Embedding 3-Polytopes on a Small Grid

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ABSTRACT

We show how to embed a 3-connected planar graph with n vertices as a 3-polytope with small integer coordinates. The coordinates are bounded by $O(2^{7.55n})$. The crucial part is the construction of a plane embedding which supports an equilibrium stress. We have to guarantee that the size of the coordinates and the stresses are small. This is achieved by applying Tutte's spring embedding method carefully.

Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Geometrical problems and computations; G.2.2 [**Graph Theory**]: Graph algorithms

General Terms

Algorithms, Theory

Keywords

Spring embedding, equilibrium stress

1. INTRODUCTION

Steinitz proved in 1922 one of the most famous results in polytope theory [13]: a graph G is an edge graph of a 3-polytope if and only if G is planar and 3-connected. A constructive approach, which uses liftings of stressed graphs, shows furthermore that rational (and thus, integer) coordinates are sufficient to realize the 3-polytope. The best previous bound on the size of the coordinates was $O(2^{18n^2})$ for embedding a graph with n vertices, by Richter-Gebert [10].

We present a construction which allows an embedding with integer coordinates not greater than $O(2^{7.55n})$. Therefore it suffices to use $O(n)$ bits to store each vertex in an embedding of a combinatorial polytope. Integer realizations with at most exponential coordinates were previously only known for polytopes whose graph contains a triangle [10].

For the special family of *stacked polytopes* a better upper bound was shown recently [18], but it is still exponential.

The key part in the construction of an integer embedding is the concept of lifting stressed graphs in equilibrium. This technique was used by Richter-Gebert to achieve the $O(2^{18n^2})$ upper bound.

In Section 2 we focus on constructing a planar straight-line embedding of the graph. In Section 3 we show how the planar embedding can be lifted into $\mathbb{R}^3$. Finally we analyze the size of the embedding in Section 4.

Little is known about a lower bound for a grid embedding. The smallest square grid that contains a convex n-gon in the plane has size $\Omega(n^{3/2}) \times \Omega(n^{3/2})$ [1, 2, 14]. Therefore the embedding of a polytope with two convex $n/2$-gonal faces, which share an edge, needs a grid where at least one dimension is $\Omega(n^{3/2})$.

Planar graphs can be embedded in the plane on a linear-size grid [12, 5]. This is also true for convex embeddings [4]. A strictly convex drawing can be realized on an $O(n^2) \times O(n^2)$ grid [3].

In higher dimension it is known that there are 4-polytopes which cannot be realized with rational coordinates at all. Moreover, a 4-polytope which can be embedded on the grid might require coordinates that are doubly exponential in the number of vertices [11, 10].

2. CONSTRUCTING A PLANE EMBEDDING

We are given a graph G with n vertices, which are denoted as $v_1, \dots, v_n$. An embedding of G assigns a point $\mathbf{p}_i = (x_i, y_i)$ to each vertex v_i of G. We denote the family of points by $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$, and the embedded graph by $G(\mathbf{p})$. We use bold letters to emphasize that a symbol refers to a vector. We assume that the first k vertices belong to the boundary face (the outer face) f_0 in cyclic order.

We use the concept of equilibrium stresses to obtain an embedding of G as 3-polytope. A *stress* ω is an assignment of scalars to the edges of G. The stress $\omega(e)$ of an edge $e = (v_i, v_j)$ is denoted as $\omega_{ij} = \omega_{ji}$. A vertex v_i in an embedding $G(\mathbf{p})$ is in *equilibrium* if

$$\sum_{1 \le j \le n : (v_i, v_j) \in E(G)} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \binom{0}{0} \tag{1}$$

The embedded graph is in equilibrium if all vertices are in equilibrium.

Our construction is based on a theorem stated by Maxwell in 1864 [7]. Whiteley [17] proved the reverse direction of Maxwell's theorem, which we apply in the construction. For our purposes, it suffices to use a weaker (special) formulation of the Maxwell-Whiteley Theorem:

THEOREM 1. *Let G be a 3-connected planar graph with a plane embedding $G(\mathbf{p})$. The following two statements are equivalent*:

- *$G(\mathbf{p})$ has an equilibrium stress ω which is positive on the interior edges and negative on the boundary edges.*
- *There exists a polytope that projects on $G(\mathbf{p})$, where the boundary of the outer face is in the xy-plane, and the polytope lies entirely in the upper half-space ($z \geq 0$).*

There is actually a one-to-one correspondence between the equilibrium stresses and the polytopes that are mentioned in the theorem. The proof is constructive. It allows to build the polytope from the equilibrium stress in an easy sequence of arithmetic operations, which are reviewed below in Section 3. An example of the correspondence between a stressed embedding and a polyhedron is shown in Figure 1, for the graph of the dodecahedron. This polytope was constructed by our method. The details are given in Section 5.

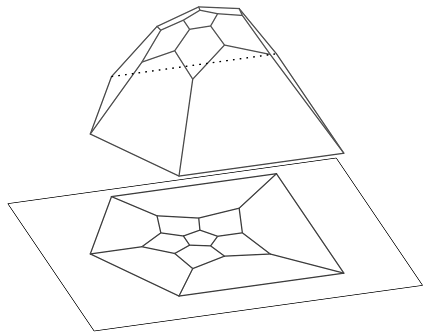


Figure 1: An example of an equilibrium embedding and its induced lifting to three dimensions.

The condition that the bottom face is horizontal at $z = 0$ is only a technical condition. It can always be achieved by a suitable affine transformation.

