Generalized Self-Approaching Curves*

Oswin Aichholzer[1], Franz Aurenhammer[1], Christian Icking[2], Rolf Klein[2], Elmar Langetepe[2], and Günter Rote[1]

[1] Technische Universität Graz, A-8010 Graz, Austria
[2] FernUniversität Hagen, Praktische Informatik VI, D-58084 Hagen, Germany.

Abstract. We consider all planar oriented curves that have the following property depending on a fixed angle φ. For each point B on the curve, the rest of the curve lies inside a wedge of angle φ with apex in B. This property restrains the curve's meandering, and for $\varphi \leq \frac{\pi}{2}$ this means that a point running along the curve always gets closer to all points on the remaining part. For all $\varphi < \pi$, we provide an upper bound $c(\varphi)$ for the length of such a curve, divided by the distance between its endpoints, and prove this bound to be tight. A main step is in proving that the curve's length cannot exceed the perimeter of its convex hull, divided by $1 + \cos\varphi$.

Keywords: Self-approaching curves, convex hull, detour, arc length.

1 Introduction

Let f be an oriented curve in the plane running from A to Z, and let φ be an angle in $[0, \pi)$. Suppose that, for every point B on f, the curve segment from B to Z is contained in a wedge of angle φ with apex in B. Then the curve f is called φ-*self-approaching*, generalizing the self-approaching curves introduced by Icking and Klein [5].

At the 1995 Dagstuhl Seminar on Computational Geometry, Seidel [2] posed the following open problems. Is there a constant, $c(\varphi)$, so that the length of every φ-self-approaching curve is at most $c(\varphi)$ times the distance between its endpoints? If so, how small can one prove $c(\varphi)$ to be?

Both questions are answered in this paper. We provide, for each φ in $[0, \pi)$, a constant $c(\varphi)$ with the above property, and prove it minimal by constructing a φ-self-approaching curve such that this factor $c(\varphi)$ is achieved.

Self-approaching curves are interesting for different reasons. If a *mobile robot* wants to get to the kernel of an unknown star-shaped polygon and continuously follows the angular bisector of the innermost left and right reflex vertices that are visible, the resulting path is self-approaching for $\varphi = \pi/2$; see [5]. Since the value of $c(\pi/2)$ is known to be ≈ 5.3331, as already shown in Icking et al. [6], one immediately obtains an upper bound for the competitive factor of the robot's

* This work was supported by the Deutsche Forschungsgemeinschaft, grant Kl 655/8-2, and by the Spezialforschungsbereich *Optimierung und Kontrolle*.

K.-Y. Chwa and O. H. Ibarra (Eds.): ISAAC'98, LNCS 1533, pp. 317–327, 1998.

strategy. Improving on this, Lee and Chwa [7] give a tight upper bound of $\pi + 1$ for this factor, and Lee et al. [8] present a different strategy that achieves a factor of 3.829, while a lower bound of 1.48 is shown by López-Ortiz and Schuierer [9].

In the construction of *spanners* for euclidean graphs, one can proceed by recursively adding to the spanning structure a point from a cone of angle φ, resulting in a sequence $p_1, p_2, \ldots, p_n$ such that for each index i, $p_{i+1}, \ldots, p_n$ are contained in a cone of angle φ with apex p_i; see Ruppert and Seidel [11] or Arya et al. [3]. (Note, however, that such a sequence of points does not necessarily define a φ-self-approaching polygonal chain because the property may not hold for every point in the interior of an edge.)

Finally, such properties of curves are interesting in their own right. For example, in the book by Croft et al. [4] on open problems in geometry, *curves with increasing chords* are defined by the property that for every four consecutive points A, B, C, D on the curve, B is closer to C than A to D. The open problem of how to bound the length of such curves divided by the distance between its endpoints has been solved by Rote [10]; he showed that the tight bound equals $\frac{2}{3}\pi$. There is a nice connection to the curves studied in this paper. Namely, a curve has increasing chords if and only if it is $\pi/2$-self-approaching in both directions.

