Quality Ratios of Measures for Graph Drawing Styles

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Abstract

When comparing two different styles to draw a graph, one can wonder how much better or worse one style can be than another one, in the worst case, for some quality measure. We compare planar straight-line drawings with fixed and free embeddings, planar circular arc drawings, and non-planar straight-line drawings, and consider the quality measures angular resolution, area requirement, edge length ratio, and feature resolution.

1 Introduction

The research area of graph drawing is concerned with placing nodes of a graph and representing its edges visually in an aesthetic or purposeful manner [4]. There are several drawing styles, like the common straight-line drawings. Here, all edges are straight-line segments. If the graph is planar, it is drawn without crossings among these line segments. Other drawing styles are orthogonal drawings, one-bend drawings (or k-bend drawings), circular-arc drawings [2] (or the special case of Lombardi drawings [6]), visibility representations [11], and contact representations [3].

For the more common graph drawing styles, several criteria exist that constitute a good drawing. Edges should not be too long nor too short, edges incident to the same vertex should not make a very small angle, and edges should not come very close to vertices they are not incident to. The area requirement when drawing the vertices on an integer grid is also studied often. Since this paper will deal with such quality criteria, we define them as "measures".

Angular resolution: The smallest angle occurring between two edges incident to the same vertex. In case of circular arcs we take the tangents at the vertex. When intersections occur we also consider the angles between intersecting edges (or the tangents at the intersection point).

Area requirement: The number of integer grid points in an axis-parallel bounding box when all vertices are on the integer grid.

Edge length ratio: The ratio in length of the longest and the shortest edge.

Feature resolution: The ratio of the length of the longest edge and the shortest distance between two vertices or between a vertex and a non-incident edge.

We define the graph drawing styles (or restricted versions of a style) that we will consider in this paper. These are:

Fixed-embedding planar straight-line drawing: edges are line segments, crossings are not allowed, and the cyclic order of the edges at each vertex is specified.

Free-embedding planar straight-line drawing: edges are line segments and crossings are not allowed.

Free-embedding planar circular-arc drawing: edges are circular arcs and crossings are not allowed.

Non-planar straight-line drawing: edges are line segments and edge crossings are allowed, but no edge contains a vertex to which it is not incident.

When we compare two drawing styles, we insist on grid drawings only when we want to compare the area requirement. We can find the optimal drawing of a graph in the four drawing styles for each of the quality measures. Figure 1 illustrates them for angular resolution for the graph K_4. Since all embeddings are isomorphic, there is no distinction between the first two styles in this case.

A *quality ratio* takes two drawing styles and a quality measure and describes how much better the one drawing style can be than the other for some graph. If M is one of the quality measures and G is a graph that can be drawn in styles S_1 and S_2, where style S_1 is more general than style S_2, then the quality ratio QR is defined as:

$$QR(S_1\!:\!S_2, M) = \sup_G \frac{M \text{ for best } S_1 \text{ drawing of } G}{M \text{ for best } S_2 \text{ drawing of } G},$$

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planar graphs	free : fixed	circular : straight	crossing : planar
angular resolution	≥ 12	≥ 4.8	∞, $\Omega(\sqrt{d/\log d})$
area requirement	∞, $\Omega(n)$	∞	∞
edge-length ratio	∞	∞	∞
feature resolution	∞, $\Omega(n)$	$\geq 3\sqrt{3}/\pi$	$\geq 2.509...$

Table 1: Quality ratio results for planar graphs with n vertices. d denotes the maximum degree.

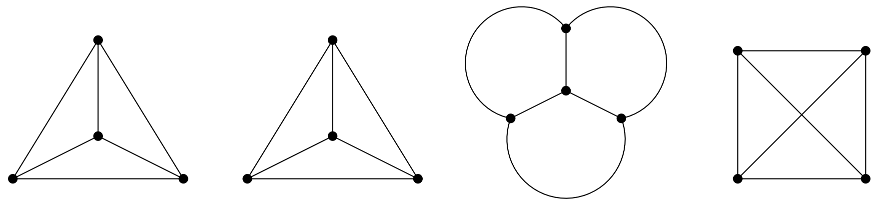

Figure 1: Optimal angular resolution drawings of K_4 for the four drawing styles.

or

$$QR(S_1 : S_2, M) = \sup_G \frac{M \text{ for best } S_2 \text{ drawing of } G}{M \text{ for best } S_1 \text{ drawing of } G},$$

depending on whether the measure should be maximized or minimized to obtain high quality. The quality ratio is always ≥ 1. Quality ratios were studied before in graph drawing to compare planar straight-line drawings and right-angle crossing drawings [12].

