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Advantage in the discrete Voronoi game

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Abstract

We study the discrete Voronoi game, where two players alternately claim vertices of a graph for t rounds. In the end, the remaining vertices are divided such that each player receives the vertices that are closer to his or her claimed vertices. We prove that there are graphs for which the second player gets almost all vertices in this game, but this is not possible for bounded-degree graphs. For trees, the first player can get at least one quarter of the vertices, and we give examples where she can get only little more than one third of them. We make some general observations, relating the result with many rounds to the result for the one-round game on the same graph.

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1 Introduction

The classic facility location problem deals with finding the optimal location for a facility (such as a supermarket, hospital, fire station) with respect to a given set of customers. Typically, we want to place the facility to minimize the distance customers need to travel to get to it. Competitive facility location is a variant of the problem when several service providers compete for the interests of the same set of customers. An example would be two supermarket chains building shops in a city – with each chain trying to attract the largest number of customers.

We study a simple model of competitive facility location called the *Voronoi game*. This game is a game played on a measurable metric space by two players. The players alternate in placing a facility on a single point in the space The game lasts for a fixed number of rounds. At the end of the game, the space is divided between the two players: each player receives the area which is closer to his or her facilities, or in other words, the sum of the areas of the corresponding regions in the Voronoi diagram. The winner is the player who controls the greater portion of the space.

The Voronoi game was first defined by Ahn, Cheng, Cheong, Golin, and van Oostrum [1], who studied it on lines and circles. Subsequently a discrete version of the game emerged, on which we shall focus in this paper; for results on the continuous game see e.g., [6, 7]. The *discrete Voronoi game* is played on the vertices of a graph G by two players called **A** and **B** for a fixed number t of rounds. Player **A** starts, and they alternatingly claim vertices of G during each round $1, \dots, t$ (we also say they put pebbles on those points). No vertex may be claimed more than once. At the end of the game, the remaining vertices are divided between the players – with each player receiving the vertices that are closer to his or her claimed vertices. If a vertex is equidistant to each players' claimed vertices then it is split evenly between **A** and **B** (each player receives half a vertex.)

This natural variant was first studied by Demaine, Teramoto and Uehara [11], who showed that it is NP-complete to determine the winner in the Voronoi game on a general graph G, even if the game lasts for only one round, (but player **B** can place more than one pebble). They also studied the game on a large k-ary tree and showed that under optimal play, the first player wins if k is odd, and that the game ends in a tie when k is even. The game on trees was studied further by Kiyomi, Saitoh, and Uehara [8] who completely solved the game on a path – they showed that the game on a path with n vertices played for $t < n/2$ rounds always ends in a draw, unless n is odd and $t = 1$, in which case **A** wins (by having one vertex more). There are many results that deal with various algorithmic questions about variations and special cases of the Voronoi game, for example for weighted graphs [3], in a planar geometric setting [5, 4], or for a "continuous" graph model [2].

The above results suggest that in general it is hard to determine the winner of the Voronoi game on a graph. Therefore, in this paper, we will not be concerned with deciding the winner of the game – rather we are interested in knowing when either player can control a large proportion of the vertices by the end of the game. One question we are interested in is: "for $\epsilon > 0$, for which graphs G does **A** have a strategy to control at least $\epsilon|G|$ in the t-round Voronoi game on G?" We are also interested in situations when a player can win the Voronoi game by a large margin i.e. "for which graphs G does **A** have a strategy to control at least $(1/2+\epsilon)|G|$ vertices by the end of the game?" The same questions are asked for Player **B** as well. These questions motivate us to make the following definition.

Definition 1 *For a given graph G define its* Voronoi ratio*, $VR(G,t)$, as the number of vertices that belong to* **A***, plus half of the number of tied vertices (if there are any) divided by the total number of vertices in G after an optimal play of t rounds.*

It is not immediately clear what range $VR(G,t)$ can take. By considering a star S_k with k leaves, it is easy to show that

$$VR(S_k,t) = 1 - \frac{t}{k+1}. \tag{1}$$

This shows that $VR(G,t)$ can be arbitrarily close to 1, and hence **A** can control almost all the vertices by the end of the game. By considering a path it is possible to show that $VR(G,t) = 1/2$ is possible as well [8]. However, constructing a graph which satisfies $VR(G,1) < 1/2$ is already non-trivial. The smallest such graph that we know of has 9 vertices. It consist of a cycle of length six, with an additional leaf attached to every other vertex of the cycle. It is easy to check that in the 1-round Voronoi game on this graph **B** can always win 5 of the 9 vertices. In Section 3 we show that, in fact, $VR(G,t)$ can be arbitrarily close to zero.

Theorem 2 *For every $\epsilon > 0$ and $t \in \mathbb{N}$, there is a graph G with $VR(G,t) < \epsilon$.*

This theorem, together with (1) shows that in general the discrete Voronoi game does not favor either player.* However there may be natural classes of graphs on which one of the players has a significant advantage.

In Section 4, we study the Voronoi game on a tree and show that every tree T satisfies $VR(T,t) \geq 1/4$ for all t. When the number of rounds is small, the first player may obtain an even larger advantage. It was noted in [11] that $VR(T,1) \geq 1/2$ for every tree T. We show that $VR(T,2) \geq 1/3$, for any tree T and construct trees whose Voronoi ratio is arbitrarily close to 1/3 for $t = 2$ moves.

In Section 5, we study the Voronoi game on a graph with bounded maximum degree. We show that every graph G with maximum degree d has $VR(G,t) \leq 1 - 1/2d$. We show that the bound in this result cannot be decreased substantially by constructing graphs G with maximum degree d whose Voronoi ratio is arbitrarily close to $1 - 1/d$.