In this paper we generalize the results of [6] to arbitrary angles $\varphi < \pi$. In Sect. 2 we prove, besides some elementary properties, the following fact. Let B, C, and D denote three consecutive points on a φ-self-approaching curve f. Then

$$CD \le BD - \cos\varphi \cdot \mathrm{length}(f[B \mathinner{..} C])$$

holds, where CD denotes the euclidean distance between points C and D and $\mathrm{length}(f[B \mathinner{..} C])$ denotes the length of f between B and C. This property accounts for the term "self-approaching"; in fact, for $\varphi \le \pi/2$ the factor $\cos\varphi$ is not negative, so that $CD \le BD$ holds: The curve always gets closer to each point on its remaining part. Although this property does not hold for $\varphi > \frac{\pi}{2}$, we will nevertheless see that our analysis of the tight upper bound $c(\varphi)$ directly applies to this case, too.

In Sect. 3 we show that the length of a φ-self-approaching curve cannot exceed the perimeter of its convex hull, divided by $1 + \cos\varphi$. This fact is the main tool in our analysis. It allows us to derive an upper bound for the curve's length by circumscribing it with a simple, closed convex curve whose length can be easily computed, see Sect. 4. Finally, in Sect. 5, we demonstrate that the resulting bound is tight, by constructing φ-self-approaching curves for which the upper bounds are achieved.

The proofs of Lemma 3 and Theorem 1, which are omitted or only sketched here due to the limitation of space, can be found in the full paper [1].

2 Definitions and properties

For two points B and C let $\boldsymbol{r}(B, C)$ denote the ray starting at B and passing through C. We simply write BC for the euclidean distance between B and C.

We consider plane curves $f\colon [0 \,..\, 1] \to \mathbb{R}^2$ parameterized by a parameter that is usually denoted by $t \in [0 \,..\, 1]$. It is considered as an oriented curve, i. e., it has a specified direction from beginning to end.

We do not make any assumptions about smoothness or rectifiability of the curve, although it will turn out that φ-self-approaching curves are rectifiable. For a point X on the curve we write t_X for the parameter value with $f(t_X) = X$ if no confusion arises. (We will see below in Lemma 1 that t_X is in fact unique.)

For two points $B = f(t_B)$ and $C = f(t_C)$ on f with $t_B \le t_C$, we write $f[B \,..\, C]$ or $f[t_B \,..\, t_C]$ for the part of f between B and C. By $f[B \,..]$ we denote the part of f from B to the end. By length(f), length($f[B \,..\, C]$), etc., we mean the length of the curve or of an arc.

Definition 1. *A curve f is called* φ-self-approaching *for $0 \le \varphi < \pi$ if for any point B on f there is a wedge of angle φ at point B which contains $f[B \,..]$. In other words, for any three points B, C, D with $t_B < t_C < t_D$, the angle CBD, if it is defined, is at most φ.*

Let f be an oriented curve from A to Z. Then the quantity

$$\frac{\text{length}(f[A \,..\, Z])}{AZ}$$

is called the detour *of f.*

The detour of a curve is the reciprocal of the *minimum growth rate* used in [10].

We wish to bound the detour of φ-self-approaching curves. The first definition of φ-self-approaching curves also makes sense for $\varphi \ge \pi$, but then any circular arc connecting two points A and Z is φ-self-approaching, which means that the detour of such curves is not bounded. Therefore, we restrict our attention to the case $\varphi < \pi$. Each 0-self-approaching curve is a line segment and its detour equals 1.

Lemma 1. *A φ-self-approaching curve does not go through any point twice.*

Proof. Suppose the curve visits point X twice. Shortly after the first visit of X there is a point on the curve for which the φ-self-approaching property is violated. □

By this lemma, we can take any point Z on the curve as the origin of a polar coordinate system and use a continuous parameterization $f(t) = (r(t), \psi(t))$ for any part $f[A \,..\, B]$ of the curve which comes before Z. For a point $X = f(t_X)$ on the curve we will use the notation $X = (r(t_X), \psi(t_X)) =: (r_X, \psi_X)$.