Since we have four drawing styles, we get six potential pairs. For the pair circular-arc and non-planar drawings, we do not have that one style is more general than the other, so five pairs remain. We will consider three of these pairs of drawing styles, the pairs we consider most interesting. They are:

- fixed embedding versus free embedding planar straight-line drawings,
- free embedding planar straight-line versus planar circular arc drawings, and
- free embedding planar straight-line versus non-planar straight-line drawings.

Figure 1 shows that, for angular resolution, the second quality ratio of the list is ≥ 4 and the third one is ≥ 1.5. If we find a graph that leads to a larger ratio, then we get better lower bounds.

Table 1 summarizes our results. The result on the angular resolution for crossing versus planar straight-line drawings follows directly from the literature. Formann et al. [7] showed that every planar graph can be drawn with angular resolution $\Omega(1/d)$, where d is the maximum degree of any vertex of the graph, but the drawing may have crossings. Garg and Tamassia [8] showed that there exists a family of planar graphs for which any planar embedding has angular resolution $O(\sqrt{\log d/d^3})$, and this family consists of graphs with arbitrarily large vertex degrees.

We will study the same questions for a more restricted class of graphs, namely trees: see Table 2. The results we get are quite different: since any tree can be drawn with unit edge length in any embedding, and any tree can be drawn with optimal angular resolution in any embedding, the table has 1s in the first and third row as tight bounds. The area requirement of a planar straight-line drawing of a tree is still a major open problem in graph drawing: for general trees the trivial lower bound is $\Omega(n)$ and the upper bound is $O(n \log n)$ [10]. A non-constant lower bound for the quality ratio for area requirement for crossing versus planar embeddings would immediately imply the first non-trivial lower bound for the area requirement of trees.

There are two open spots in Table 2. Here we do not have any results, although we could have written the trivial ≥ 1 bounds.

2 Fixed vs. free embedding straight-line drawings

We consider the four quality ratios for planar straight-line drawings with fixed and free embedding.

We first show that the quality ratio for *angular resolution* is at least 12. Refer to Figure 2. We begin with an embedding of K_4 with outer face v_1, v_2, v_3 and we add $k \geq 0$ neighbours to each v_i. We choose k such as to increase the degree of each v_i to a multiple of 12 ($k = 9$ in the figure). Then we can embed the original K_4 as an equilateral triangle with a vertex in its center and the neighbours of each v_i with perfect angular resolution around v_i. The smallest angle in this drawing is thus $360/(k+3)^\circ$. In the fixed embedding setting, we force all new neighbours to the inside of the K_4 as shown in Figure 2. This gives a smallest angle of at most $30/(k+1)^\circ$. Hence, the quality ratio is at least $360/30 \cdot (k+1)/(k+3) = 12 \cdot (k+1)/(k+3)$, which

trees	free : fixed	circular : straight	crossing : planar
angular resolution	1	1	1
area requirement	$\geq 16/15$	≥ 1.5	$\geq 22/21$
edge-length ratio	1	1	1
feature resolution	$> 1+\varepsilon$		

Table 2: Quality ratio results for trees. $\varepsilon > 0$ is some unknown constant.

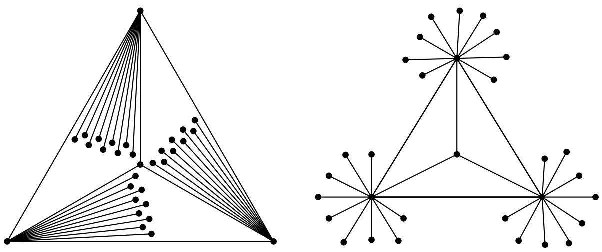

Figure 2: Angular resolution for two different embeddings of the same graph.