*In fact the construction easily generalizes to the continuous Voronoi game as well, but here we focus only on the discrete version.

In order to prove some of the above results, we first establish bounds on the Voronoi ratio which hold for *all* graphs. In Section 2 we show that for any t, $VR(G,t)$ can be bounded in terms of the quantity $VR(G,1)$:

Theorem 3 *For every graph G and $t \geq 1$ we have*

$$\frac{1}{2}VR(G,1) \leq VR(G,t) \leq \frac{1}{2}(VR(G,1)+1).$$

Thus, to a limited extent, the outcome of the Voronoi game is determined just by the outcome of the one-round game. In particular, if the Voronoi ratio for one round it close to 1, then it cannot be close to 0 for more rounds, and vice versa. This theorem is useful for finding good bounds on the Voronoi ratio of various classes of graphs beyond those considered in this paper.

2 General bounds on $VR(G,t)$

In this section we give bounds for $VR(G,t)$ for a graph G in terms of $VR(G,1)$. We prove Theorem 3.

Proof: Both inequalities are proved by strategy stealing arguments. Let $n = |V(G)|$.

First we prove the left-hand inequality, $VR(G,1)/2 \leq VR(G,t)$. Suppose that **B** has a strategy in the t-round game that gives him more than $1 - VR(G,1)/2$ of vertices in G.

Let v be the optimal vertex to pick for **A** in the one-round game. Player **A**'s strategy for the t-round game is as follows. First she picks v. Then she pretends that she has not picked it and follows **B**'s strategy (in case she has to pick a vertex that she has taken already, e.g. v, she can pick arbitrarily), which would give her a fraction $1 - VR(G,1)/2$, except that she cannot play the last move. The vertices that **A** could have controlled by playing the last move u, but does not control having played v are contained in $S = \{x \in G : \mathrm{dist}(x,u) \leq \mathrm{dist}(x,v)\}$. By the definition of $VR(G,1)$, we have $|S| \leq (1 - VR(G,1))n$. So at the end of the game **A** controls at least $(1 - VR(G,1)/2)n - |S| \geq VR(G,1)n/2$ vertices, proving the lower bound.

Now we prove the right-hand inequality, $VR(G,t) \leq \frac{1}{2}(VR(G,1)+1)$. Suppose A plays v_A in her first move, and let v_B be the best response of **B** if he were playing the one-round game. Let $H = \{\, h \in G : \mathrm{dist}(h,v_B) < \mathrm{dist}(h,v_A) \,\}$ and $K = \{\, k \in G : \mathrm{dist}(k,v_B) = \mathrm{dist}(k,v_A) \,\}$. By definition of $VR(G,1)$, we have that $|H| + |K|/2 \geq (1 - VR(G,1))n$.

For the remainder of the game **B** is only interested in controlling as much of $H \cup K$ as possible. In order to do this, we consider an auxiliary game called *the new game* played on the graph $G - v_A$. The following are the rules of the new game.

- Two players, named X and Y, alternate. Player X goes first.
- The game lasts for $t-1$ rounds.

- Before the start of play, the vertex v_B is occupied by player Y.
- At the end of the game, the players score a point for each vertex of H that they control and half a point for each vertex of K that they control. Accordingly, tied vertices in H give half a point to each player and tied vertices in K give a quarter point to each player. The winner is the player with the most points.

The winner of the new game scores at least $|H|/2 + |K|/4$ points. We will show that **B** can always end up controlling at least $|H|/2 + |K|/4$ vertices at the end of the original game. This proves the upper bound of the theorem since $|H|/2 + |K|/4 \geq (1 - VR(G, 1))n/2$.

Player **B**'s strategy in the original game depends on which player wins under optimal play in the new game.

Case 1: Suppose that player Y wins the new game. In this case, in the original game, player **B** occupies v_B on his first move, and then follows player Y's strategy for the new game. At the end of the game, the situation is as in the new game except that **A** has an extra pebble on v_A. The inequality $\text{dist}(v_B, h) < \text{dist}(v_A, h)$ for all $h \in H$ ensures that this extra pebble makes no difference for the outcome in H: player **B** controls everything in H which was controlled by player Y at the end of the new game, and ties are preserved in the same way. Since $\text{dist}(v_B, k) = \text{dist}(v_A, k)$ for all $k \in K$, **B** gets at least half a vertex for every vertex in K which was controlled (scoring $\frac{1}{2}$) or tied (scoring $\frac{1}{4}$) by Y at the end of the new game. Therefore, **B**'s score of vertices within $H \cup K$ is at least the number of points obtained by Y at the end of the new game. Since player Y won the new game, player **B** must control at least $|H|/2 + |K|/4$ vertices in the original game.

Case 2: Suppose that player X wins the new game, or the new game ends in a draw. In this case, player **B** plays player X's strategy for the new game. If player **A** ever occupies v_B (such a move was not possible for Y in the new game), then **B** wastes his following move by playing arbitrarily. If **B** ever needs to play on a vertex that he already occupies (from a previous wasted move), then he plays arbitrarily again, as in the usual strategy stealing argument. **B** also wastes his last move, which was not part of the new game. At the end of the game, the difference from the situation in the new game is that (i) **A** has a pebble on v_A, (ii) **A** has possibly *no pebble* on v_B, whereas Y had a pebble there, and (iii) **B** has some extra pebbles (in fact, one or two) from wasted moves. The changes (ii) and (iii) are obviously in **B**'s favor, hence it suffices to discuss the effect of (i). We can also assume that **A** has a pebble on v_B, like player Y.