Lemma 2. *If $t_A < t_B < t_Z$ and $\psi_A \equiv \psi_B \pmod{2\pi}$, then $r_B \le r_A$.*

Proof. The assumption $r_B > r_A$ would lead to an angle $BAZ = \pi$, violating the φ-self-approaching property. (The case $r_B = r_A$ is also excluded, by Lemma 1, but we do not use this fact.) □

The following lemma shows, roughly speaking, that a φ-self-approaching curve $f[A \,..\, Z]$ is enclosed by two oppositely winding φ-logarithmic spirals through A with pole Z, see Fig. 1. For $\varphi > \pi/2$, this is true only locally, as long as the curve does not leave the vicinity of A. In polar coordinates (r, ψ), a φ-logarithmic spiral is the set of all points with $r = e^{\psi \cot \varphi}$ or $r = e^{-\psi \cot \varphi}$.

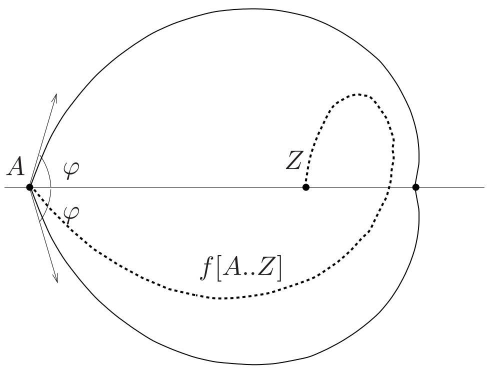

Fig. 1. For $0 < \varphi < \pi/2$ the curve is enclosed by two congruent arcs of a φ-logarithmic spiral.

Each ray through the origin intersects the spiral in the same angle φ. For $\varphi \neq \pi/2$, the length of an arc $S[A \,..\, B]$ of a φ-logarithmic spiral S with pole Z is given by $\frac{1}{\cos\varphi}(AZ - BZ)$. For $\varphi = \pi/2$, the spiral degenerates to a circle. The properties of Lemma 3 are used in Sect. 4 for the circumscription of a φ-self-approaching curve by a convex area whose perimeter is easy to determine.

Lemma 3. *Let* $t_A \leq t_B < t_Z$.

(a) $$r_B \leq r_A \cdot e^{-|\psi_B - \psi_A| \cot \varphi} .$$

(b) $$r_B \leq r_A - \cos\varphi \cdot \mathrm{length}(f[A \,..\, B]) .$$

Note that Lemma 3 also applies to any subsection of the curve f.

3 φ-self-approaching curves and the perimeter of their convex hull

In this section we show that the length of a φ-self-approaching curve is bounded by the perimeter of its convex hull, divided by $1 + \cos\varphi$.

Let $\mathrm{conv}(P)$ denote the convex hull of a point set P and $\mathrm{peri}(P)$ the length of the perimeter of $\mathrm{conv}(P)$.

Theorem 1. *For a φ-self-approaching curve f with* $0 \leq \varphi < \pi$,

$$(1 + \cos\varphi)\, \mathrm{length}(f) \leq \mathrm{peri}(f) .$$

The proof can only be sketched here. First, the claim is shown for a polygonal chain whose vertices lie on the curve f. This is done by induction on the number of vertices. But the length of a curve f is, by definition, the supremum of the lengths of all polygonal chains which are obtained as polygonal subdivisions of f. Therefore, an upper bound for the length of all such chains is also an upper bound for the length of f.