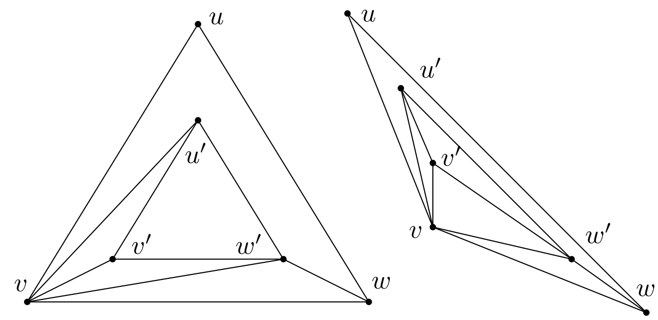

Figure 4: Gadget for edge length ratio.

approaches 12 as k approaches infinity.

For the *area requirement* we use $n/3$ nested triangles with an edge between two corners as in Figure 3. It is known that a drawing of a graph with $n/3$ nested triangles requires $\Omega(n^2)$ area, and it is clear that the embedding shown in Figure 3 to the right needs only $O(n)$ area. Hence, the quality ratio for the area requirement is unbounded.

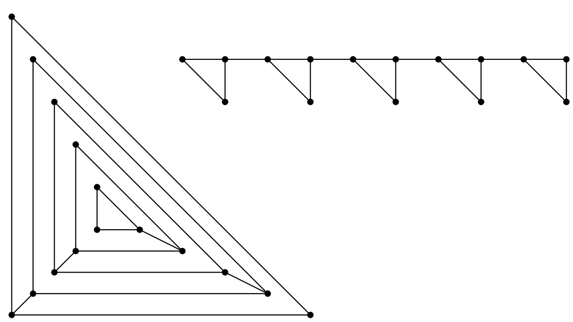

Figure 3: Area requirement for two different embeddings of the same graph.

For *edge length ratio* we show unboundedness, based on analyzing the example in Figure 4.

We will argue that if the ten edges have a constant edge length ratio c, then the perimeter of the outer triangle uvw is longer by at least a constant factor $\gamma > 1$ than the perimeter of the inner triangle $u'v'w'$.

Assume first that the angle of $\triangle uvw$ at v is at most some constant $\alpha < \pi$. Since none of u', v', w' can be very close to v, we immediately conclude that $\triangle u'v'w'$ has a significantly smaller perimeter than uvw. The factor γ depends on c and α and is a constant > 1 because c and α are constants. So the case to be analyzed is when the angle at v is close to π. To keep the perimeter of $\triangle u'v'w'$ close to that of $\triangle uvw$, two vertices of $\triangle u'v'w'$ must be very close to u and w. However, if w' is very close to w then the edge ww' is very short. If v' is very close to w, then the existence of $\triangle vv'w'$ implies that w' is right of the directed line through v and v'. This implies that w' is closer to w than v', or the sidedness implied by triangles $\triangle u'v'w'$ and $\triangle vv'u'$ cannot be kept. Finally, if u' is very close to w, then the sidedness argument leads to the fact that no vertex of $\triangle u'v'w'$ is close to u.

In all cases, the perimeter of $\triangle u'v'w'$ is significantly smaller than that of $\triangle uvw$, or the embedding was not kept.

The idea is now to repeat the construction with a triangle $\triangle u''v''w''$ that is inside $\triangle u'v'w'$, but with $u'w'$ in the role of vw. In the general embedding version, we will place $\triangle u'v'w'$ flipped outside $\triangle uvw$ by mirroring at edge vw. Since $\triangle u''v''w''$ is attached to $u'v'$ and $u'v'$ is now an outer edge, we can mirror $\triangle u''v''w''$ at $u'w'$ and bring it to the outside as well. It is easy to repeat the construction. For the fixed embedding version, we get $\Theta(n)$ nested triangles where the perimeter to the inside decreases by a constant factor < 1 per nested triangle. This gives an unbounded edge length ratio. For the general embedding version it is easy to keep the edge-length ratio bounded by 2.

The example in Figure 3 also shows that the *feature resolution* is unbounded. Consider any drawing with the left embedding. Normalize so that the smallest distance between any two consecutive nested triangles is 1. Then the outermost triangle must have edges of at least linear length. Hence, any drawing with the left embedding has feature resolution at least linear in n while it is constant in the right drawing.

