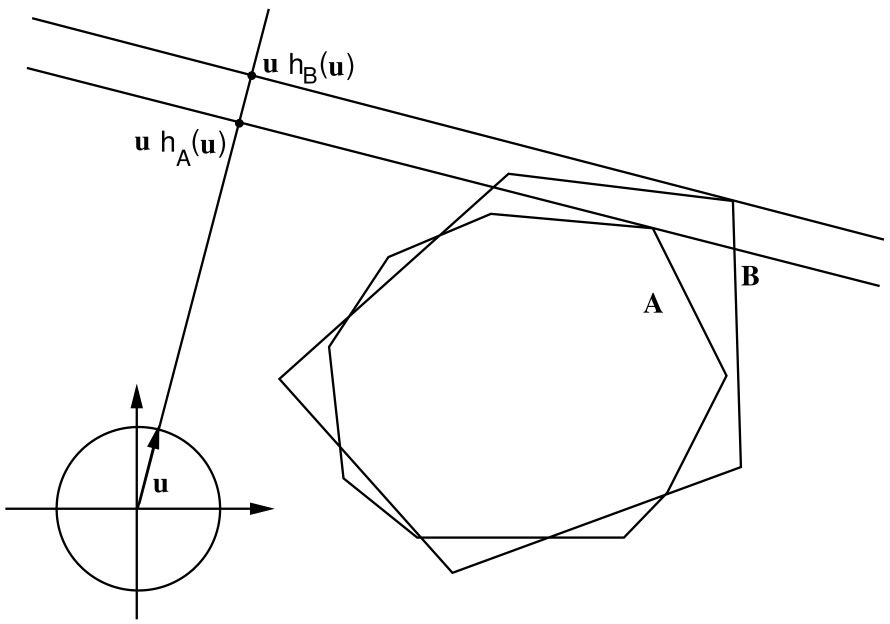

Figure 3: The support functions $h_A(\boldsymbol{u})$, $h_B(\boldsymbol{u})$ of two convex bodies A, B.

Also in the field of functional analysis there is a series of articles concerned with mappings that assign points to convex bodies (for a survey of the results see Przesławski and Yost [PY]). Let X be a Banach space and $\mathcal{C}(X)$ the set of closed, bounded, convex, non-empty subsets of X. Then using the Hausdorff distance as a metric on $\mathcal{C}(X)$, a continuous mapping $r: \mathcal{C}(X) \to X$ is called a *selector* if $r(A) \in A$ for all $A \in \mathcal{C}(X)$. *Michael's selection Theorem* [M] states the existence of a selector for any Banach space. Several authors raised the question whether there exist *Lipschitz-continuous* selectors and showed that they do if X is finite-dimensional. In this context Przesławski rediscovered the Steiner point [P].

The value of the Lipschitz constant of the Steiner point is well-known, see for example Daugavet [D]. To keep our paper self-contained we include the elementary calculation here.

Let

$$\chi_d = \frac{2d\,\mathrm{Vol}(B^{d-1})}{\mathrm{Vol}(S^{d-1})}.$$

Then by the formulas for surface and volume of d-dimensional spheres

$$\chi_d = \frac{2\Gamma(d/2+1)}{\sqrt{\pi}\Gamma(d/2+1/2)} \sim \sqrt{\frac{d}{2\pi}} \tag{6}$$

Altogether, we have, combining results from Grünbaum [G], Schneider [Sch], and Daugavet [D]:

Theorem 7 *The Steiner point is a reference point for similarity transformations in arbitrary dimension $d \geq 2$. Its quality is χ_d, which for $d = 2$ is $4/\pi$, for $d = 3$ it is $3/2$, for arbitrary d it lies between $\sqrt{2/\pi}\sqrt{d}$ and $\sqrt{2/\pi}\sqrt{d+1}$.*

Proof: The equivariance of the Steiner point under similarity transformations is well known [G, Sch]. For the bound on the quality, we observe that for two convex bodies A and B, $\|h_A(\boldsymbol{u}) - h_B(\boldsymbol{u})\| \leq \delta_H(A, B) =: \delta$ for any $\boldsymbol{u} \in S^{d-1}$ (see Figure 3).