Spring Embedding. If we fix the position of the outer vertices, it is easy to obtain an embedding with equilibrium at the interior vertices: We may take any positive stresses ω_{ij} and solve the system of linear equations (1) for the interior vertices. This system has an unique solution, and it embeds the graph without crossings. (This is the essence of Tutte's spring embedding method [15, 16] for obtaining strictly convex drawings of planar graphs.)

For simplicity, we will always choose $\omega_{ij} = 1$ for all interior edges. This gives, after fixing the position of the k boundary vertices, a plane embedded graph that is in equilibrium at every interior vertex. The boundary points are not in equilibrium. For a boundary point $\mathbf{p}_i$, the sum in (1) will sum up to a nonzero *force vector* $\mathbf{F}_i$. The problem is now to obtain equilibrium at the boundary vertices. We have control over the stresses of the k boundary edges. These stresses must fulfill a system of $2k$ linear equations (two equilibrium equations at every vertex).

If the outer face is a triangle, it turns out that the stress can *always* be completed to an equilibrium stress at the boundary vertices (in a unique way, and with negative boundary stresses).

If the outer face is a quadrilateral or a face with more sides, however, it is not always possible to balance the forces from the interior edges by stresses on the boundary edges. (This dichotomy between the triangular case and the remaining cases has also a geometric interpretation: by a variation of the Maxwell Theorem (Theorem 1), a stress that is in equilibrium at the interior vertices gives rise to a polyhedral terrain, i. e., a surface with boundary that projects on $G(\mathbf{p})$. Now, if the outer face is triangular, it always lies in a plane and there is no problem to put the "bottom" under the surface to obtain a closed surface bounding a polytope. If the outer face has 4 or more vertices, they are not necessarily coplanar, and thus it may not be possible to close the surface.)

Richter-Gebert [10] solved this problem by choosing a triangle at the outer face if the graph contains a triangular face. Otherwise, the dual graph must contain a triangular face. Realizing the dual polytope and applying a polarity to it yields a realization of the original polytope. However, the arithmetic operations that are involved in producing the polar lead to a growth of the coordinates. After an appropriate scaling, the resulting integer coordinates are bounded by $O(2^{18n^2})$.

We follow a different approach. By choosing the outer polygon carefully, we ensure that the forces at the outer vertices can be canceled by appropriate stresses on the boundary edges.

The Substitution Lemma. To analyze how we should position the boundary vertices, we need to find out how their placement affects the resulting boundary forces. It turns out that, for this purpose, we can replace the whole graph by a complete graph on the boundary vertices only, with appropriate stresses.

Let A be the adjacency matrix of G (possibly weighted according to the stresses ω), and let D be the diagonal matrix of row sums of A. We subdivide A and D into block matrices indexed by the sets $B = \{1, \dots, k\}$ and $I = \{k+1, \dots, n\}$. The matrix $D - A$ is called the *Laplacian matrix* L of G and the matrix $D_I - A_{II}$ is called the *reduced Laplacian matrix* $\bar{L}$ of G. Furthermore let $\mathbf{x}_B = (x_1, \dots, x_k)^T$ and $\mathbf{x}_I = (x_{k+1}, \dots, x_n)^T$, and similarly for the y-coordinates.

LEMMA 1 (SUBSTITUTION LEMMA). *There are nonnegative weights $\tilde{\omega}_{ij} = \tilde{\omega}_{ji}$, for $i, j \in B$, independent of $\mathbf{p}$, such that the resulting forces at the boundary vertices $i \in B$ are obtained by*

$$\mathbf{F}_i = \sum_{j \in B: j \neq i} \tilde{\omega}_{ij}(\mathbf{p}_i - \mathbf{p}_j). \tag{2}$$

The weights $\tilde{\omega}$ are multiples of $1/\det \bar{L}$.

PROOF. Let $\mathbf{F}_x$ denote the vector $(F_1^x, \dots, F_k^x)^T$, where F_i^x is the x-coordinate of the force $\mathbf{F}_i$. In block matrix form, the equilibrium equations for the x-coordinates are written as follows

$$\begin{pmatrix} D_I - A_{II} & -A_{IB} \\ -A_{BI} & D_B - A_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{x}_I \\ \mathbf{x}_B \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{F}_x \end{pmatrix} \tag{3}$$

From this we can obtain

$$\mathbf{F}_x = A_{BI}(D_I - A_{II})^{-1}A_{IB}\mathbf{x}_B - D_B\mathbf{x}_B =: \tilde{A}\mathbf{x}_B.$$

For the y-coordinates, we obtain a similar formula with the same matrix $\tilde{A}$. We define $\tilde{\omega}_{ij}$ as the entries $\tilde{a}_{ij}$ of $\tilde{A}$. Since $A_{BI} = (A_{IB})^T$, the matrix $\tilde{A}$ is symmetric and therefore $\tilde{\omega}_{ij} = \tilde{\omega}_{ji}$ holds.

To show that the expression $\mathbf{F}_x = \tilde{A}\mathbf{x}_B$ has the form stated in (2) we have to check that all row sums in $\tilde{A}$ equal 0. Let $\mathbf{1}$ denote the vector where all entries are 1. We know that $A_{II}\mathbf{1} + A_{IB}\mathbf{1} = D_I\mathbf{1}$ and therefore $(D_I - A_{II})^{-1}A_{IB}\mathbf{1} = \mathbf{1}$. Plugging this expression into $\tilde{A}\mathbf{1} = A_{BI}(D_I - A_{II})^{-1}A_{IB}\mathbf{1} - D_B\mathbf{1}$ gives us $\tilde{A}\mathbf{1} = A_{BI}\mathbf{1} - D_B\mathbf{1}$, which is $(0, \ldots, 0)^T$.