4 An upper bound for the detour

Theorem 2. *The length of a φ-self-approaching curve is not greater than $c(\varphi)$ times the distance of its endpoints, where*

$$c(\varphi) := \begin{cases} 1, & \text{for } \varphi = 0, \\ \max\limits_{\beta\in[0..\frac{\pi}{2}]} \dfrac{2\beta+\pi+2}{\sqrt{5-4\cos\beta}}, & \text{for } \varphi = \pi/2, \\ \max\limits_{\beta\in[0..\varphi]} h(\varphi,\beta), & \text{otherwise,} \end{cases}$$

with $h(\varphi,\beta) := \dfrac{\left((1+e^{\pi\cot\varphi})e^{\beta\cot\varphi} - \frac{2}{1+\cos\varphi}\right)/\cos\varphi}{\sqrt{((1+e^{\pi\cot\varphi})e^{\beta\cot\varphi}-\cos\beta)^2+\sin^2\beta}}$.

The function $c(\varphi)$ is strictly monotone and continuous for $\varphi \in [0,\pi)$. It tends to infinity if φ tends to π, see Fig. 2 for its values up to $\varphi = 1.8$.

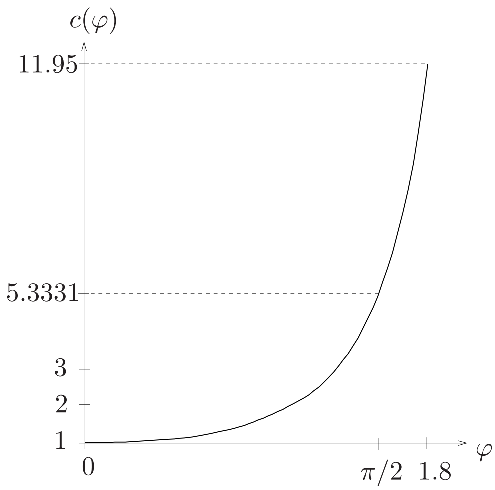

Fig. 2. The graph of $c(\varphi)$.

Proof. For $\varphi = 0$ we have a straight line segment, and the theorem is obvious. So let us assume $0 < \varphi < \pi$ from now on.

Let f be a φ-self-approaching curve from A to Z. W.l.o.g., we may assume that f does not cross the segment AZ. Otherwise we apply the bound $c(\varphi)$ successively to each subcurve between to successive curve points on AZ and add up the results. Due to the self-approaching property the curve points on AZ appear in the same order as on f, so the overall bound is less than or equal to $c(\varphi)$.

Assume w. l. o. g. that the curve starts by leaving A on the left side of the directed line AZ. Let AC be the right tangent from A to the curve, touching the curve in the point C. (The curve is on the left side of AC.) We select C as far as possible from A, and we denote the angle ACZ by β, see Fig. 3. (For $C = Z$ we set $\beta = 0$.) Note that $\beta \leq \varphi$ holds, otherwise there is a point C_0 before C on f such that the angle CC_0Z is also greater than φ which contradicts the self-approaching property at C_0.

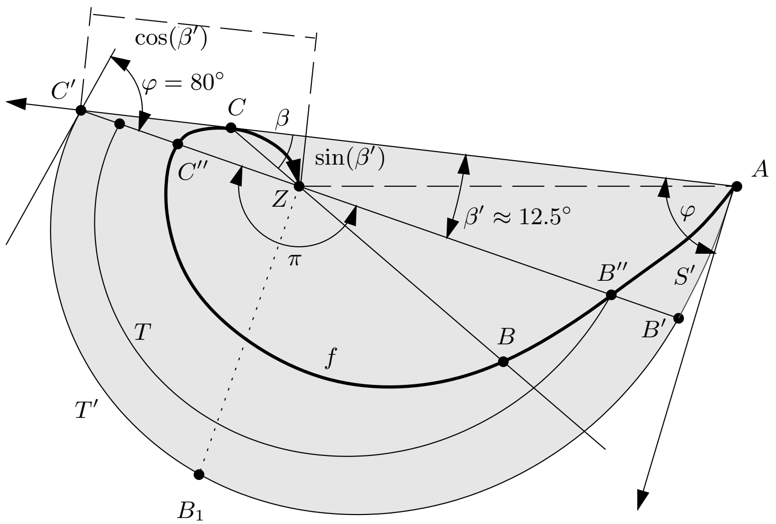

Fig. 3. A φ-self-approaching curve must lie in this area. The angle β' shown here maximizes $h(\varphi, \beta)$.