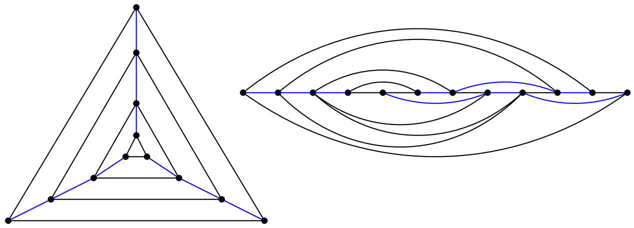

Figure 5: A graph that requires grid size $\Omega(n^2)$ in a planar straight-line drawing and $O(n)$ in a planar circular-arc drawing.

3 Straight-line versus circular-arc drawings

We discuss how much better circular-arc drawings can be than straight-line drawings. For *angular resolution*, the example that gives the biggest ratio is not the tetrahedron skeleton from the introduction (ratio 4) but the icosahedron. A circular-arc drawing can have a perfect angular resolution of 72° (it has a Lombardi drawing [6]), whereas a straight-line drawing has an angular resolution of at most 15°, giving a ratio of 4.8.

Figure 5 shows that the quality ratio for the *area requirement* is unbounded. Furthermore, the circular arcs can be made arbitrarily flat.

We will show that the quality ratio for the measure *edge length ratio* is unbounded. To this end we use a construction similar to the one in Figure 4, except that we use one extra edge to make the embedding with nested triangles fixed. We already argued that the edge length ratio is unbounded for straight-line drawings; the extra edge can only make it worse. A circular arc drawing allows a constant edge-length ratio for the same graph, as shown in Figure 6. The circles on which the triangles lie can be placed arbitrarily close together, also if we have many circles, leading to an edge length ratio arbitrarily close to 3.

To ensure that the straight-line version cannot use a different embedding of the nested triangles, we make two copies of the construction that have one vertex of their outer triangles in common. In any embedding, one of the two copies will appear in the embedding assumed before. In the circular arc version the two copies can easily be made with two circular constructions sharing a vertex on the outside.

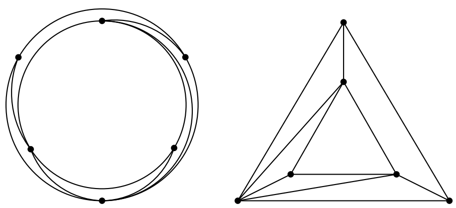

Figure 6: Edge length ratio construction for circular-arc and straight-line drawings.

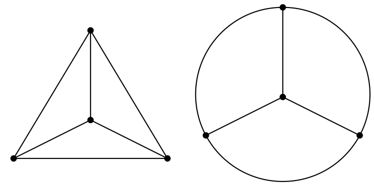

Figure 7: K_4 with optimal feature resolution in two drawing styles.

The construction cannot be used for *feature resolution*, since vertices will be very close to non-incident circular arcs. The example in Figure 7 will give a lower bound > 1. The circular arcs together form the smallest enclosing circle of the triangle. The feature resolutions are $2\sqrt{3}$ and $2\pi/3$ for the straight-line and circular-arc versions, and hence the quality ratio is $3\sqrt{3}/\pi$.

4 Planar versus non-planar straight-line drawings

The quality ratio has been studied before for planar straight-line drawings and right-angle crossing (RAC) drawings [12]; RAC drawings were introduced in [5]. Three of these results carry over to the case of general crossings. The *feature resolution* quality ratio was not studied before.

We observe that if a straight-line drawing has an angle $\alpha \leq \pi/3$, then the feature resolution of that drawing is at most $1/\sin\alpha$. If we take the example that shows that RAC drawings can have an angular resolution that is 2.75 times better than the best planar drawing [12], then we also get a bound on the quality ratio for feature resolution. The RAC drawing has a feature resolution of $\sqrt{2}$ and the best planar drawing has a feature resolution of at most $1/\sin(\pi/11)$. So the quality ratio is at least $2.509\ldots$.

5 Results for trees

We study the interesting cases only: area requirement and feature resolution.

We show that the *area requirement* for a fixed embedding of a tree can be higher than for a free embedding. The tree consists of a star graph where two leaves are replaced by degree-2 internal nodes attached to the center and to a leaf, see Figure 8. The fixed embedding is the one on the top left. This embedding cannot be placed on a 5×3 grid: only the middle grid points admit a degree-12 vertex, but in both cases we cannot place both of the leaves not incident to the center on the free grid points (Figure 8, right), and we need a grid with at least 16 points to draw the tree. The free embedding can be drawn on a 5×3 grid as shown in Figure 8, bottom left.