Since $\text{dist}(v_A, h) \geq \text{dist}(v_B, h)$ for all $h \in H \cup K$, the additional pebble on v_A has no effect on the outcome for the vertices from $H \cup K$. Ties remain ties, and vertices under **B**'s control remain so. Player X had at least $|H|/2 + |K|/4$ points at the end of the new game. It follows that **B** gets at least this many vertices under the scoring rules of the original game, since the score can only increase when going back to the original game: for vertices in K is doubled; for vertices in H it is unchanged. □

3 There is no lower bound on the Voronoi ratio

The goal of this section is to prove Theorem 2. In fact we prove the following stronger version.

Theorem 4 *For every $t_0 \geq 1$ and $\varepsilon > 0$, there is a graph G for which player* **B** *has a strategy for the Voronoi game ensuring him control over at least a fraction $1 - \varepsilon$ of the vertices after* each *of the rounds $1, \ldots, t_0$.*

This is slightly stronger than Theorem 2, which requires for each *fixed* number of rounds $t \leq t_0$, that a winning strategy exists (possibly a different strategy for each t). We will need this stronger statement when we consider graphs of bounded degree in Section 5.

Proof: We first illustrate the idea for the one-round game ($t = 1$). The construction is based on a continuous Voronoi game played on a d-dimensional regular simplex with the Euclidean metric and $\frac{1}{d+1}$ weight on each vertex.* In this game, no matter where **A** places her pebble, **B** can take a facet of the simplex that does not contain this pebble and place his pebble on the projection of **A**'s pebble to the facet. In this way, **A** gets $\frac{1}{d+1}$ and **B** gets $\frac{d}{d+1}$.

Consider the point set

$$\{ (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d \mid x_i \geq 0,\ x_1 + x_2 + \cdots + x_d = d^2 \}$$

and connect two points by an edge if their Manhattan distance is 2, see Figure 1. This graph models a regular $(d-1)$-dimensional simplex in d dimensions, and the distances in the graph are $\frac{1}{2}$ times the L_1-distance on $\mathbb{Z}^d$. The corners C are the points $(d^2, 0, \ldots, 0)$, $(0, d^2, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, d^2)$. Attach N leaves to each corner. The distance from $x = (x_1, \ldots, x_d)$ to the i-th corner is $d^2 - x_i$. Let $\pi_i(x)$ denote the point obtained by subtracting $d - 1$ from x_i and adding 1 to all remaining coordinates. This operation corresponds to projecting x to the simplex facet opposite the i-th corner, except that x is moved only by a fixed step size. As long as all coordinates of $\pi_i(x)$ are nonnegative, moving from x to $\pi_i(x)$ brings us closer to all corners except the i-th corner. Suppose **A** takes vertex $(x_1, \ldots, x_d)$, and let its largest coordinate be x_i. Then $x_i \geq d$, and **B** can take $\pi_i(x)$. This vertex is closer to all corners except the i-th. This ensures that **B** controls at least Nd vertices, which for sufficiently large N is within ϵ of $\frac{|G|}{d+1}$.

We now prove the theorem for the general case of t_0 moves. We start with the following set of points.

$$S := \{ (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d \mid x_i \geq 0,\ x_1 + x_2 + \cdots + x_d = d^2 t_0 \}$$

As before, we attach N leaves to each vertex.

Now we try to play against **A** as in the case of a single move. If **A** takes vertex $(x_1, \ldots, x_d)$, we find the largest coordinate x_i, and try to move to $\pi_i(x)$.

*We could get rid of the weights by starting a long, narrow path from each vertex of the simplex, giving a construction with uniform weight distribution, but not convex.

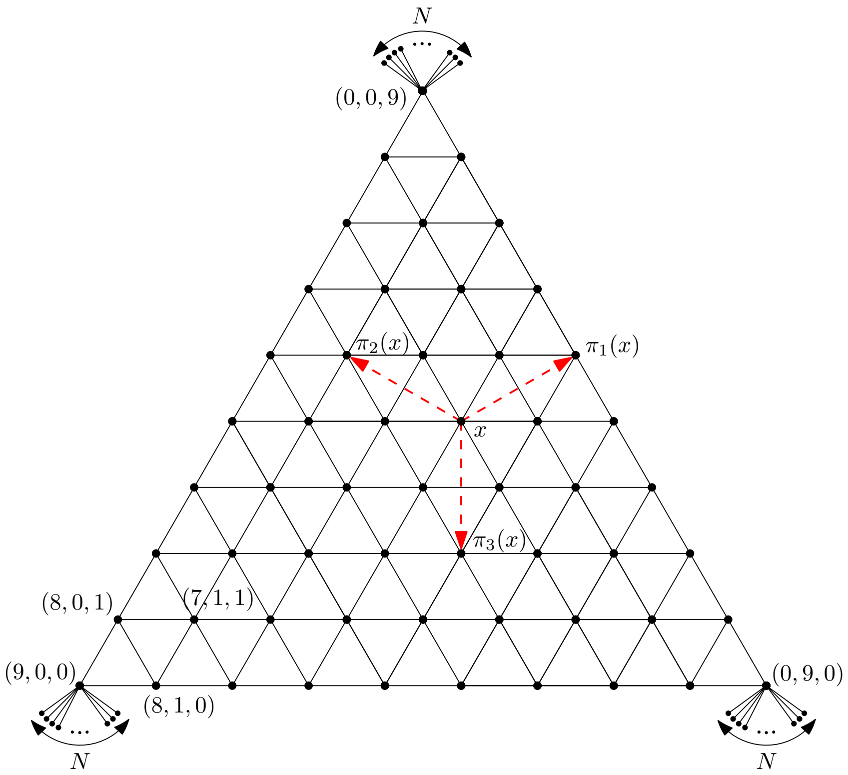

Figure 1: For $d = 3$, the graph becomes a triangular grid with edge length 9. For $d = 4$, the graph becomes a grid of degree 12 filling a tetrahedron of side length 16 in the manner of a densest sphere packing.