The matrix $\tilde{A}$ can be written as a rational expression whose denominator is the determinant of $D_I - A_{II} = \bar{L}$, and thus the weights $\tilde{\omega}$ are multiples of $1/\det \bar{L}$. □

We observe that the stress $\tilde{\omega}$ is independent of the shape of the outer face. It only depends on the graph G. In other words, the stress $\tilde{\omega}$ stores all the necessary information about the combinatorial structure of G. Thus we have a compact (constant-size) description of the structure of G, as far as it determines the non-resolving forces at the boundary. We name the stress $\tilde{\omega}$ *substitution stress* to emphasize that it is used as a substitution for the stress ω on the whole graph G.

The above lemma holds for any stress on the graph G. We apply it to the stress that is uniformly equal to 1 on the interior edges, and zero on the boundary edges. In other words, A is the adjacency matrix after removing the boundary edges.

For the later analysis of the grid size it is necessary to bound the substitution stresses.

LEMMA 2. *The substitution stresses $\tilde{\omega}_{ij}$ are bounded by*

$$0 \leq \tilde{\omega}_{ij} < n - k.$$

PROOF. The substitution stresses are independent of the location of P. Therefore we can choose the positions for the boundary points freely. We place vertex $\mathbf{p}_i$ at position $(0,0)^T$, and all other boundary vertices at $(1,0)^T$. By (2), the stress $\tilde{\omega}_{ij}$ is the x-component of $\mathbf{F}_j$ and therefore $\tilde{\omega}_{ij} = \sum_{k \in I} \omega_{jk}(x_j - x_k)$. The sum consists of $|I|$ nonnegative terms less than 1 (under the assumption that all ω's are 1). Thus we have $0 \leq \tilde{\omega}_{ij} < n - k$. □

We always choose a face with the smallest number of sides as the outer face f_0, and thus we have to distinguish three cases for the outer face: a triangle, a quadrilateral, or a pentagon. By Euler's formula, a face of one of these three types must always exist.

2.1 The Plane Embedding for Graphs with a Triangular Face

The triangular case is easy: we can position its vertices at any convenient position. We place the three boundary vertices as follows:

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4)$$

LEMMA 3. *If the smallest face of G is a triangle and we place the boundary vertices like stated in* (4) *then the boundary forces can be resolved.*

PROOF. We embed G as spring embedding and calculate the substitution stresses. After setting $\omega_{12} = -\tilde{\omega}_{12}, \omega_{23} = -\tilde{\omega}_{23}, \omega_{13} = -\tilde{\omega}_{13}$ all points are in equilibrium. □

2.2 The Plane Embedding for Graphs with a Quadrilateral Face

In the quadrilateral case, the Substitution Lemma helps us to reduce the problem to the consideration of the stresses $\tilde{\omega}_{ij}$ between $k = 4$ vertices. It turns out that the substitution stresses $\tilde{\omega}_{ij}$ between adjacent vertices (on the boundary) are irrelevant, since their resulting forces can be directly cancelled by the corresponding stresses ω_{ij} (see formula (6) below). Thus, we only have to look at $\tilde{\omega}_{13}$ and $\tilde{\omega}_{24}$. (Or more precisely, only the *ratio* between $\tilde{\omega}_{13}$ and $\tilde{\omega}_{24}$ matters.)

We need to position $\mathbf{p}_1, \ldots, \mathbf{p}_4$ in such a way that they form a convex quadrilateral, and the linear system for the four canceling stresses ω_{12}, ω_{23}, ω_{34}, ω_{14} on the boundary edges is solvable. The positions and the boundary stresses can be computed as the solution of a non-linear equation system which consists of the equations

$$\omega_{i,suc(i)}(\mathbf{p}_i - \mathbf{p}_{suc(i)}) + \omega_{i,pre(i)}(\mathbf{p}_i - \mathbf{p}_{pre(i)}) = -\mathbf{F}_i, \quad (5)$$

for all $i \in B$. where $suc(i)$ denotes the successor of v_i and $pre(i)$ denotes the predecessor of v_i at f_0 in cyclic order. The variables $\mathbf{F}_i$ can be expressed by (2) in terms of the substitution stresses.

To obtain a solution we fix some of the positions of the boundary vertices, namely

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_3 = \begin{pmatrix} 2 \\ y_3 \end{pmatrix}, \mathbf{p}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (6)$$

Under this assumption, there is the unique solution

$$\begin{aligned}
\omega_{12} &= -2\tilde{\omega}_{13} - \tilde{\omega}_{12}, \\
\omega_{23} &= \tilde{\omega}_{24} - 2\tilde{\omega}_{13} - \tilde{\omega}_{23}, \\
\omega_{34} &= -\frac{\tilde{\omega}_{24}}{2} - \tilde{\omega}_{34}, \\
\omega_{14} &= \frac{\tilde{\omega}_{24}\tilde{\omega}_{13} - \tilde{\omega}_{24} + 2\tilde{\omega}_{24}\tilde{\omega}_{13} - \tilde{\omega}_{14}\tilde{\omega}_{24}}{\tilde{\omega}_{24} - 2\tilde{\omega}_{13}}, \\
y_3 &= \frac{\tilde{\omega}_{24}}{2\tilde{\omega}_{13} - \tilde{\omega}_{24}}.
\end{aligned} \quad (7)$$

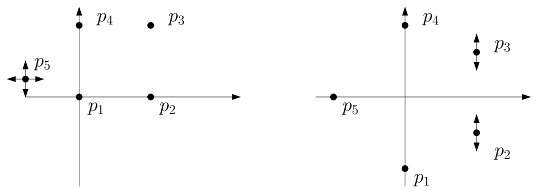
We assume that $\tilde{\omega}_{13} \geq \tilde{\omega}_{24}$. (Otherwise we cyclically relabel the vertices of f_0.) Thus $y_3 > 0$ and f_0 forms a *convex* face.