Now let $\boldsymbol{p} = s(A) - s(B)$, and consider the inner product of $\boldsymbol{p}$ with an arbitrary unit vector $\boldsymbol{e}$. Without loss of generality we assume that $\boldsymbol{e} = (0, \ldots, 0, 1)$ is the unit vector in the d-th coordinate direction.

$$\begin{aligned}
\langle \boldsymbol{p}, \boldsymbol{e} \rangle &= \langle s(A) - s(B), \boldsymbol{e} \rangle \\
&= \frac{d}{\mathrm{Vol}(S^{d-1})} \int\limits_{S^{d-1}} (h_A(\boldsymbol{u}) - h_B(\boldsymbol{u})) \langle \boldsymbol{u}, \boldsymbol{e} \rangle \, d\omega(\boldsymbol{u}) \\
&\leq \frac{d}{\mathrm{Vol}(S^{d-1})} \cdot \left[\int\limits_{\substack{S^{d-1} \\ u_n \geq 0}} (+\delta) \langle \boldsymbol{u}, \boldsymbol{e} \rangle \, d\omega(\boldsymbol{u}) + \int\limits_{\substack{S^{d-1} \\ u_n \leq 0}} (-\delta) \langle \boldsymbol{u}, \boldsymbol{e} \rangle \, d\omega(\boldsymbol{u}) \right] \\
&= \delta d \cdot \left[\frac{1}{\mathrm{Vol}(S^{d-1})/2} \cdot \int\limits_{\substack{S^{d-1} \\ u_n \geq 0}} \langle \boldsymbol{u}, \boldsymbol{e} \rangle \, d\omega(\boldsymbol{u}) \right]
\end{aligned}$$

The expression in brackets in the last line is nothing but the d-th coordinate of the center of gravity of the upper unit half-sphere. If we compute the integral by projecting away the d-th coordinate and integrate over $(x_1, x_2, \ldots, x_{d-1}) \in B^{d-1} \subseteq \mathbb{R}^{d-1}$, a straightforward calculation gives that the surface element is transformed by

$$d\omega(\boldsymbol{u}(x_1, x_2, \ldots, x_{d-1})) = \frac{1}{\langle \boldsymbol{u}, \boldsymbol{e} \rangle} dx_1 dx_2 \cdots dx_{d-1}.$$

Thus the integral turns out to be just the volume of B^{d-1}, and we get the following bound on $\langle \boldsymbol{p}, \boldsymbol{e} \rangle$:

$$\langle \boldsymbol{p}, \boldsymbol{e} \rangle \leq \delta d \cdot \frac{\mathrm{Vol}(B^{d-1})}{\mathrm{Vol}(S^{d-1})/2} = \delta \chi_d$$

This last expression is a bound on the length of $\boldsymbol{p}$ since $\boldsymbol{e}$ was in fact an arbitrary unit vector and therefore can be substituted by $\boldsymbol{p}/\|\boldsymbol{p}\|$. Considering the definition of χ_d the values $\Gamma(3/2) = \sqrt{\pi}/2$, $\Gamma(2) = 1$, and $\Gamma(5/2) = 3\sqrt{\pi}/4$ give the claimed quality bounds for $d = 2$ and $d = 3$. The quotient of the two Γ-functions in (6) is between $\sqrt{d/2}$ and $\sqrt{(d+1)/2}$, and this gives the general bound.