However, this point may already be occupied by a previous pebble of **A**. Thus we try the points $\pi_i(x)$, $\pi_i(\pi_i(x))$, $\pi_i(\pi_i(\pi_i(x)))$, ... in succession. Since **A** has played at most $t_0 - 1$ previous pebbles, one of the first t_0 points of this sequence is free, and since $x_i \geq dt_0$, it is an element of S.

Thus, after each round, **A** can own at most one additional corner. If **A** plays one of the N leaves incident to a corner, we can treat this case as if **A** had played the corresponding corner. Thus, by making N large enough so that the vertices of S become negligible, **A** will never get more than a fraction $\frac{t_0}{d} + \varepsilon'$ of the vertices, where $\varepsilon' > 0$ can be made as small as we want. The statement of the theorem follows by setting $d := 1 + \lceil t_0/\varepsilon \rceil$. □

4 Trees

In this section we investigate the quantity $VR(T, t)$ when T is a tree. We provide tight lower bounds on $VR(T, t)$ for $t = 1$ and $t = 2$ moves. For one round, it is well-known that **A** can always claim half the vertices of any tree, see for example [11, Section 6]:

Proposition 5 *For all trees T, we have $VR(T, 1) \geq \frac{1}{2}$. This bound is tight, because for the path P_n with n vertices, we have $VR(P_n, 1) \leq \frac{1}{2} + \frac{1}{2n}$.*

Proof: The optimal strategy is to put a pebble on a central vertex. Since our proof for two moves will extend the proof of this fact and of the existence of central vertices, we include this easy proof here.

An edge of the tree splits it into two parts of size $x \leq n/2$ and $n - x$. We assign the smaller size x as the *weight* of this edge and direct it from the smaller side to the larger side. A tree may have a single undirected edge (of weight $n/2$), which is called the *central edge* c_1c_2. It is easy to show that every vertex has at most one out-going arc. There can only be one or two vertices without outgoing arcs (roots). If there is a single root, it is called the *central vertex* c of the tree; otherwise the two roots are the two vertices of the central edge. We can view the tree as a directed tree oriented towards a single root c or two adjacent roots c_1, c_2.

In any tree T, the optimal strategy for **A** is to play the central vertex or one of the two vertices incident to the central edge. Removal of this vertex v splits the graph into components of size at most $n/2$. In one move, **B** can get at most one component, and thus, **A** keeps at least half of the vertices.

A path on an even number of vertices, or more generally, any tree which has a central edge, shows that the bound cannot be improved. □

Combining Proposition 5 with Theorem 3 implies the following.

Corollary 6 *For every tree T and every t, $VR(T, t) \geq \frac{1}{4}$.*

For the case of two moves, we will improve this lower bound to $VR(T, 2) \geq \frac{1}{3}$, which cannot be improved. We need the following lemma:

Lemma 7 *Let T be a tree with n vertices. Either, the central vertex c has the following property:*

C_1: *All components of the graph $T - \{c\}$ have at most $n/3$ vertices,*

or there are two distinct vertices u, v with the following properties:

C_2: *All components of the graph $T - \{u, v\}$ have at most $n/3$ vertices.*

C_2': *After removing the edges on the path from u to v, the component T_u containing u and the component T_v containing v contain more than $n/3$ vertices each.*

Proof: We use the orientation and weight labeling from the proof of Proposition 5. We will try to find our vertices u and v as the vertices which have the following *threshold property*:

(i) All incoming edges have weight $\leq n/3$.

(ii) No outgoing edge has weight $\leq n/3$.

Part (ii) of the condition means generally that the outgoing edge has weight $> n/3$, but it includes the case that there is no outgoing edge at all (the vertex is the central vertex c or it is incident to the central edge (of weight $n/2$)). We call a vertex with properties (i) and (ii) a *threshold vertex*.

Claim 8 *There is at least one threshold vertex, and there can be at most two threshold vertices.*

Proof: To see that a threshold vertex exists, start from a root (c or c_1 or c_2). If it has an incoming edge of weight $> n/3$ proceed along this edge, and repeat. Eventually, a threshold vertex must be reached.

Since weights are strictly increasing towards the root, no threshold vertex can be an ancestor of another threshold vertex. Thus, the subtrees of different threshold vertices must be disjoint. On the other hand, the subtree rooted at a threshold vertex u must contain more than $n/3$ vertices: if u has an outgoing arc, this follows from property (ii). If u is the central vertex c or one of the endpoints c_1, c_2 of the central edge, the subtrees have size n and $n/2$ respectively. It follows that there cannot be more than 2 threshold vertices. □

We note that a tree with a central vertex and two incoming arcs of weight $> n/3$ must have two threshold vertices, by the argument in the first part of the proof. We will need this fact later.

Now we can complete the proof of the lemma. If there is a single threshold vertex u which coincides with the central vertex c, all components of $G - c$ have size $\leq n/3$, and we have established condition C_1.

Otherwise, there are either (a) two threshold vertices u, v, or (b) a single threshold vertex $u \neq c$.

Case (a): There are two threshold vertices $u \neq v$. Since no threshold vertex is the ancestor of another threshold vertex, the path from u to v uses the outgoing arc from u (or if u is incident to the central edge, it uses that central edge.) The weight of this arc is the size of T_u, and by the definition of threshold vertices, it is $> n/3$. The same argument holds for v, and thus we have established property C_2'.

There are two types of components of $T - \{u, v\}$. There can be an "inner component" that contains the path from u to v (unless u and v are adjacent). The remaining components are the *outer components*: they are connected by edges that are directed into u and v. Again, by the definition of threshold vertices, their size is $\leq n/3$. The inner component contains everything except T_u and T_v, and hence its size is at most $|T| - |T_u| - |T_v| < n - n/3 - n/3 = n/3$, thus giving property C_2.