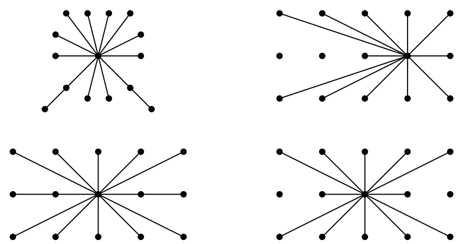

Figure 8: The tree on the top left admits a straight-line embedding on a grid of 5×3 only if we change the embedding.

The quality ratio for *area requirement* for straight-line versus circular-arc drawings also has a non-trivial lower bound. We choose the star graph with 36 nodes. With a circular arc embedding it can of course be embedded on a 6×6 grid (or another rectangular grid with 36 grid points), but a straight-line drawing cannot use all grid points due to alignments with the center of the star. The best we can do is a 3×17 grid: see Figure 11 for some grid sizes that allow the embedding of a star graph with 36 nodes. So the quality ratio is at least 51/36.

In an attempt to improve this lower bound we generalize this example. This gives rise to the following question: given n and m, in an $n \times m$ integer grid, what is the maximum number of straight-line segments we can draw from one grid point (to be chosen) to other grid points such that these line segments do not contain any grid points in their interior? In the geometry of numbers, these vectors are called *primitive vectors.* For $2 \times m$ grids and $3 \times m$ grids, these values are easily seen to be $m + 2$ and $2m + 2$, respectively. For square grids of size $n \times n$ it is known that the value approaches $6n^2/\pi^2 \approx 0.6079 \cdot n^2$ in the limit, since it relates to the fraction of relative primes (coprimes) on the integer grid [9]. Hence, in the limit, a $3 \times m$ grid allows a larger star graph to be embedded planar and with straight lines than an $n \times n$ grid with the same number of grid points. It turns out that $3 \times n$ grids are indeed the best we can do, in the limit, to embed star graphs, see the following lemma. Since circular-arc drawings can always use all grid points, the quality ratio is at least $\lim_{n\to\infty} \frac{3n}{2n+3} = 1.5$.

Lemma 1 *A rectangular $m \times n$ grid containing the origin contains at most $\frac{2}{3} \cdot mn + 2$ primitive vectors.*

Proof. (Sketch.) Every 6×6 box contains 12 vectors which are multiples of 2 or 3, see Figure 9. Thus, the fraction of primitive vectors is at most $24/36 = 2/3$. By removing such boxes, the given rectangular grid can be reduced to a cross shape straddling the coordinate axes, see Figure 10, and it is sufficient to prove the claim for such a shape. The width of the arms is bounded: the points have distance at most 5 from the respective coordinate axis. Thus, there is a finite number of possibilities for the set of horizontal and vertical lines that are used by the cross. The length of the arms is unbounded, but the primitive vectors are periodic in this direction, with period $\mathrm{lcm}(2, 3, 4, 5) = 60$. Thus, by removing boxes of size $w \times 60$, where $w \leq 11$, the problem is reduced to a finite number of remaining crosses, which can be checked individually. For the finitely many different types of $w \times 60$ boxes, it turns out that they always contain at most $40w$ primitive vectors, a fraction of 2/3. □

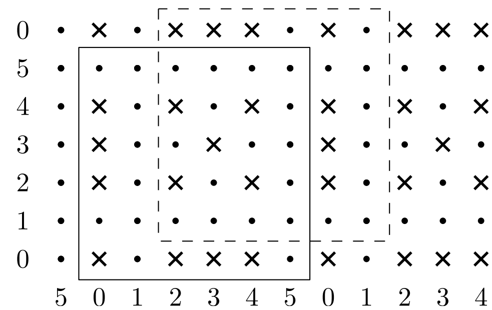

Figure 9: Every 6×6 square contains 12 vectors which are multiples of 2 or 3. These are marked with a $\times$ symbol. The row and column labels are the coordinates modulo 6.

A *lower bound* on the fraction of primitive vectors in rectangles has been worked out, with more effort, in [1].