From the proof of the upper bound one can see how to construct an example showing that it cannot be improved: S^{n-1} must be divided into two half-spheres, and $h_A(\boldsymbol{u}) - h_B(\boldsymbol{u})$ will ideally always be equal to $+\delta$ or $-\delta$, depending on the half-sphere in which $\boldsymbol{u}$ lies. Figure 4 shows two two-dimensional point sets A_δ and B_δ. A_δ is just a circle of radius r,

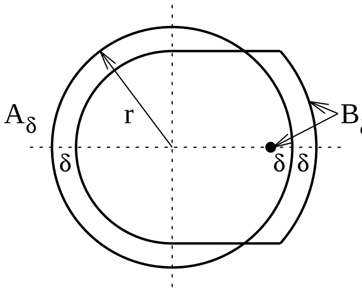

Figure 4: $\|s(A_\delta) - s(B_\delta)\|$ is close to $4/\pi \cdot \delta$.

and B_δ consists of a "distorted" circle and an additional point. If we allow to apply any similarity to B_δ in order to minimize the Hausdorff distance from A_δ, the optimal position is as shown in Figure 4, and the Hausdorff distance is δ. The distance of the Steiner point $s(B_\delta)$ from the center $s(A_\delta)$ of the circle can be calculated as $2/\pi \cdot \sqrt{r\delta} \arccos \frac{r-\delta}{r+\delta}$, which approaches $4/\pi \cdot \delta$ as r goes to ∞. If one lets the two Steiner points coincide, the Hausdorff distance rises by this amount, showing that $1 + 4/\pi$ is indeed the loss factor of Algorithm T. Since A_δ is rotation-symmetric, this holds also for Algorithm R. The above construction generalizes easily to higher dimensions. □

The following theorem is well-known [G, Sh].

Theorem 8 *The Steiner point of a convex polytope is the weighted sum of its vertices, where the weight of vertex v is that fraction of the surface of the unit sphere that lies between the unit vectors normal to the hyperplanes meeting at v (the normalized exterior angle at v). (For a two-dimensional example see Figure 5.)*

For smooth convex bodies, the Steiner point can also be defined as the centroid of a non-uniform mass distribution on the boundary, where the density is the (Gaussian) curvature.

Combining Theorems 3, 7, and 8 we get:

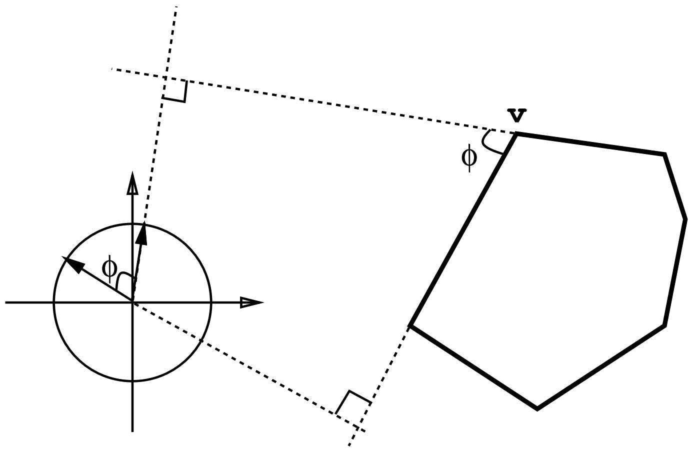

Figure 5: The weight of vertex v of the polygon is $\frac{\phi}{2\pi}$.

Theorem 9 *Let A and B be sets of n and m line segments in $d = 2$ dimensions or n and m triangles in $d = 3$ dimensions. Then approximately optimal matchings can be found for A and B applying the corresponding algorithms of Section 2 as indicated in the following table.*

$\mathcal{T}$	running time	loss factor
translations		
$d = 2$	$O((n+m)\log(n+m))$	$4/\pi + 1$
$d = 3$	$O(H(n,m))$	2.5
rigid motions		
$d = 2$	$O(nm\log(nm)\log^*(nm))$	$4/\pi + 1$
$d = 3$	$O((nm)^3 H(n,m))$	2.5
similarities		
$d = 2$	$O(nm\log(nm)\log^*(nm))$	$4/\pi + 3$
$d = 3$	$O((nm)^3 H(n,m))$	4.5