Case (b): There is a single threshold vertex $u \neq c$. In this case, we set $v := c$. The path from u to $v = c$ is directed from u to v. As in case (a), the first edge has weight $> n/3$, and thus $|T_u| > n/3$. The last edge is directed towards v; therefore $|T_v| > n/2$, and property C_2' is established. As above, this implies the bound of $n/3$ on the size of the inner component.

The outer components that are incident to u are treated as in case (a). Let us consider the outer components incident to $v = c$. If there were such a component with $> n/3$ vertices, it would mean that another threshold vertex could be found by following this edge down the tree, as we remarked after the proof of Claim 8. This is excluded in case (b), and thus we have established property C_2. □

Theorem 9 *1. For every tree T, $VR(T, 2) > \frac{1}{3}$.*

2. For every $\varepsilon > 0$ and every $t \geq 2$, there is a tree T with $VR(T, t) < \frac{1}{3} + \varepsilon$.

Proof: Lower bound. If Lemma 7 produces a single vertex c, **A**'s strategy is obvious: take c. All components of $T - c$ have size $\leq n/3$. With two moves, **B** can take at most 2 components, and thus **A** keeps at least $n/3$ vertices, even without placing her second pebble.

If Lemma 7 produces two points u, v, then **A** tries to put pebbles on them. If this succeeds, we are done: as above, after placing two pebbles, **B** can own at most two components of $T - \{u, v\}$, and thus have at most $2n/3$ vertices in total.

However, **B** might occupy u or v in his first move. Therefore, **A** has to use a more refined strategy. Let T_u and T_v denote the components of u and v after removing the edges on the path between u and v. By property C_2', we know that $|T_u|, |T_v| > n/3$. We call the neighbors of u and v that are not on the path from u to v the *children* of u and v. Each child x corresponds to an (outer) component T_x of $T - \{u, v\}$, and we pick the child for which this component is largest. Suppose w.l.o.g. that this is a child u' of u. Then **A** begins by placing a pebble on v. If **B** does not take u as a response, **A** takes it, and we are done, as we have seen above. So let us assume that **B** takes u. Then **A** takes u' in her second move.

Case 1. **B** does not take a vertex in T_v in his final move. Then **A** still owns T_v, and we are done.

Case 2. **B** takes a vertex in a component $T_{v'}$, for a child v' of v. Then **A** still owns the rest of T_v, excepting $T_{v'}$, plus all of $T_{u'}$, giving in total at least

$$|T_v| - |T_{v'}| + |T_{u'}| \geq |T_v| > n/3,$$

by the choice of u'. This concludes the proof of the lower bound.

Upper bound. We construct a tree so that **B** has a strategy to gain approximately $\frac{2}{3}$ of the vertices for any number of turns $t \geq 2$. Observe the following tree and strategy. First we need to introduce a couple of definitions.

A vertex together with x neighbors of degree 1 forms a *broom* of size x. Take a path and attach a broom at successive distances $1, 2, 4, 8, \ldots, 2^{m-1}$ from each other. We call such a path *a leg* if it contains k brooms of size N. Numbers k and N will be specified later. If N is very large, the vertices of the path become negligible, and the mass of the graph is concentrated in the brooms.

Construction: Take a center point c that will be of degree three. We attach two legs to c and a vertex h forming a broom of size kN, which we call the head, see Figure 2. If N is large, each component of $G - \{c\}$ has about $\frac{1}{3}$ of the vertices.

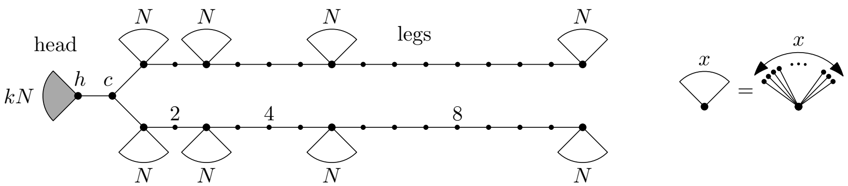

Figure 2: Player **B** can ensure to get $2/3 - 1/k$ in $t = 2$ moves. A circular sector represents a large number x of leaves incident to one vertex (a "broom"). In the example, each leg has $k = 4$ brooms.

The longest path of a leg will be called *the path* of a leg. We define a natural ordering on the path of a leg. Vertex c will be on the top and all other *over* and *below* relations of the vertices on the path of a leg we correlate according to that.

As a straightforward consequence of the exponentially increasing distances of a leg, we obtain the following observation.

Observation 10 *Suppose player* **B** *claims a vertex v on the path of a leg and w is the closest vertex below v such that w or a leaf adjacent to w is claimed by a player (either by* **A** *or* **B***). Then* **B** *controls all the brooms lying below v up to and including w, except at most one.*

As a consequence, we get.

Observation 11 *Suppose player* **B** *has claimed c or the highest vertex of a leg l and player* **A** *has claimed i vertices of this leg. In addition, suppose for each vertex w on the path for which w or a leaf adjacent to w is claimed by player* **A***, some player has claimed the vertex w' immediately below it (unless w is the lowest vertex, for which w' does not exist). Then* **A** *owns at most i brooms, plus possibly i individual leaves in brooms which are otherwise taken by* **B**.

If this condition is fulfilled, we say that **B** *dominates* the leg. When **B** has claimed c or the highest vertex of a leg l, then he can ensure that he dominates l if he can place as many pebbles into l as **A**, in addition to the pebble placed at c or the highest vertex, by following the strategy suggested by Observation 11.