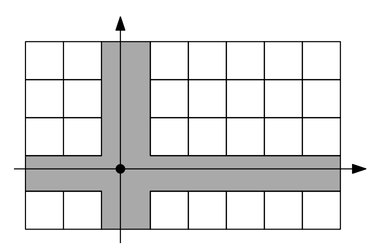

Figure 10: The cross shape (shaded) that remains after removing 6×6 boxes from an $m \times n$ rectangle.

The quality ratio for *area requirement* for non-planar straight-line tree drawings with respect to planar ones is also greater than 1. Let an (m, n)-star be a tree with two non-leaf nodes whose degrees are m and n, so it has $m + n$ nodes in total. A $(10, 11)$-tree can be embedded non-planarly on a 3×7 grid (see Figure 12), but a planar embedding on this grid does not exist. There is a planar embedding on a 2×11 grid, so we get a quality ratio of at least 22/21.

Finally we consider the measure *feature resolution*. The tree shown in Figure 13 cannot be drawn with feature resolution 1 in the embedding on the left, but if we change the embedding we can make a drawing with feature resolution 1. Hence, when comparing the drawing

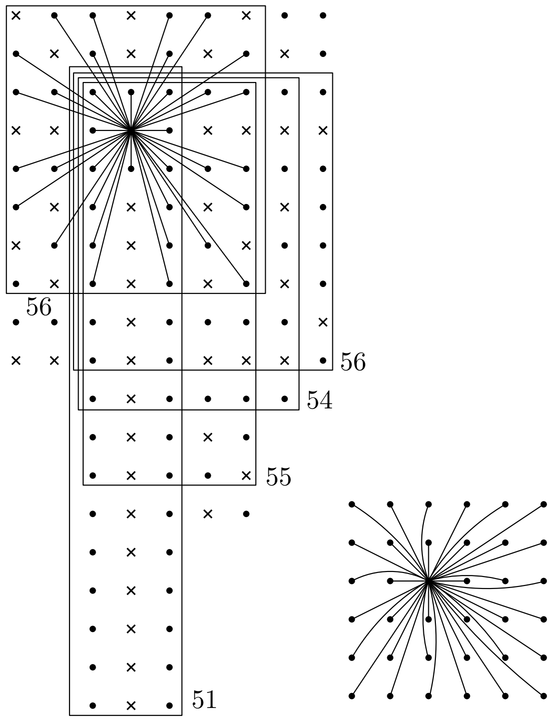

Figure 11: Circular-arc drawings of trees may use less area. Left, various minimal rectangular grids that allow the embedding of a star graph with 36 nodes. The size of the grid is shown with each rectangle. Right, a circular arc plane embedding on the grid.

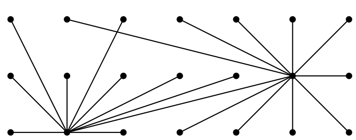

Figure 12: Non-planar embedding of a $(10, 11)$-star on a 3×7 grid. No planar embedding exists.

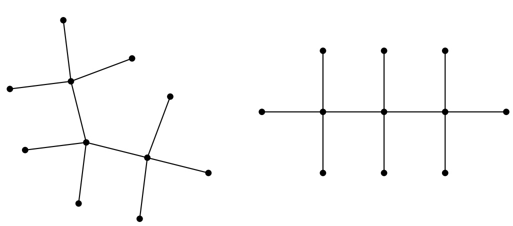

Figure 13: Two embeddings of the same tree that result in different feature resolutions.

styles with a fixed embedding and with a free embedding for trees, the feature resolution is strictly greater than 1.

6 Conclusions

A quality ratio of two drawing styles of a graph shows how much better a drawing can be in one style than in the other style, where the quality is measured using some value. We have compared four drawing styles, and from these, three pairs of styles to obtain quality ratios for four different quality measures. We considered the graph classes of planar graphs and trees separately. The results are shown in Tables 1 and 2. In the tables we see that some entries are ∞. In these cases it may be interesting to study the rate of growth of the ratio in the size of the graph. Other entries contain a constant, which is usually a lower bound on the ratio. We do not have upper bounds other than 1 for the feature resolution ratio in the straight-line vs. circular arc and the planar vs. non-planar straight-line settings. Establishing constant upper bounds, or proofs that these ratios are unbounded, are obviously interesting questions that remain to be answered.

In graph drawing there are other drawing styles that have received considerable attention, like one-bend or k-bend drawings. It is interesting to study how much bends can help to improve a quality measure.

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