Proof: For the proof note that the Steiner point for a convex polygon or polytope can be computed in linear time because of Theorem 8, after the convex hulls have been constructed in $O(n \log n + m \log m)$ time. The bound of $O(H(n,m))$ (cf. Theorem 3) for translations in three dimensions comes from the final computation of the Hausdorff distance. Just finding the approximately optimal translation takes only $O(n \log n + m \log m)$ time. □

4 Lower Bounds

In this section we will prove lower bounds for the quality of reference points with respect to the set of all *translations*. Of course, these bounds carry over to every set of transformations $\mathcal{T}$ which includes all translations, i. e., to every interesting set for which we proposed an algorithm in Section 2. We first show an easy lower bound for one dimension, whose proof already gives the flavor of the proof of our two-dimensional lower bound.

Theorem 10 *Reference point based matching for translations (Algorithm T) cannot yield a loss factor better than 2 in the worst case. This holds for arbitrary dimension $d \geq 1$.*

Proof: To see this, consider the 1-dimensional sets of 2 or 3 points A_1, A_2, A_3, A_4, A_5 shown in Figure 6. Clearly $\delta_H(A_{i-1}, A_i) = \delta$ for $i = 2, 3, 4, 5$. Suppose we have a reference

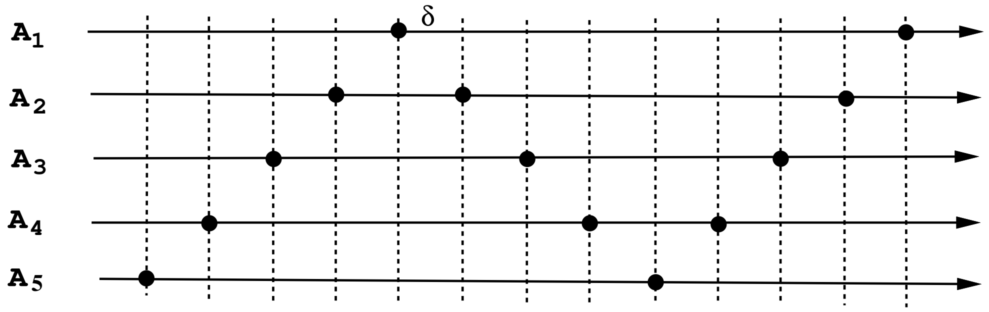

Figure 6: Sets A_1, A_2, A_3, A_4, A_5.

point based matching algorithm and match A_2 onto A_1, A_3 onto the new position A'_2 of A_2, A_4 onto the new one A'_3 of A_3, and A_5 onto A'_4. Since all reference points are matched onto each other and A_1 and A_5 are congruent A'_5 must coincide with $A'_1 := A_1$. So for some i, $1 \leq i \leq 4$, the left endpoint of A'_i must lie not to the right of the left endpoint of A'_{i+1}. It is easy to check that this implies $\delta_H(A'_i, A'_{i+1}) \geq 2\delta$. ◻

We may augment each set A_i in the above proof by a point which is at a large distance M from its rightmost point. This prevents reflections and scalings from possibly improving the Hausdorff distance between A_i and A_{i+1}, and so the above lower bound remains true even if reflections and scalings are allowed.

Note that in one dimension, reference points with loss factor at most 2 exist. For example, the left-most point, or the right-most point, or any fixed convex combination of these two points will do. The Steiner point is just the midpoint between the two extremes.