If f_0 is convex, the boundary stresses must necessarily be negative, since otherwise the boundary vertices could not be in equilibrium, with all interior stresses being positive. Thus we need not explicitly check the sign of the boundary stresses.

LEMMA 4. *If the smallest face of G is a quadrilateral and we place the boundary vertices as stated in* (6) *and* (7)*, then f_0 forms a convex polygon and the boundary stresses in* (7) *cancel the boundary forces.* □

2.3 The Plane Embedding for Graphs with a Pentagonal Face

The case of a pentagon is more complicated. We have $\binom{5}{2} = 10$ substitution stresses $\tilde{\omega}_{ij}$, but the adjacent ones do not count. So we are left we five "diagonal" substitution stresses $\tilde{\omega}_{ij}$. Again, we managed to find suitable positions



(a) Case A (b) Case B

Figure 2: Placement of $\mathbf{p}_1, \ldots, \mathbf{p}_5$.

for $\mathbf{p}_1, \ldots, \mathbf{p}_5$ that will allow the forces to be cancelled by boundary stresses.

We mimic the approach of Section 2.2. The location of f_0 will be again computed as solution of a non-linear equation system. The system consists of the equations given in (2),(5) and constraints for boundary vertices. However, we have to make more effort to guarantee the convexity of f_0. The following lemma helps us here:

LEMMA 5. *We can relabel the boundary points for any stress $(\tilde{\omega}_{ij})_{1\le i,j\le 5}$ such that*

$$\tilde{\omega}_{35} \ge \tilde{\omega}_{24} \quad \textit{and} \quad \tilde{\omega}_{25} \ge \tilde{\omega}_{13}.$$

PROOF. Without loss of generality we assume that the largest stress on an interior edge is $\tilde{\omega}_{35}$. If $\tilde{\omega}_{25} \ge \tilde{\omega}_{13}$ we are done. Otherwise we relabel the vertices by exchanging $\mathbf{p}_3 \leftrightarrow \mathbf{p}_5$ and $\mathbf{p}_1 \leftrightarrow \mathbf{p}_2$. $\square$

For the rest of this section we label the vertices such that Lemma 5 holds. The way we embed the f_0 depends on the substitution stresses $\tilde{\omega}_{ij}$.

CASE A: We assume that

$$\tilde{\omega}_{35}\tilde{\omega}_{14} + \tilde{\omega}_{14}\tilde{\omega}_{25} + \tilde{\omega}_{25}\tilde{\omega}_{24} + \tilde{\omega}_{13}\tilde{\omega}_{35} > \tilde{\omega}_{35}\tilde{\omega}_{25}. \tag{8}$$

In this case we assign

$$\mathbf{p}_1 = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{p}_3 = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{p}_4 = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{p}_5 = \begin{pmatrix}x_5\\y_5\end{pmatrix}.$$

Figure 2a illustrates the location of the points. Together with (2) and (5) we obtain as the solution for $\mathbf{p}_5$:

$$x_5 = \frac{(\tilde{\omega}_{13} - \tilde{\omega}_{25} - \tilde{\omega}_{24})(\tilde{\omega}_{35} + \tilde{\omega}_{13} - \tilde{\omega}_{24})}{\tilde{\omega}_{35}\tilde{\omega}_{14} + \tilde{\omega}_{14}\tilde{\omega}_{25} + \tilde{\omega}_{25}\tilde{\omega}_{24} + \tilde{\omega}_{13}\tilde{\omega}_{35} - \tilde{\omega}_{35}\tilde{\omega}_{25}},$$

$$y_5 = \frac{\tilde{\omega}_{35} + \tilde{\omega}_{13} - \tilde{\omega}_{24}}{\tilde{\omega}_{35} + \tilde{\omega}_{25}}.$$

We have to check that the points $\mathbf{p}_1, \ldots, \mathbf{p}_5$ form a convex polygon. Clearly $y_5 > 0$ since the $\tilde{\omega}_{ij}$'s are positive and $\tilde{\omega}_{35} \ge \tilde{\omega}_{24}$. Moreover $y_5 < 1$, because $\tilde{\omega}_{25} \ge \tilde{\omega}_{13}$. The numerator of x_5 is negative and due to (8) the denominator of x_5 is positive. Therefore $x_5 < 0$ and the outer face is embedded as a convex face.

CASE B: We assume the opposite of (8). The coordinates for the boundary vertices are chosen as

$$\mathbf{p}_1 = \begin{pmatrix}0\\-1\end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix}1\\y_2\end{pmatrix}, \mathbf{p}_3 = \begin{pmatrix}1\\y_3\end{pmatrix}, \mathbf{p}_4 = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{p}_5 = \begin{pmatrix}-1\\0\end{pmatrix}.$$