Let B denote the first point of intersection of the curve with $\boldsymbol{r}(C, Z)$. (For $C = Z$ we take $B = A$.) Since the curve neither crosses $\boldsymbol{r}(A, C)$ nor the segment AZ, Z must lie between B and C. We apply Lemma 3a to the arc $f[B \,..\, C]$, considering Z as the origin, and we get

$$CZ \leq BZ \cdot e^{-\pi \cot \varphi} .$$

The application of the lemma is justified because, by Lemma 2, C is the first point of intersection of the curve with $\boldsymbol{r}(Z, C)$. Applying the lemma to $f[A \,..\, B]$ with C as the origin, we get

$$CB \leq CA \cdot e^{-\beta \cot \varphi} .$$

All in all, since $CB = CZ + BZ$, we obtain

$$CA \geq CZ \cdot (1 + e^{\pi \cot \varphi}) e^{\beta \cot \varphi} . \tag{1}$$

Now we select a point C' on $\boldsymbol{r}(A, C)$ with $C'A \geq CA$ such that the equation

$$C'A = C'Z \cdot (1 + e^{\pi \cot \varphi}) e^{\angle AC'Z \cdot \cot \varphi} \tag{2}$$

is fulfilled. Such a point C' exists, since, as we move C' further away from A, the ratio $C'A : C'Z$ converges to 1, the angle $\beta' := \angle AC'Z$ decreases towards 0 and hence $e^{\beta' \cot \varphi}$ also converges to 1, whereas $1 + e^{\pi \cot \varphi}$ is a constant bigger than 1. Therefore the inequality (1) changes direction as $AC' \to \infty$. We have $\beta' \leq \beta \leq \varphi$, and also $C' \neq Z$.

In the following let B' be the point on $\boldsymbol{r}(C', Z)$ with $B'Z = C'Z \cdot e^{\pi \cot \varphi}$ and $\angle C'ZB' = \pi$, see Fig. 3.

Lemma 4. *A φ-self-approaching curve from A to Z is contained in the convex region bounded by the following three curves, see Fig. 3:*

(1) *a φ-logarithmic spiral from A to B' of polar angle β' with pole C';*
(2) *a φ-logarithmic spiral from B' to C' of polar angle π with pole Z;*
(3) *the segment AC'.*

Proof. Let B'' be the first intersection point of f with $\boldsymbol{r}(C', Z)$. By Lemma 3a applied to the pole C, the arc $f[A \mathinner{..} B'']$ lies inside the φ-logarithmic spiral S with pole C with polar angle β starting at A. The φ-logarithmic spiral S' with pole C' through A is obtained from S by stretching it about the center A, and hence $f[A \mathinner{..} B'']$ is also contained inside S', and in part (1) of the region boundary. In particular, $B''Z \leq B'Z$. It follows with the help of Lemma 2 that, between the rays $\boldsymbol{r}(Z, A)$ and $\boldsymbol{r}(Z, B')$, no point of the whole curve f can lie outside S'.

Now let C'' be the first intersection point of f with $\boldsymbol{r}(Z, C')$. By Lemma 3a applied to the pole Z, the arc $f[B'' \mathinner{..} C'']$ lies inside the φ-logarithmic spiral T around pole Z with polar angle π starting at B''. Since $B''Z \leq B'Z$, the arc lies inside the logarithmic spiral T' which forms part (2) of the region boundary. Again, it follows with the help of Lemma 2 that, below the line $C'B'$, no point of the whole curve f can lie outside T'.