Researchers in functional analysis also investigated lower bounds for Lipschitz constants of selectors. For this purpose they considered the embeddings

$$\mathbb{R}^d \overset{f}{\hookrightarrow} \mathcal{K}^d \overset{g}{\hookrightarrow} C(S^{d-1}),$$

where $\mathcal{K}^d$ is the set of *convex* and compact subsets of $\mathbb{R}^d$, $C(S^{d-1})$ the set of continuous functions $S^{d-1} \to \mathbb{R}$, $f(x) = \{x\}$ for all $x \in \mathbb{R}^d$, and $g(A) = h_A$ for all $A \in \mathcal{K}^d$.

These embeddings are compatible with the vector addition on $\mathbb{R}^d$, the Minkowski sum on $\mathcal{K}^d$ and the standard addition of functions in $C(S^{d-1})$. They are also compatible with the Euclidean distance in $\mathbb{R}^d$, the Hausdorff distance in $\mathcal{K}^d$ and the supremum norm in $C(S^{d-1})$. Observe that $\mathbb{R}^d$ and $C(S^{d-1})$ are Banach spaces with these operations, whereas $\mathcal{K}^d$ is not, since it is not a group with respect to Minkowski addition. Let $r: \mathcal{C}^d \to \mathbb{R}^d$ be a reference point with respect to translations. We may assume without loss of generality that $r(\{x\}) = x$ for all $x \in \mathbb{R}^d$. If this does not hold, we may select some arbitrary point o and subtract the constant vector $r(\{o\}) - o$ from r. This clearly does not change the quality of r, and it does not violate the equivariance with respect to translations. By the same equivariance property, $r(\{o\}) = o$ implies then that $r(\{x\}) = x$ for all $x \in \mathbb{R}^d$. In other words, we may regard r as a *retract*, i. e., a function, which, restricted to its range, equals the identity. Linear retracts between Banach spaces are called *projections*. Rutovitz [R] and Daugavet [D] investigated lower bounds on the Lipschitz constant (i. e. the norm $\|P\| = \sup_{x \neq 0} \|P(x)\|/\|x\|$) of projections between Banach spaces. They (implicitly) established a lower bound of χ_d for the Lipschitz constant of any projection $C(S^{d-1}) \to \mathbb{R}^d$, $d \geq 2$. Przesławski and Yost [PY] could extend this lower bound from (linear) projections to arbitrary retracts from $\mathcal{K}^d$ to $\mathbb{R}^d$. Consequently, the lower bound holds also for retracts $r: \mathcal{C}^d \to \mathbb{R}^d$, and therefore for reference points. So we can state (cf. Proposition 4.5 in [PY]):

Theorem 11 *The quality of any reference point with respect to translations from $\mathcal{C}^d$ into $\mathbb{R}^d$, $d \geq 2$ cannot be better (i.e. smaller) than χ_d.*

In this sense Theorem 7 shows that the Steiner point is an optimal reference point.

Rutovitz' proof implicitly shows that for any given projection $P: C(S^{d-1}) \to \mathbb{R}^d$ bad examples can be constructed where the Lipschitz constant exceeds or gets arbitrarily close to χ_d. However, the extension to retracts by Przesławski and Yost [PY] uses the existence of invariant means on abelian semigroups. This is based on the Hahn-Banach Theorem, which, in turn, is based on (a weaker version of) the Axiom of Choice. Consequently, the proof by Przesławski and Yost is nonconstructive, i. e., it does not yield bad examples for given retracts.

In contrast, we will present for the two-dimensional case a "universally bad" example in the following theorem. However, the lower bound does not quite match the upper one.