See Figure 2b for an illustration. This leads to the solution $y_2 =$

$$\frac{\tilde{\omega}_{24}\tilde{\omega}_{13} + \tilde{\omega}_{24}\tilde{\omega}_{35} + \tilde{\omega}_{25}\tilde{\omega}_{13} + 2\tilde{\omega}_{25}\tilde{\omega}_{35} - \tilde{\omega}_{13}^2 - 2\tilde{\omega}_{13}\tilde{\omega}_{35} - \tilde{\omega}_{35}\tilde{\omega}_{14}}{(\tilde{\omega}_{24}\tilde{\omega}_{35} + 2\tilde{\omega}_{25}\tilde{\omega}_{13} + 2\tilde{\omega}_{25}\tilde{\omega}_{35})/2},$$

and $y_3 =$

$$\frac{\tilde{\omega}_{24}\tilde{\omega}_{13} + \tilde{\omega}_{24}\tilde{\omega}_{35} + \tilde{\omega}_{25}\tilde{\omega}_{13} + 2\tilde{\omega}_{25}\tilde{\omega}_{35} - \tilde{\omega}_{24}^2 - 2\tilde{\omega}_{24}\tilde{\omega}_{25} - \tilde{\omega}_{14}\tilde{\omega}_{25}}{(\tilde{\omega}_{24}\tilde{\omega}_{35} + \tilde{\omega}_{25}\tilde{\omega}_{13} + 2\tilde{\omega}_{25}\tilde{\omega}_{35})/2}.$$

(The boundary stresses ω_{ij} are more complicated expressions and not shown here.) The outer face is convex if $-2 < y_2 < y_3 < 2$. The inequalities $-2 < y_2$ and $y_3 < 2$ are equivalent to

$$-\tilde{\omega}_{13}^2 - \tilde{\omega}_{13}\tilde{\omega}_{35} - \tilde{\omega}_{35}\tilde{\omega}_{14} + \tilde{\omega}_{13}(\tilde{\omega}_{24} - \tilde{\omega}_{35}) < 0 \quad \text{and}$$
$$-\tilde{\omega}_{24}^2 - \tilde{\omega}_{24}\tilde{\omega}_{25} - \tilde{\omega}_{14}\tilde{\omega}_{25} + \tilde{\omega}_{24}(\tilde{\omega}_{13} - \tilde{\omega}_{25}) < 0.$$

These inequalities hold, because we add only negative terms on the left side. It remains to check $y_2 - y_3 < 0$. First we get rid of the denominator and bring all negative terms on the right side. This leads to the equivalent inequality

$$\begin{aligned} &\tilde{\omega}_{13}^2 + \tilde{\omega}_{24}^2 + 2\tilde{\omega}_{13}\tilde{\omega}_{35} + 2\tilde{\omega}_{24}\tilde{\omega}_{25} + \tilde{\omega}_{25}\tilde{\omega}_{14} + \tilde{\omega}_{35}\tilde{\omega}_{14} \\ &\quad < 2\tilde{\omega}_{24}\tilde{\omega}_{35} + 2\tilde{\omega}_{25}\tilde{\omega}_{13} + 4\tilde{\omega}_{25}\tilde{\omega}_{35} + 2\tilde{\omega}_{24}\tilde{\omega}_{13}. \end{aligned} \tag{9}$$

We observe that $\tilde{\omega}_{13}^2 \le \tilde{\omega}_{25}\tilde{\omega}_{13}$ and $\tilde{\omega}_{24}^2 \le \tilde{\omega}_{24}\tilde{\omega}_{35}$. Because of the assumption for case B we have $4\tilde{\omega}_{35}\tilde{\omega}_{25} > 2\tilde{\omega}_{13}\tilde{\omega}_{35} + \tilde{\omega}_{24}\tilde{\omega}_{25} + \tilde{\omega}_{25}\tilde{\omega}_{14} + \tilde{\omega}_{35}\tilde{\omega}_{14}$. Therefore the right side of (9) is greater than its left side which shows that $y_2 < y_3$ and the outer face is embedded as convex pentagon. This finishes the case distinction and we can conclude:

LEMMA 6. *If the smallest face of G is a pentagon and we place the boundary vertices as discussed above, then the outer face will be embedded as a convex polygon and the computed boundary stresses cancel the boundary forces.* $\square$

We defined four different ways to embed G. The selected embedding depends on the combinatorial structure G. If G contains a triangle we say it is of *type 3*. If it contains a quadrilateral but no triangle G it is of *type 4*. Otherwise it is of *type 5A* or *type 5B*, depending on the case (Case A or Case B).

3. LIFTING TO THREE DIMENSIONS

We now review how the lifting is obtained from an equilibrium stress [6, 17, 10]. (This is one direction of Theorem 1.) Every lifted face f_i of G lies on a plane H_i. Together with the 2d coordinates $\mathbf{p}_i$, the set of planes H_i determines the embedding in space. We describe each plane H_i as a function which assigns to every xy-coordinate a height:

$$H_i\colon \mathbf{q} \mapsto z_i(\mathbf{q}) = \langle \mathbf{a}_i, \mathbf{q} \rangle + d_i.$$

The plane H_i is defined by its two-dimensional *gradient vector* $\mathbf{a_i}$ and a scalar d_i.