Finally, in the region between the rays ZC' and ZA, the curve cannot escape across the segment AC', and the proof is complete. □

Now we can prove Theorem 2. We treat only the case $\varphi \neq \pi/2$, where we have proper spirals. The case $\varphi = \pi/2$, where we have circular arcs, has already been treated in [6].

We choose a scale such that $C'Z$ equals 1. Now $AC' = (1+e^{\pi \cot \varphi})e^{\beta' \cot \varphi}$ and $B'Z = e^{\pi \cot \varphi}$ holds by construction. Therefore the lengths of the three curves in Lemma 4, using the formula for the arc length of a φ-logarithmic spiral, are given by

$$L_1 := \frac{1}{\cos \varphi}(AC' - B'C') = \frac{1}{\cos \varphi}\left((1 + e^{\pi \cot \varphi})e^{\beta' \cot \varphi} - (1 + e^{\pi \cot \varphi})\right)$$
$$L_2 := \frac{1}{\cos \varphi}(B'Z - C'Z) = \frac{1}{\cos \varphi}(e^{\pi \cot \varphi} - 1)$$
$$L_3 := AC' = (1 + e^{\pi \cot \varphi})e^{\beta' \cot \varphi},$$

and for the distance of the endpoints of f we have

$$AZ = \sqrt{((1 + e^{\pi \cot \varphi})e^{\beta' \cot \varphi} - \cos \beta')^2 + \sin^2 \beta'}\,.$$

Altogether we conclude from Theorem 1

$$\frac{\text{length}(f)}{AZ} \leq \frac{L_1 + L_2 + L_3}{AZ} \cdot \frac{1}{1 + \cos \varphi}\,.$$

The right term can be transformed to

$$\frac{(1 + e^{\pi \cot \varphi})e^{\beta' \cot \varphi} - \frac{2}{1+\cos \varphi}}{\cos \varphi \sqrt{((1 + e^{\pi \cot \varphi})e^{\beta' \cot \varphi} - \cos \beta')^2 + \sin^2 \beta'}}\,.$$

□

5 Tightness of the upper bound

Theorem 3. *The upper bound $c(\varphi)$ given in Theorem 2 for the detour of φ-self-approaching curves is tight.*

Proof. We construct a φ-self-approaching curve f from A to Z as in the proof of Lemma 4, the construction is shown in Fig. 4. The curve starts with two

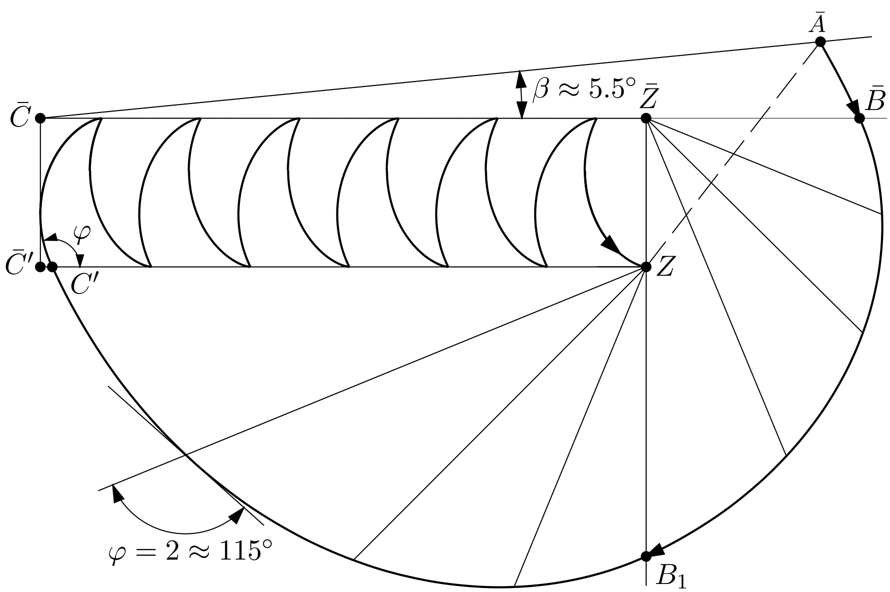

Fig. 4. The construction of the φ-self-approaching curve with maximal detour for φ=2.