Strategy:

- If **A** takes the center vertex c in the first turn, then **B** takes h. In the second turn **A** can either take a leaf from the broom at h or a vertex from one of the legs. In either case there is a leg l where **A** did not put any pebble yet. In his second turn **B** takes the closest vertex to c on l. Therefore at this point of the game **B** owns the whole leg l completely. In his further moves, **B** will *defend* l, ensuring that he dominates l according to the condition of Observation 11: If **A** claims a vertex v on the path of l, then **B** claims the vertex below v if it is defined and available. If it is not available, then v is either the lowest vertex or it is above an already claimed vertex. In either case **B** can claim any available vertex. If **A** claimed a leaf belonging to a broom on l, **B** claims the neighbor of v on the path of l if it is available. Otherwise, **B** can claim any available vertex. If **A** claims a vertex not belonging to l, then **B** claims any available vertex.

- If **A** does not take the center vertex c, then **B** takes it. From now on, **B** will try to defend both legs, as in the strategy above. The problem is that **B** may be one move short in his defensive strategy, if **A** has moved to a leg in his first move. If **A** takes a vertex from the head in any of her turns (including her first move), then **B** can catch up with **A** and dominate both legs from then on. If **A** never takes a vertex from the head, then **B** can successfully defend only one leg, but he owns the whole head.

  Now we describe the strategy more precisely. There are two possibilities. Suppose **A** takes a vertex from the broom formed by h in the first turn. Then **B** claims c, and in all his forthcoming turns, **B** will defend both legs, see the strategy above. More precisely, when **A** claims a vertex from the leg l_1, then **B** defends l_1. When **A** claims a vertex from the leg l_2, then **B** defends l_2.

  Consider the other case, when **A** claims a vertex from a leg in her first turn. Then **B** claims the center c, and in all remaining turns, if **A** claims a vertex from a leg, **B** will defend that leg. If in a turn **A** claims a vertex from the broom formed by h, then **B** will claim the vertex which is right below the vertex taken by **A** in her first turn if it is defined and available. If the taken vertex by **A** in her first turn was a leaf in a broom, **B** takes the broom if available. In all other cases, **B** is free to choose any available vertex.

Analysis of the strategy: In the first case **B**'s strategy was to gain h and as much as possible from a leg. As a result of this strategy, by Observation 11, **B** ensures himself the whole leg except of those brooms in which **A** claimed a vertex. In the end of the game, **B** will control all vertices of the broom formed by h except at most t leaves and the leg l without at most t brooms.

When **B**'s strategy was to defend both legs by Observation 11 he ensures himself both legs except those brooms in which **A** claimed a vertex, which is at most t.

In the last case **B**'s strategy was to defend a leg, while he controls the large broom of h, and if **A** claimed a vertex from the broom formed by h, then **B** defended both legs. Thus either **B** obtained both legs except of those brooms in which **A** claimed a vertex, or **B** gained the broom formed by h and a leg possibly without at most t brooms.

Counting the gain: By our construction the tree contains kN vertices in the brooms of each of the three subtrees connected to the center vertex c. Hence, there are $3kN$ vertices in the brooms of the tree. In each of the three cases **B** gains at least $2kN - tN$ vertices in brooms. There are more vertices in the tree outside the brooms but we achieve that the number of those is negligible by increasing N. Therefore, **B** gets $\frac{2}{3} - \frac{t}{3k}$, and for big k this amount is close to $\frac{2}{3}$. Hence, the statement of the theorem follows. □

5 Graphs with bounded degree

In this section, we investigate when player **B** is able to obtain some positive proportion of the vertices, i.e., for a fixed $\varepsilon > 0$ we are interested in knowing for which graphs G we have $VR(G,t) \le 1-\varepsilon$. For every $\varepsilon > 0$ and t, there are certainly graphs for which $VR(G,t) > 1-\varepsilon$. For example, we could take G to be a star with more than $\frac{t}{\varepsilon}$ leaves. However if G is not allowed to have vertices of high degree, then the situation changes.

Lemma 12 *In a connected graph G with n vertices and maximum degree Δ, we have*

$$VR(G,1) \le 1 - \frac{1}{\Delta} + \frac{1}{n\Delta}.$$

Proof: Let v be the vertex chosen by player **A** on her first move, and let $x_1, \dots, x_k$ be the neighbors of v, with $k \le \Delta$. Let $H(x_i)$ be the set of vertices which are closer to x_i than to v. Obviously every vertex of G belongs to at least one $H(x_i)$. **B** picks the neighbor x for which $|H(x)|$ is largest and will control at least $|H(x)| \ge (n-1)/\Delta = n/\Delta - 1/\Delta$ vertices. This implies $VR(G,1) \le 1 - \frac{1}{\Delta} + \frac{1}{n\Delta}$. □

Combining Lemma 12 with Theorem 3 we obtain the following.

Corollary 13 *In a connected graph G with n vertices and maximum degree Δ, we have*

$$VR(G,t) \le 1 - \frac{1}{2\Delta} + \frac{1}{2n\Delta}.$$

Let $S_{k,N}$ be the spider graph formed from a star with k leaves by replacing every leaf with a path of length N. Since player **A** can always choose the center of $S_{k,N}$ on her first move, it is easy to see that $VR(S_{k,N},1) \to 1 - \frac{1}{k}$ as $N \to \infty$. This shows the bound in Lemma 12 cannot be substantially improved.

For $t \ge 2$, we were not able to determine whether the bound in Corollary 13 can be improved or not. However, we were able to find graphs which show that

the bound in Corollary 13 cannot be decreased by more than $\frac{1}{2\Delta}$, by proving the following.