The following relation lies at the heart of the correspondence of Theorem 1. If two faces f_s and f_t share an edge (v_i, v_j) and f_s lies left of this edge, then

$$\omega_{ij}\begin{pmatrix}y_i - y_j\\x_j - x_i\end{pmatrix} = \mathbf{a}_t - \mathbf{a}_s \tag{10}$$

The planes H_i are computed incrementally. We start with the boundary face f_0, whose plane H_0 is the x-y-plane: $\mathbf{a}_0 = (0,0)^T$ and $d_0 = 0$. Let us assume we have computed the planes for some connected region R of faces in G. In the next step we select a face f_{new} which shares an edge (v_i, v_j) with R, but is not a member of R. Let f_{old} be the face in R which contains (v_i, v_j). With help of (10) we compute $\mathbf{a}_{new}$. The scalar d_{new} is then given by the condition that

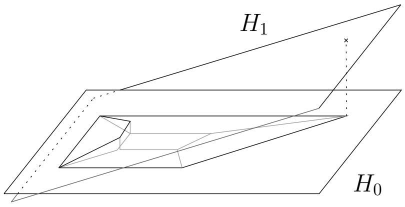


Figure 3: Lifting the face f_1.

the lifting $(\mathbf{p}_i, z_{old}(\mathbf{p}_i))$ of $\mathbf{p}_i$ lies on H_{new}, which leads to $d_{new} = z_{old}(\mathbf{p}_i) - \langle \mathbf{a}_{new}, \mathbf{p}_i \rangle$.

The following lemma helps to guarantee that the lifting produces integer z-coordinates.

LEMMA 7. *If every $\mathbf{a}_i$ used in the lifting is integral and all $\mathbf{p}_j$ are integral then the lifting yields integer z-coordinates for every $\mathbf{p}_j$ as well.*

PROOF. Since the operations described above involve only multiplications and additions of integers, an inductive argument shows that all d_i values and all heights $z(\mathbf{p_j})$ are integral. □

Let f_1 be a plane adjacent to the outer face f_0. Then the lifted (convex) polytope lies completely between the planes H_0 and H_1 (see Figure 3). Therefore we can bound the maximal z-coordinate by $\max_{j \in B} z_1(\mathbf{p}_j)$.

4. BOUNDING THE GRID SIZE

4.1 Size of an Integral Spring Embedding

So far we obtained an embedding of G which supports a lifting. The coordinates of the computed plane embedding are rational. A quantitative analysis of the spring embedding method allows us to bound the denominator of the interior points.

Let us review the *spring embedding* method shortly. The coordinates of the boundary vertices were obtained in Section 2. From the equilibrium equations (3), the positions of the of the interior points $\mathbf{x}_I$ and $\mathbf{y}_I$ can be computed by

$$\begin{aligned} \mathbf{x}_I &= -\bar{L}^{-1} A_{IB} \mathbf{x}_B, \\ \mathbf{y}_I &= -\bar{L}^{-1} A_{IB} \mathbf{y}_B. \end{aligned} \tag{11}$$

As shown by Tutte [15, 16], this yields a non-crossing embedding (see also [10]).

LEMMA 8. *If the boundary points are integral, the spring embedding yields coordinates which are multiples of $1/\det \bar{L}$.*

PROOF. By Cramer's rule every coordinate can be expressed as

$$x_i = \det \bar{L}^{(i)} / \det \bar{L},$$

where $\det \bar{L}^{(i)}$ is obtained from $\bar{L}$ by replacing the i-th column of $\bar{L}$ by $A_{IB}\mathbf{x}_B$. Since $\det \bar{L}^{(i)}$ is integral, $\det \bar{L}$ is the denominator of x_i. The same holds for y_i. □

4.2 An Upper Bound for the Determinant of the Reduced Laplacian Matrix

For bounding the grid size it is necessary to take a closer look at $\det \bar{L}$, because it appears as the denominator of the stresses and the coordinates. This quantity is connected to the number of certain spanning forests of G.

DEFINITION 1. *A subgraph F_B of G is called spanning B-forest if*

- *$\mathcal{F}_B$ consists of $|B|$ vertex-disjoint trees covering all vertices of G,*
- *each tree contains one vertex of B.*

A generalization of the Matrix-Tree-Theorem is given in [8]. It states that the number $\#F_B(G)$ of spanning B-forests of G is $\det \bar{L}$. The classical Matrix-Tree-Theorem is the case $|B| = 1$.

Moreover, the number $\#F_B(G)$ is related to the number $\#T(G)$ of spanning trees of G.

LEMMA 9. *Let G be a planar graph with a distinguished set of vertices B. The number of spanning B-forests of G is bounded from above by*

$$\#F_B(G) \le \binom{n-1}{|B|-1} \cdot \#T(G).$$

PROOF. Let T be a spanning tree of G. To get a spanning B-forest, we have to remove $|B| - 1$ edges from T. Every spanning B-forest can be obtained in this way. □

It is easy to give an exponential upper bound for $\#T(G)$:

PROPOSITION 1 ([8]). *1. The number of spanning trees in a graph is bounded by the product of all vertex degrees:*

$$\#T(G) \le \prod_i \deg(v_i)$$

2. For a planar graph, we have $\prod_i \deg(v_i) < 6^n$.

PROOF. 1. Pick an arbitrary vertex v_1. Consider all $\prod_{i \ne 1} \deg(v_i)$ directed graphs that are obtained by choosing an outgoing edge in G out of every vertex except v_1. By ignoring the edge orientations, one obtains all spanning trees (and many graphs that are not spanning trees).

2. This follows from the arithmetic-geometric-mean inequality and $\sum_i \deg(v_i) < 6n$, which is a consequence of Euler's formula. □

Sharper bounds for $\#T(G)$ have been given by Ribó Mor [8], see also [9]. These bounds also take into account whether G contains triangles or quadrilaterals:

$$\begin{aligned} \text{if } G \text{ has type 3:} \quad \#F_B(G) &\le \binom{n-1}{2} 5.\bar{3}^n, \\ \text{if } G \text{ has type 4:} \quad \#F_B(G) &\le \binom{n-1}{3} 3.529988^n, \\ \text{if } G \text{ has type 5A/5B:} \quad \#F_B(G) &\le \binom{n-1}{4} 2.847263^n. \end{aligned}$$

4.3 Scaling

Finally we scale the embedded graph to get integer coordinates. For the sake of compact statements we abbreviate $\det \bar{L}$ with Δ. Remember that Lemma 7 implies that integral xy-coordinates and integral vectors $\mathbf{a}_i$ are sufficient conditions for integer z-coordinates.