logarithmic spirals similar to part (1) and part (2) in the proof of Lemma 4, except that part (2) is split into two parts. The segment $C'Z$ of length 1 is replaced by a φ-self-approaching zigzag curve of length $L_3' = \frac{2}{1+\cos\varphi}$ from C'

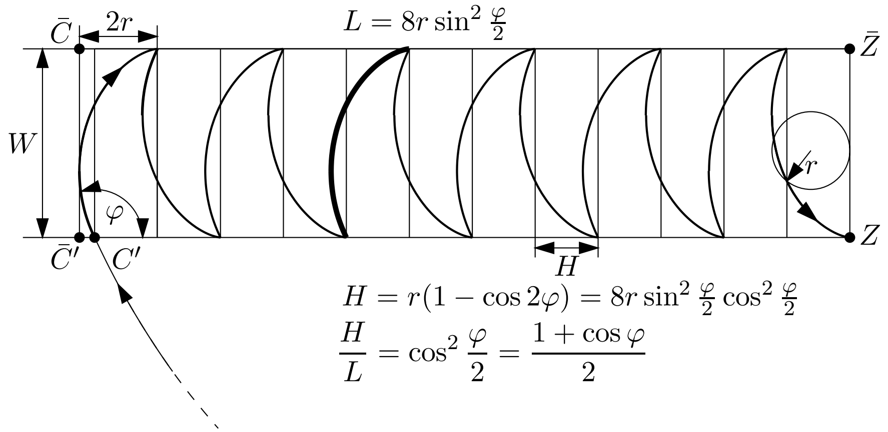

Fig. 5. A sequence of cycloid parts forming a φ-self-approaching curve inside a thin rectangle.

to Z, which moves inside a thin rectangle along the segment $C'Z$. This last part of the curve between Z and C', see Fig. 5, is obtained by "stacking" small cycloids $(x, y) = (r(\alpha - \sin\alpha), r(1-\cos\alpha))$, for $0 \le \alpha \le 2\varphi$. One piece of cycloid has "height" $H = r(1-\cos 2\varphi) = 8r\sin^2\varphi\cos^2\varphi$ and length $L = 8r\sin^2\frac{\varphi}{2}$. We must choose the size parameter r in such a way that $1/H$ is an even integer n; then the curve will precisely connect the points C' and Z. The length of the curve is then $L/H = \frac{2}{1+\cos\varphi}$, which is what we need. The width of the construction is $W = r(2\varphi - \sin 2\varphi)$. We arrange the cycloid pieces on the left side of the segment $C'Z$, as indicated in Fig. 5; then the curve is contained in a rectangle $\bar{C}'Z\bar{Z}\bar{C}$ of width $W = Z\bar{Z}$. The long side $Z\bar{C}'$ slightly extends the segment ZC' by the amount $2r - H$.

Now we can describe the whole construction of the curve, in reverse direction. The curve consists of the following parts (numbered in accordance with Lemma 4, using the notation from there):

- (3′) The curve ends with the φ-self-approaching curve from C' to Z described above, whose length is $\frac{2}{1+\cos\varphi}$;
- (2) before, there is a φ-logarithmic spiral of polar angle $\pi/2$ with pole Z connecting C' to some point B_1.
- (2′) then, a φ-logarithmic spiral of polar angle $\pi/2$ with pole $\bar{Z}$ connecting B_1 to $\bar{B}$;
- (1′) the curve starts with a φ-logarithmic spiral of polar angle β with pole $\bar{C}$ between a point $\bar{A}$ and $\bar{B}$, where $\beta \le \varphi$ is the value for which the maximum in the definition of $c(\varphi)$ is attained.