Theorem 14 *For every $\Delta, t \ge 1$ and $\varepsilon > 0$, there is a connected graph G with maximum degree Δ satisfying*

$$VR(G,t) \ge 1 - \frac{1}{\Delta} - \varepsilon.$$

In order to prove Theorem 14, we first need to show that for every t, there are graphs with maximum degree 3 on which **B** can claim almost all the vertices after t rounds. We prove the following.

Lemma 15 *For every $t \ge 1$ and $\varepsilon > 0$, there is a graph $G_{t,\varepsilon}$ with maximum degree 3 and the following property: Player **B** has a strategy for the Voronoi game on $G_{t,\varepsilon}$ such that after each round $1, \dots, t$, he will control a fraction $1-\varepsilon$ of the vertices after* each *of the rounds $1, \dots, t$.*

Proof: The proof is an extension of Theorem 4. We set $d = \lceil \frac{2t}{\varepsilon} \rceil$. Instead of a hyperplane in $\mathbb{Z}^d$, we will take a full cube of side length $L = d^2 t$ from which the lowest corner has been cut off: the graph H has vertex set

$$\{ (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d \mid 0 \le x_i \le L,\ x_1 + x_2 + \dots + x_d \ge L \}.$$

Two vertices are connected in H whenever their L_1 distance is 1, i.e. they differ in one coordinate and the difference is 1. Then the distance between any two vertices equals their L_1 distance. As before, the corners C are the points $(L,0,\dots,0)$, $(0,L,0,\dots,0)$, …, $(0,\dots,0,L)$. The distance from a vertex $(x_1,\dots,x_d)$ to the j-th corner can be calculated as

$$L + \sum_{i=1}^{d} x_i - 2x_j.$$

The strategy of Theorem 4 must be adapted to account for the fact that H has additional vertices: Suppose **A** takes vertex $x = (x_1, \dots, x_d)$, and suppose w.l.o.g. that $x_1 \ge L/d = dt$ is the largest coordinate. Then **B** calculates the response point $\pi_1(x) = (x'_1, x'_2, \dots, x'_d)$, where $x'_i = \min\{x_i + 1, L\}$ for $i = 2, \dots, d$, and $x'_1 = x_1 - (d-1) \ge (t-1)d$. These formulas ensure that $x' \in H$, and one can easily show that every corner except the first is closer to x' than to x.

By an argument analogous to the proof of Theorem 4, one can find a strategy $\mathcal{S}$ for **B** in the Voronoi game on H which ensures that after each round $1, \dots, t$, there are at least $d - t$ corners c satisfying

$$\mathrm{dist}(c,B) < \mathrm{dist}(c,A),$$

where A and B are the sets of vertices chosen by **A** and **B** respectively.

To cut down the maximum degree, we use a variation of the *cube-connected cycles* of Preparata and Vuillemin [9]. We construct "grid-connected cycles".

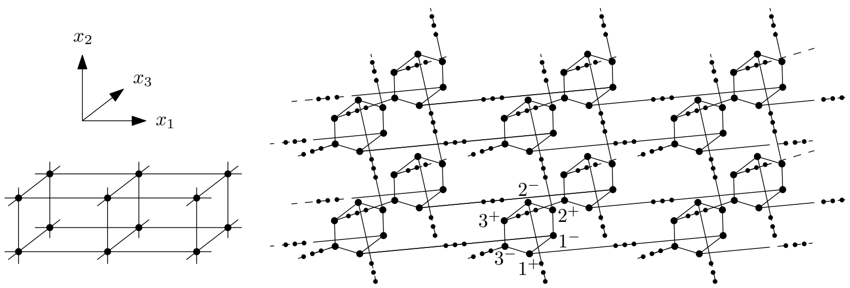

Figure 3: Representing the a $3 \times 2 \times 2$ section of the grid $\mathbb{Z}^3$ (shown on the left) by grid-connected cycles. The connection paths should have $6d - 2 = 16$ intermediate vertices, but only 3 are shown.

Each point $x \in \mathbb{Z}^d$ is replaced by a circular ring of $2d$ nodes that are labeled $1^+, 1^-, 2^+, 2^-, \ldots, d^+, d^-$ in cyclic order. We denote them by $x(1^+)$, $x(1^-)$ etc. For $i = 1, \ldots, d$, node $x(i^+)$ is connected to node $x'(i^-)$ by a *connection path* of $6d - 1$ edges, where $x' = x + e_i$ and e_i is the i-th unit vector. Figure 3 shows a three-dimensional example. The resulting graph has maximum degree 3. Nodes on the boundary of the cube have unused connections. For each corner, we pick one of these degree-2 nodes and attach a very long path of length N to it. As usual, we make N so big that the original graph becomes only a negligible fraction of the whole graph. This gives us the graph $G = G_{t,\varepsilon}$.

The following proposition implies that playing the Voronoi game on $G_{t,\varepsilon}$ is approximately the same as playing it on H. The distances are preserved up to a multiplicative factor with an additive error.

Proposition 16 *Let $x(p^\pm)$ and $y(q^\pm)$ be two vertices of G corresponding to grid points $x, y \in \mathbb{Z}^d$. Then their distance* $\operatorname{dist}(x(p^\pm), y(q^\pm))$ *in the graph is bounded as follows:*

$$6d \cdot \|x - y\|_1 - 1 \le \operatorname{dist}(x(p^\pm), y(q^\pm)) \le 6d \cdot \|x - y\|_1 + 5d$$

Proof: Lower bound. The connection paths that connect different rings correspond to neighbouring points in $\mathbb{Z}^d$. Hence the path between $x(p^\pm)$ and $y(q^\pm)$ needs at least $\|x - y\|_1$ of these connection paths. But since these paths are not directly adjacent, a path in G has has to contain at least one ring edge between any two connection paths.