Getting integer coordinates for the boundary vertices is our first objective. Let S_x be the scaling factor for the x-coordinates and S_y the scaling factor for the y-coordinates.

We will use integral scaling factors; therefore, no integer coordinate will be scaled to a non-integer.

If G is of type 3 then we need not scale, since all boundary coordinates are 0 or 1 (thus we set $S_x = S_y = 1$). If G is of type 4 we have to scale only the y-coordinates. We multiply every y-coordinate by $S_y := (2\tilde{\omega}_{13} - \tilde{\omega}_{24})\Delta$, which is an integer, by Lemma 1. Observe that the only non-integer coordinate y_3 is scaled to $y_3 S_y = \tilde{\omega}_{24}\Delta$, and is therefore, by Lemma 1, integral.

If G is of $5A$ we have to scale such that $S_x x_5$ and $S_y y_5$ are integral. This will be achieved by

$$S_x = (\tilde{\omega}_{35}\tilde{\omega}_{14} + \tilde{\omega}_{14}\tilde{\omega}_{25} + \tilde{\omega}_{25}\tilde{\omega}_{24} + \tilde{\omega}_{13}\tilde{\omega}_{35} - \tilde{\omega}_{35}\tilde{\omega}_{25}) \cdot \Delta^2$$
$$S_y = (\tilde{\omega}_{35} + \tilde{\omega}_{25}) \cdot \Delta.$$

It can be checked that these factors are integral, and so are $S_x x_5$ and $S_y y_5$.

It remains to introduce scaling factors when G is of type $5B$. Since the non-integer boundary coordinates are y_2 and y_3, we need to scale in y-direction only. We choose

$$S_y = (\tilde{\omega}_{24}\tilde{\omega}_{35} + \tilde{\omega}_{25}\tilde{\omega}_{13} + 2\tilde{\omega}_{25}\tilde{\omega}_{35}) \cdot \Delta^2.$$

Again we can observe that, by Lemma 1, S_y is integral as well as $S_y y_2$ and $S_y y_3$.

We have found for every type of G a pair of scaling factors such that the scaled boundary points are integral. To obtain integral coordinates for all points we extend the scaling factors S_x and S_y to $\bar{S}_x := S_x \cdot \Delta$ and $\bar{S}_y := S_y \cdot \Delta$. Because of Lemma 8, the extended scaling factors will yield integer coordinates for all $\mathbf{p}_i$.

Furthermore we observe

LEMMA 10. *We choose $\bar{S}_x$ and $\bar{S}_y$ as discussed above depending on the type of G. In the plane embedding of G, we multiply all x-coordinates by $\bar{S}_x$ and all y-coordinates by $\bar{S}_y$. The z-coordinates of the lifting of $G(\mathbf{p})$ induced by ω will be integral.*

PROOF. It is sufficient to show that the gradient $\mathbf{a}_1$ of some face f_1 that is adjacent to the outer face f_0 is integral. This implies that the remaining vectors $\mathbf{a}_i$ are also integral, since we can use the integral stresses on interior edges to compute them.

To prove the integrality of $\mathbf{a}_1$ we make a case distinction on the type of G. Assume G has type 3. Let f_1 be the face which shares the edge (v_1, v_2) with f_0. Calculating $\mathbf{a_1}$ with help of (10) yields $\mathbf{a}_1 = -\omega_{12}(0, \bar{S}_x)$. Since ω_{12} is a multiple of $1/\Delta$, and in our case $\Delta = \bar{S}_x$, $\mathbf{a}_1$ is integral.

Next we assume G has type 4. The face f_1 is selected as in the previous case. We obtain $\mathbf{a}_1 = -\omega_{12}(0, \bar{S}_x)^T$. The stress ω_{12} is part of the solution (7), namely $-2\tilde{\omega}_{13} - \tilde{\omega}_{12}$. Since $\bar{S}_x$ was chosen as Δ, we have $\mathbf{a}_1 = (0, \Delta(2\tilde{\omega}_{13} - \tilde{\omega}_{12}))^T$. Therefore $\mathbf{a}_1$ is integral.

If G has type $5A$, f_1 is again chosen as the face which shares the edge (v_1, v_2) with f_0. We obtain $\mathbf{a}_1 = -\omega_{12}(0, \bar{S}_x)^T$. The system we solved to determine $\mathbf{p}_5$ gives

$$-\omega_{12} = \frac{\Omega_1}{\tilde{\omega}_{25}\tilde{\omega}_{13} + \tilde{\omega}_{23}\tilde{\omega}_{25} + \tilde{\omega}_{24}\tilde{\omega}_{35} + \tilde{\omega}_{35}\tilde{\omega}_{23} - \tilde{\omega}_{25}\tilde{\omega}_{35}},$$

where Ω_1 is a polynomial in $\tilde{\omega}$ of degree 3 which is less than $12(n-5)^3$. We notice that $\bar{S}_x \omega_{12}$ equals $\Omega_1 \Delta^3$. Due to Lemma 1, $\mathbf{a}_1$ is integral.