It can be checked that the parts which are logarithmic spirals are always φ-self-approaching, as the line from any point X to the current pole contains the whole curve on one side. So obviously the whole curve is φ-self-approaching. Since r can be made as small as we like, we have $\bar{C} \to C'$, $\bar{Z} \to Z$, $\bar{A} \to A$, $\bar{B} \to B'$ as $r \to 0$, and the logarithmic spirals that we use will approach the "ideal" logarithmic spirals that appear in Lemma 4, see Fig. 3. This means that

$$\frac{\text{length}(f)}{\bar{A}Z} \to \frac{L_1 + L_2 + L_3'}{AZ} = \frac{(1 + e^{\pi \cot \varphi}) e^{\beta \cot \varphi} - 2 + \frac{2}{1+\cos\varphi} \cdot \cos\varphi}{\cos\varphi \sqrt{((1 + e^{\pi \cot \varphi}) e^{\beta \cot \varphi} - \cos\beta)^2 + \sin^2\beta}}$$

which equals $c(\varphi)$. □

6 Conclusions

Here we analyze the maximum length of curves with an upper bound on the angular wedge at A. This condition is not symmetric since it distinguishes the source and the target of the curve. One might also consider a symmetric situation, where curves are φ-self-approaching in both directions.

Generalizations to three dimensions are also completely open.

References

1. O. Aichholzer, F. Aurenhammer, C. Icking, R. Klein, E. Langetepe, and G. Rote. φ-self-approaching curves. Technical Report 226, Department of Computer Science, FernUniversität Hagen, Germany, 1997. Submitted for publication. 318
2. H. Alt, B. Chazelle, and R. Seidel, editors. *Computational Geometry*. Dagstuhl-Seminar-Report 109. Internat. Begegnungs- und Forschungszentrum für Informatik, Schloss Dagstuhl, Germany, March 1995. 317
3. S. Arya, G. Das, D. M. Mount, J. S. Salowe, and M. Smid. Euclidean spanners: short, thin, and lanky. In *Proc. 27th Annu. ACM Sympos. Theory Comput.*, pages 489–498, 1995. 318
4. H. P. Croft, K. J. Falconer, and R. K. Guy. *Unsolved Problems in Geometry*. Springer-Verlag, 1990. 318
5. C. Icking and R. Klein. Searching for the kernel of a polygon: A competitive strategy. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 258–266, 1995. 317
6. C. Icking, R. Klein, and E. Langetepe. Self-approaching curves. Technical Report 217, Department of Computer Science, FernUniversität Hagen, Germany, 1997. To appear in Mathematical Proceedings of the Cambridge Philosophical Society. 317, 318, 323
7. J.-H. Lee and K.-Y. Chwa. Tight analysis of a self-approaching strategy for on-line kernel-search problem. Technical report, Department of Computer Science, KAIST, Taejon, Korea, 1998. 318
8. J.-H. Lee, C.-S. Shin, J.-H. Kim, S. Y. Shin, and K.-Y. Chwa. New competitive strategies for searching in unknown star-shaped polygons. In *Proc. 13th Annu. ACM Sympos. Comput. Geom.*, pages 427–429, 1997. 318

9. A. López-Ortiz and S. Schuierer. Position-independent near optimal searching and on-line recognition in star polygons. In *Proc. 13th Annu. ACM Sympos. Comput. Geom.*, pages 445–447, 1997. 318
10. G. Rote. Curves with increasing chords. *Mathematical Proceedings of the Cambridge Philosophical Society*, 115(1):1–12, 1994. 318, 319
11. J. Ruppert and R. Seidel. Approximating the d-dimensional complete Euclidean graph. In *Proc. 3rd Canad. Conf. Comput. Geom.*, pages 207–210, 1991. 318