Upper bound. Consider two nodes $x(p^\pm)$ and $y(q^\pm)$ that we want to connect by a path. Let $u = (u_1, \ldots, u_d)$ be the elementwise maximum of x and y: $u_i = \max\{x_i, y_i\}$. Then we have $\|x - y\|_1 = \|x - u\|_1 + \|y - u\|_1$ To get from $x(p^\pm)$ to $y(q^\pm)$, we go via the ring u. We connect $x(p^\pm)$ to the node $u((p-1)^-)$ by sequentially increasing each coordinate value $i = p, p+1, \ldots, p-1$ from x_i to u_i. This procedure works because the graph represents a subcube of $\mathbb{Z}^d$, from which some "lower" part has been removed. It is always possibly to increase a coordinate, up the maximum L.

We make possibly one initial step to $x(p^+)$. The coordinate move in direction i goes from some node $z(i^+)$ to some node $z'(i^-)$ in $6d|x_i - u_i| - 1$ steps, strictly alternating between connection paths and ring edges. One more step brings us to $z'((i+1)^+)$ to get ready for the next coordinate direction, for a total of $6d|x_i - u_i|$ steps. This bound does not work for $x_i = u_i$: there we need 2 steps from $z(i^+)$ to $z((i+1)^+)$. In total, we can bound the number of steps to at most $\sum_{i=1}^d (6d|x_i - u_i| + 2) = 6d\|x - u\|_1 + 2d$. Similarly, $y(q^\pm)$ is connected to some vertex on the ring u in at most $6d\|y - u\|_1 + 2d$, steps, and we need at most d additional steps on the ring u. □

We continue the proof of Lemma 15. Let $V' \subset V(G)$ denote the nodes on rings, and let $f\colon V' \to H \subset \mathbb{Z}^d$ denote the function which maps every node $x(i^\pm)$ to its grid point x. As a consequence of the previous proposition, for two vertices $u, v \in H$, we can recover the L_1 distance of their corresponding grid points from their distance in the graph:

$$\|f(u) - f(v)\|_1 = \left\lfloor \frac{\operatorname{dist}(u, v) + 1}{6d} \right\rfloor$$

This means that strict equalities between distances in H carry over to corresponding vertices of V'.

Player **B**'s strategy on $G_{t,\varepsilon}$ is as follows: If **A** moves to a node u on one of the rings, **B** interprets this as a move to $f(u)$ in H, calculates his response x according to the strategy $\mathcal{S}$ on H, and chooses an arbitrary node $x(q^\pm)$ on the corresponding ring. If **A** selects several nodes on the same ring, they are interpreted as wasted moves in H. If **A** plays on one of the long paths, this is interpreted as a move to the corresponding corner vertex. Finally, **A** might move to a node w on a connection path between vertices u and u' on two rings. Then $f(u)$ and $f(u')$ differ in exactly one coordinate x_j. Let us assume that $f(u)$ has the smaller x_j-coordinate. Then **B** interprets this as a move to $f(u)$ and responds as above. To analyze the error incurred by this interpretation, let us imagine that **A** had covered *both* u and u'. This would certainly be more advantageous for **A** than covering w alone. However there is only one corner which is closer to $f(u')$ than to $f(u)$: the j-th corner. All other corners are closer to $f(u)$. Thus, by allowing **A** to cover the vertex u' in addition to u, she can win at most one additional corner. It follows that after the k-th round $(1 \le k \le t)$, **A** owns at most $2k$ corners. This implies the lemma. □

We can now prove Theorem 14.

Proof: [Proof of Theorem 14.] For given ε and t, we construct the graph G from Δ disjoint copies of $G_{t,\varepsilon}$ called $G_1, \ldots, G_\Delta$ and an extra vertex v, by adding exactly one edge between v and an arbitrary vertex of G_i for each i.

On her first move **A** claims the vertex v. Subsequently **A** always claims a vertex from the same G_i as **B** in the previous move. **A** treats $G_1, \ldots, G_\Delta$ as separate games, and plays the strategy of the second player given by Lemma 15. Hence she controls at least $(1-\epsilon)|V(G_i)|$ vertices of G_i after her move. However, she cannot answer the very last move of **B**.

This ensures that at the end she controls at least $(1-\varepsilon)|V(G_i)|$ of the vertices of each G_i except for one. Since for $i \neq j$ there are no edges between G_i and G_j, Player **B** can capture at most $|G_i| = \frac{1}{\Delta}(|V(G)|-1)$ vertices on his last move. Therefore **A** controls at least at least $\left(1-\varepsilon-\frac{1}{\Delta}\right)|V(G)|$ vertices at the end of the game, proving the result. □

Remarks and acknowledgment

Several questions are left open. Are there trees T for which $VR(T,t)$ is close to $\frac{1}{4}$ or can the first player always get at least $\frac{1}{3}$ of the vertices, for $t \geq 3$? How much can she get if they play on a planar graph, e.g., on a grid? What about biased versions of the game, where the players play different amounts of pebbles?

A very interesting special case is the one-round game where the first player claims t vertices, then the second player claims one. Denote by $VR_{t:1}(G,1)$ the fraction that the first player gets after an optimal play on G and the minimum over all G for such a game by $VR_{t:1} = \inf_G VR_{t:1}(G,1)$. We know very little of $VR_{t:1}$, it is even possible that $VR_{t:1} = 0$ for every t or maybe already $VR_{2:1} > 0$. Recently it was shown, by constructing a family of functions that represent a combinatorial abstraction of the game, by David Speyer [10] that $VR_{2:1} \leq \frac{10}{21}$ and then by Sam Zbarsky [12] that $VR_{t:1} \leq \frac{t-1}{t+1}$. Is this bound sharp?

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