Finally we assume G has type $5B$, the face f_1 is now chosen as the face which shares (v_2, v_3) with f_0. We have $\mathbf{a}_1 = \omega_{23}(\bar{S}_y(y_3 - y_2), 0)^T$. The stress ω_{23} is part of the solution of the equation system we solved to determine y_2 and y_3, namely

$$\omega_{23} = \frac{-\Omega_2}{(\tilde{\omega}_{24}\tilde{\omega}_{35} + \tilde{\omega}_{25}\tilde{\omega}_{13} + 2\tilde{\omega}_{25}\tilde{\omega}_{35})(y_3 - y_2)}.$$

The term Ω_2 denotes a polynomial in $\tilde{\omega}$ of degree 3 which is less than $17(n-5)^3$. Evaluating $\mathbf{a}_1$ gives $(-\Omega_2\Delta^3, 0)^T$, hence $\mathbf{a}_1$ is integral. □

It remains to analyze the necessary grid size for the lifting. We have already calculated the values for $\mathbf{a}_1$ in the proof of the previous Lemma. We observe that $\delta_1 = \Omega_2\Delta^4$ if G is of type $5B$, otherwise $\delta_1 = 0$. The different equations for H_1, depending on the type of G, are listed in Table 1. Table 2 contains the maximal z-coordinate of the lifted boundary vertices, depending on the type of G.

type of G	$z_1(x, y)$
3	$\tilde{\omega}_{12}\Delta^2 y$
4	$(\tilde{\omega}_{12} + 2\tilde{\omega}_{13})(2\tilde{\omega}_{13} - \tilde{\omega}_{24})\Delta^3 y$
$5A$	$\Omega_1(\tilde{\omega}_{25} + \tilde{\omega}_{35})\Delta^5 y$
$5B$	$\Omega_2\Delta^4(1 - x)$

Table 1: Equations for H_1, for each type of G.

type of G	$\max_{i \le k} z_1(\mathbf{p}_i)$
3	$z_1(\mathbf{p}_3) = \tilde{\omega}_{12}\Delta^2$
4	$z_1(\mathbf{p}_3) = (\tilde{\omega}_{12} + 2\tilde{\omega}_{13})\tilde{\omega}_{24}\Delta^3$
$5A$	$z_1(\mathbf{p}_4) = \Omega_1(\tilde{\omega}_{25} + \tilde{\omega}_{35})\Delta^5$
$5B$	$z_1(\mathbf{p}_5) = 2\Omega_2\Delta^4$

Table 2: Maximal z-coordinates on the plane H_1, depending on the type of G.

We observe that exponentially large coordinates suffice to embed G as 3-polytope. Lemma 2 implies that the exponential growth of the size of the coordinates is at most Δ^5. The analysis of the polynomial factors is straightforward. Details are given in the full version of the paper. Table 3 summarizes the upper bounds for the grid size.

type of G	u.b. for $\lvert x \rvert$	u.b. for $\lvert y \rvert$	u.b. for $\lvert z \rvert$
3	$n^2\ 5.3^n$	$n^2\ 5.3^n$	$n^5\ 28.4^n$
4	$n^3\ 3.53^n$	$n^7\ 12.46^n$	$n^{11}\ 46.38^n$
$5A/5B$	$n^{14}\ 23.08^n$	$n^{10}\ 8.10^n$	$n^{24}\ 187.12^n$

Table 3: Upper bounds (u.b.) for the grid size for the different types of G.

THEOREM 2. *Every planar 3-connected graph G with n vertices can be embedded as 3-polytope with integer coordinates that are bounded by $O(2^{7.55n})$.*

5. AN EXAMPLE

A dodecahedron is one of the five platonic solids. It has 20 vertices, 30 edges and 12 faces. Figure 1 shows the edge graph and a 3-dimensional realization of the dodecahedron. This example is motivated by the fact, that all faces of the dodecahedron are pentagons. Thus we have to apply the more involved methods for an integer embedding. Since the dodecahedron is symmetric it makes no difference which face we choose as the outer face. We start with the computation with calculating the substitution stresses. We obtain for all the stresses $\tilde{\omega}_{13}, \tilde{\omega}_{14}, \tilde{\omega}_{24}, \tilde{\omega}_{25}$ and $\tilde{\omega}_{35}$ the value 36/449. The fact that all these stresses have the same value is again due to the symmetry of the dodecahedron. Because the outer face is a pentagon, we have to check if the graph of the dodecahedron is of type $5A$ or $5B$. Evaluating (8), shows that the graph is of type $5A$. With help of the substitution stresses we can compute the coordinates of the boundary points. We obtain

$$\mathbf{p}_1 = (0,0)^T, \mathbf{p}_2 = (1,0)^T, \mathbf{p}_3 = (1,1)^T$$

$$\mathbf{p}_4 = (0,1)^T, \mathbf{p}_5 = (-1/3, 1/2)^T.$$

As the next step we apply Tutte's method to compute the coordinates of the interior points. The stress ω_{12} is computed with help of the substitution stresses – we obtain $\omega_{12} = -151/449$. We scale the embedded graph as defined in Section 4.3. We obtain $\Delta = 403202$. This yields the scaling factors

$$\begin{aligned} \bar{S}_x &= 1264158727403904, \\ \bar{S}_y &= 26069428512. \end{aligned}$$

Next we compute the planes H_i. Finally we obtain the coordinates of the polytope by plugging the coordinates into the equation for the corresponding plane. The result is shown in Figure 1. We have scaled the z-coordinates down to obtain a nicer picture. The greatest coordinate is

$$z_{max} = 3{,}845{,}325{,}824{,}461{,}495{,}633{,}711{,}104 \approx 2^{81.67},$$

which is quite large, but much smaller than the bound 2^{151} of Theorem 2.

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