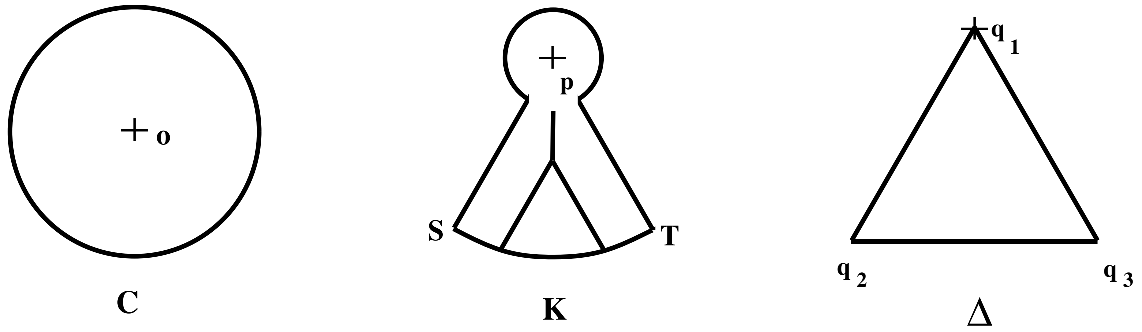

Figure 7: C, K, and Δ. The origin is marked by a cross.

In fact, we consider three point sets (see Figure 7): A circle C with center o and radius 5; a keyhole-shaped figure K with two circular arcs centered at p of radii 2 and 8 and opening angles of $5\pi/3$ and $\pi/3$, respectively, and with three line segments forming an upside-down Y inside; and finally an equilateral triangle $\Delta = q_1 q_2 q_3$ with side length 10.

The dimensions given in the middle part of Figure 8 exhibit the position of the Y in the keyhole K. The origin is at the points o, p, and q_1, respectively.

For these three sets we have

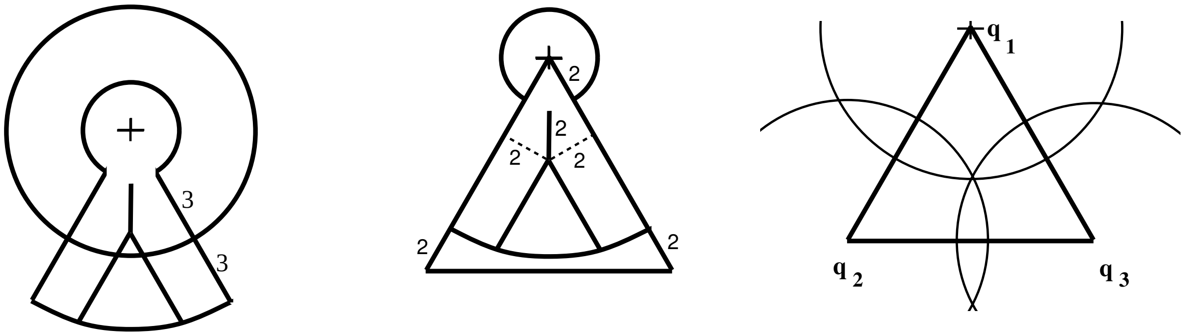

Figure 8: Optimal matchings of K with C and Δ.

Theorem 12 *For any reference point $r: \mathcal{C}^2 \to \mathbb{R}^2$ with respect to translations*

$$\begin{aligned} \textit{either} \quad \|r(K) - r(C)\| \; &\geq \sqrt{4/3}\, \delta_H(K, C) \\ \textit{or} \quad \|r(K) - r(\Delta)\| \; &\geq \sqrt{4/3}\, \delta_H(K, \Delta). \end{aligned}$$

To see this we first show:

Lemma 13 *The optimal matchings between C and K and K and Δ are achieved by superposing the origins, as shown in Figure 8, and we have $\delta_H(K, C) = 3$ and $\delta_H(K, \Delta) = 2$.*

Proof: Consider any placement of K such that the Hausdorff distance to C is not greater than 3. Then K must lie within the 3-neighborhood of C, i.e. the set of all points having distance ≤ 3 to C, which is an annulus A with inner radius 2 and outer radius 8 (see Figure 9).

Especially K must not intersect the inner hole H of A. So if K is not placed as in Figure 8, the Y prevents H from lying inside the outer boundary of K. In particular, the equilateral triangle pST cannot contain H. Since the triangle and H are convex there is a straight line l separating them. So triangle pST must lie in the intersection of A with a halfplane not containing H, consequently in a strip U of width 6. This is not possible since the minimum width of triangle pST is its height which is slightly greater than 6.92. Similarly, by considering the possible positions of K with respect to the 2-neighborhood of Δ, it can be shown that the position between Δ and K given in Figure 8 is optimal. □

Proof of Theorem 12: Let $r: \mathcal{C}^2 \to \mathbb{R}^2$ be any reference point, $L_C = \|r(C) - r(K)\| \,/\, \delta_H(C, K)$, $L_\Delta = \|r(\Delta) - r(K)\| \,/\, \delta_H(\Delta, K)$, and $c = \max\{L_C, L_\Delta\}$.

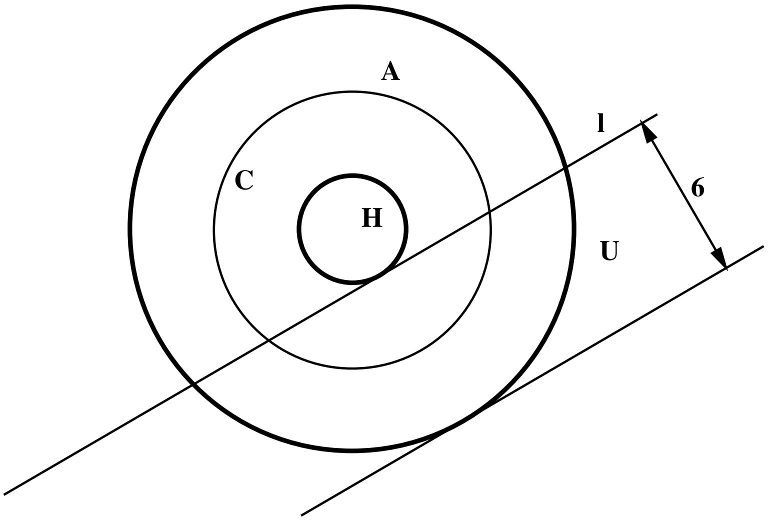

Figure 9: 3-neighborhood of C.

Let us first assume that the reference point $r(C)$ of the circle is its center o. Since $\delta_H(C, K) = 3$, the reference point $r(K)$ of K must lie in a circle of radius $3c$ around p. Similarly, $r(\Delta)$ must lie in a circle of radius $2c$ around $r(K)$, i.e., in a circle of radius $5c$ around q_1. If we turn K by 120° in both ways, we can conclude in a similar way that $r(\Delta)$ must lie in circles of radius $5c$ around q_2 and around q_3. By equivariance with respect to translations, the triangle Δ has only a single reference point regardless where it is placed, and therefore the three circles must intersect, as shown in the right part of Figure 8. This means that c must be at least $\sqrt{4/3} \approx 1.155$.

If the reference point $r(C)$ is not the center o, the only difference is that the centers of the final three circles will be translated by the respective amount. ◻

5 Open Problems

Our example in Theorem 12 proves a lower bound for the Lipschitz constant of a reference point in a completely elementary way, in contrast to the proof of Theorem 11 by Przesławski and Yost [PY], which uses rather deep analytical tools. On the other hand, our bound of $\sqrt{4/3} \approx 1.155$ is not as strong as the true lower bound of $4/\pi \approx 1.272$. Furthermore, we use non-convex point sets, whereas the lower bound holds even when restricted to convex sets. It is thus challenging to find better constructions that either give a better bound or that use convex sets only.

In an abstract graph-theoretic model of the problem, we could recently obtain a slightly larger bound than $\sqrt{4/3}$, but as yet we have not been able to translate this into concrete geometric examples.

We believe that our example of Figure 7 generalizes to three and higher dimensions, using a ball, a simplex, and some kind of higher-dimensional keyholes interpolating between them, but the proof should certainly be more complicated.

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